A Simple Control Variate Method for Options Pricing with Stochastic Volatility Models

Guo Liu, Qiang Zhao, and Guiding Gu

***Abstract*—In this paper we present a simple control variate method, for options pricing under stochastic volatility models by the risk-neutral pricing formula, which is based on the order moment of the stochastic factor Y_t of the stochastic volatility for choosing a non-random factor $Y(t)$ with the same order moment. We construct the control variate using a stochastic differential equation with a deterministic diffusion coefficient as the price process of the underlying asset. Numerical experiment results show that our method achieves better variance reduction efficiency, than that of the constant volatility control variate method, and simpler computation, than that of the martingale control variate method[4], and it has a promising wider-range application than the previous method proposed by Ma and Xu(2010)[10], and Du et al.(2013)[2].**

***Index Terms*—control variates, Monte Carlo method, options pricing, stochastic volatility.**

I. Introduction

OPTIONS pricing has been being a topic in the field of mathematical finance since Black and Scholes(1973)[1] gave the Black-Scholes formula for the European option under some perfect assumptions. However, these assumptions are not perfect suitable for the real market data. Numerous works have been carried out on relaxing the assumptions of the Black-Scholes model. For example, Merton(1973)[11], Roll(1977)[12], Geske(1979)[5], Whaley(1981)[15] priced the options with the stock paying dividend. Hull and White(1987)[8], Scott(1987)[13], Stein and Stein(1991)[14], Heston(1993)[7] priced the options with stochastic volatility models.

The increasing complexity of the models of the underlying asset renders the option valuation very difficult. In fact, there are few options which can be priced analytically. Then the numerical method is a wiser choice in options pricing. The classical numerical methods, like the lattice method (including binary tree method and ternary tree method), the finite difference method, are limited to the problems in which the number of state variables are less than there (or including three). Because the computation grows exponentially as the number of state variables increases. Monte Carlo method, for its easy and flexible computation, is suitable for the complex problems with over three state variables. But its convergence rate is slow. So Monte Carlo method is usually needed to be accelerated when it is applied to options valuation, where variance reduction method is the principle one used to accelerate Monte Carlo method, usually including antithetic method, control variate method and important sampling method.

In this paper we consider the control variate method for accelerating the Monte Carlo method to price options under stochastic volatility models. There are four kinds of control variate methods, appeared in the previous works, including: (a) the control variate method constructed by the constant volatility model, like Hull and White(1987)[8], John and Shanno(1987)[9], (b) the martingale control variate method proposed by Fouque and Han(2007)[4], (c) the control variate method combining the first and second order moment of the underlying asset proposed by Ma and Xu(2010)[10], and (d) the control variate method constructed with the order moment of the stochastic volatility proposed by Du, Liu and Gu(2013)[2]. The first method is the simplest one but with low variance reduction efficiency. The martingale method is difficult for the computation of the invariant distribution of the stochastic volatility, while the last two methods are more efficient in variance reduction and simpler than the martingale method. Here we propose a new control variate method, which is more efficient than the constant volatility method, much simpler than the martingale control variate method, and has a wider-range application than those proposed by Ma and Xu(2010), and Du et al.(2013), respectively. The idea of the new control variate method is that we derive an auxiliary process with a non-stochastic volatility which is constructed by a non-stochastic factor having the same order moment to the stochastic factor. Then we construct an instrument option by an auxiliary process with the non-stochastic volatility above as the new control variate. We deduct the new control variate method in European options and Asian options pricing with Hull-White model.

The rest of this paper is organized as follows. First we provide the new control variate method in the general options pricing under the stochastic volatility model, especially for Hull-White model(1987), Heston model(1993) and Stein-Stein model(1991). Then we compare our new control variate method with other two methods by Ma and Xu(2010), and Du et al.(2013). In Section IV we present the numerical experiences for pricing European options and Asian options with the new control variate method. Finally we give some conclusions in Section V.

II. New control variate method

In this section we present the new control variate method in the general case.

Suppose with the probability space $(\Omega, \mathcal{F}, P)$, the underlying asset price processes of the option satisfy the following stochastic differential equations (here we suppose the probability P is the risk-neutral probability measure, and

Manuscript received Apr. 13, 2014; revised Feb. 2, 2015. This work is partly supported by NSFC(11371105), Shanghai Colleges Outstanding Young Teachers Scientific Research Project(ZZCD12007), Research Innovation Foundation of Shanghai University of Finance and Economics(CXJJ-2013-323).

Guo Liu, and Qiang Zhao are with School of Finance, Shanghai University of Finance and Economics, Shanghai, SH, 200433 CH e-mail: (Email: guoliu819@163.com, zqpku@126.com).

Guiding Gu is with the Department of Applied Mathematics, Shanghai University of Finance and Economics.

ignore the market price of the volatility risk)

$$\begin{aligned} dS_t &= S_t(rdt + \sigma_t dW_{1t}), \\ \sigma_t &= f(Y_t), \\ dY_t &= \alpha(Y_t)dt + \beta(Y_t)dW_{2t}, \end{aligned} \tag{1}$$

where $S_0 = s_0$, $Y_0 = y_0$, r is a constant, both W_{1t} and W_{2t} are standard Brownian motions, which satisfy $cov(dW_{1t}, dW_{2t}) = \rho dt$, that means we can get $W_{2t} = \rho W_{1t} + \sqrt{1-\rho^2}W_t$, where W_t is a standard Brownian motion and it is independent with W_{2t}.

The new control variate method is presented as follows. First we construct an auxiliary process $S(t)$ satisfying

$$\begin{aligned} dS(t) &= S(t)(rdt + \sigma(t)dW_{1t}), \\ S(0) &= s_0, \end{aligned} \tag{2}$$

where r, W_{1t}, and s_0 are the same as (1). $\sigma(t)$ is a non-stochastic and square-integrable function, which is different with σ_t. A good control variate for an option pricing must be as close as possible to the option. Here the problem becomes how we can choose $\sigma(t)$ as close as possible to σ_t, to make $S(t)$ be closer to S_t. Here we first choose the non-random factor $Y(t)$ such that

$$Y^m(t) = E[Y_t^m],$$

where $m \in R$, R is the real number set. Then replacing Y_t with $Y(t)$ in the σ_t, we have

$$\sigma(t) = f(Y(t)), \tag{3}$$

The auxiliary process becomes

$$dS(t) = S(t)(rdt + f(Y(t))dW_{1t}). \tag{4}$$

Finally the option based on the underlying asset with the auxiliary process is the new control variate, which can be priced analytically.

Several popular stochastic volatility models are collected as follows.

TABLE I
MODELS OF STOCHASTIC VOLATILITY

Model	$f(y)$	Y_t process	correlation
Hull-White(1987)	$\sqrt{y}$	lognormal	ρ=0
Scott(1987)	e^y	Mean-reverting O-U	ρ=0
Stein-Stein(1991)	$\lvert y\rvert$	Mean-reverting O-U	ρ=0
Ball-Roma(1994)	$\sqrt{y}$	CIR process	ρ=0
Heston(1993)	$\sqrt{y}$	CIR process	$\rho \neq 0$

It is worthy to be mentioned that the stochastic factors in all stochastic volatility models satisfy only three kinds of processes as listed in Table I(some multi-factors stochastic volatility models are also driven by these processes). Their expectations for these stochastic factors can be easily obtained. Here, we apply our aforementioned method, for options pricing with these stochastic volatility models, which can achieve more variance reduction ratios than the control variate method of constant volatility, and can have a potentially wider application due to its simpler implementation compared with the methods proposed by Ma and Xu(2010), Du et al.(2013), Fouque and Han(2007). Therefore, this aforementioned control variate method will be applied to pricing European options and Asian options with the most typical stochastic volatility model including Hull-White model, Heston model and Stein-Stein model in the following subsections.

A. Hull-White model

The Hull-White stochastic volatility model is first proposed by Hull and White(1987), which provides the closed form price formula of European option with the Hull-White stochastic volatility, just when the correlation coefficient between the underlying asset price and the stochastic factor of the volatility is zero. The model is

$$\begin{aligned} \sigma_t &= \sqrt{Y_t}, \\ dY_t &= Y_t(\mu dt + \sigma dW_{2t}), \end{aligned} \tag{5}$$

where μ and σ are constant. Then we can easily derive

$$E[Y_t^m] = y_0^m \exp\{mt(\mu + \frac{1}{2}(m-1)\sigma^2)\}. \tag{6}$$

According to the new control variate method, we choose $Y(t)$ such that

$$E[Y^m(t)] = E[Y_t^m],$$

that is

$$Y(t) = (E[Y_t^m])^{\frac{1}{m}} = y_0 \exp\{t(\mu + \frac{1}{2}(m-1)\sigma^2)\}. \tag{7}$$

Then we derive the deterministic volatility

$$\sigma(t) = \sqrt{Y(t)} = y_0^{\frac{1}{2}} \exp\{\frac{1}{2}t(\mu + \frac{1}{2}(m-1)\sigma^2)\}. \tag{8}$$

B. Heston model

The Heston stochastic volatility model is first presented by Heston(1993), which prices the European option analytically. But the representation is very difficult to calculate the accurate price. Then accelerated Monte Carlo method is the most useful one to price options. The model is

$$\sigma_t = \sqrt{Y_t}, \tag{9}$$

$$dY_t = k(\theta - Y_t)dt + \sigma\sqrt{Y_t}dW_{2t}, \tag{10}$$

where k, θ, and σ are constant. It is difficult to derive the closed formula solution for Y_t, but we can derive its expectation, that is the first order moment

$$E[Y_t] = e^{-kt}y_0 + \theta(1 - e^{-kt}), \tag{11}$$

and the m-th order moment $E[Y_t^m]$ by the $m-1$, $m-2$,...,1-th order moment. We omit them here for simplicity. Then we have

$$\sigma(t) = \sqrt{E[Y_t]} = \sqrt{e^{-kt}y_0 + \theta(1 - e^{-kt})}. \tag{12}$$

C. Stein-Stein model

The Stein-Stein model is proposed by Stein and Stein(1991). The model is

$$\begin{aligned} \sigma_t &= |Y_t|, \\ dY_t &= \alpha(\beta - Y_t)dt + \sigma dW_{2t}, \end{aligned} \tag{13}$$

where α, β and σ are constant.

Then we can easily have

$$E[Y_t] = e^{-\alpha t}y_0 + \beta(1 - e^{-\alpha t}).$$

By the new control variate method, we choose $Y(t)$ as

$$E[Y(t)] = E[Y_t],$$

that is

$$\sigma(t) = |Y(t)| = |e^{-\alpha t} y_0 + \beta(1 - e^{-\alpha t})|. \tag{14}$$

Theorem 1. *Suppose that the stochastic volatility σ_t in (1) is replaced by a deterministic square-integrable volatility $\sigma(t) = f(Y(t))$, there is an analytic solution for European put option,*

$$\begin{aligned} X_p|_{t=0} &= e^{-rT} E[(K - S(T))^+] \\ &= e^{-rT} K N(d_1) - s_0 N(d_1 - b), \end{aligned} \tag{15}$$

where

$$d_1 = \frac{\ln K - a}{b}, \tag{16}$$

$$a = \ln s_0 + rT - \frac{1}{2}\int_0^T \sigma^2(t)dt, \quad b = \sqrt{\int_0^T \sigma^2(t)dt}. \tag{17}$$

For the Hull-White model, the non-random volatility is (8), and the value of the European put option as the control variate is as follows

$$V_p|_{t=0} = K e^{-rT} N(d_1) - s_0 N(d_1 - b),$$

where

$$d_1 = \frac{\ln k - a}{\sqrt{b}}, \tag{18}$$

$$a = \ln s_0 + rT + b, \quad b = y_0 \frac{e^{ct} - 1}{c}, \tag{19}$$

$$c = \mu + \frac{1}{2}(m-1)\sigma^2. \tag{20}$$

For the Heston model, the non-random volatility is (12), and the value of the European option as the control variate is as follows

$$V_p|_{t=0} = K e^{-rT} N(d_1) - s_0 N(d_1 - b),$$

where

$$d_1 = \frac{\ln k - a}{\sqrt{b}}, \tag{21}$$

$$a = \ln s_0 + rT - \frac{b}{2}, \tag{22}$$

$$b = \theta T + \frac{1}{k}(y_0 - \theta)(1 - e^{-kT}). \tag{23}$$

For the Stein-Stein model, the non-random volatility is (14), and the value of the European option as the control variate is as follows

$$V_p|_{t=0} = K e^{-rT} N(d_1) - s_0 N(d_1 - b), \tag{24}$$

where

$$\begin{aligned} d_1 &= \frac{\ln K - a}{\sqrt{b}}, \\ b &= \beta^2 + 2\beta(y_0 - \beta)^2 \frac{e^{-\alpha T} - 1}{-\alpha} + (y_0 - \beta)^2 \frac{e^{-2\alpha T} - 1}{-2\alpha}, \\ a &= \ln s_0 + rT - \frac{b}{2}. \end{aligned}$$

III. Comparing with other two control variate methods

In this section we will compare the new control variate method with other two control variate methods, including the control variate constructed from the m-th order moment($m \in R$) of the stochastic volatility σ_t by Du, Liu and Gu(2013), and the control variate constructed from the second order moment of the underlying asset price S_t by Ma and Xu(2010), which are called as Method 1 and Method 2, respectively.

A. Method 1

This method is presented by Du, Liu and Gu(2013), which gives a class of control variates for Asian options with fixed strike price and floating strike price. They also used this method for multi-asset options pricing[3]. Here for comparing it with our new method, we price the European option with stochastic volatility models using Method 1.

First we choose $\sigma(t)$ such that

$$\sigma^m(t) = E[\sigma_t^m],$$

where $m \in R$. The the control variate is the option that based on the underlying asset price satisfying $S(t)$ with the non-random volatility $\sigma(t)$.

For the Hull-White stochastic volatility model, we have

$$\begin{aligned} E[\sigma_t^m] &= E[Y_t^{\frac{m}{2}}] \\ &= E[Y_0^{\frac{m}{2}} \exp\{\frac{mt}{2}(\mu - \frac{1}{2}\sigma^2) + \frac{1}{2}m\sigma W_{2t}\}] \\ &= Y_0^{\frac{m}{2}} \exp\{\frac{m}{2}t(\mu + \frac{1}{4}(m-2)\sigma^2)\}. \end{aligned}$$

Then we can choose $\sigma(t)$ such that

$$\begin{aligned} E[\sigma^m(t)] &= E[\sigma_t^m] = E[Y_t^{\frac{m}{2}}], \\ \sigma(t) &= Y_0^{\frac{1}{2}} \exp\{\frac{1}{2}t(\mu + \frac{1}{4}(m-2)\sigma^2)\}. \end{aligned} \tag{25}$$

This is similar to our new control variate method for calculating $E[Y_t^{\frac{m}{2}}]$ first. It is easy to see that for European options with the Hull-White model, the non-random volatility constructed from $2m$-order moment of the stochastic volatility using Method 1 is equal to that constructed from m-order moment of the stochastic factor by our new control variate method.

For the Heston model, we cannot derive the first order moment of the stochastic volatility σ_t, but the second order moment.

$$E[\sigma_t^2] = E[Y_t] = e^{-kt} y_0 + \theta(1 - e^{-kt}). \tag{26}$$

Then we choose $\sigma(t)$ such that $E[\sigma^2(t)] = E[\sigma_t^2]$, that is

$$\begin{aligned} E[\sigma^2(t)] &= E[\sigma_t^2], \\ \sigma(t) &= \sqrt{e^{-kt} y_0 + \theta(1 - e^{-kt})}. \end{aligned} \tag{27}$$

This is the same as that by the first order moment of the stochastic factor with our new method. It is easy to get the $2n$-th order moment of σ_t, where n is any non-zero positive integer. We know that they are the same as that by the n-th order moment of Y_t with our new method.

For the Stein-Stein model, we can calculate the first moment of the stochastic volatility,

$$E[|Y_t|] = \frac{2\varrho}{\sqrt{2\pi}} \exp\{-\frac{\nu^2}{2\varrho^2}\} + \nu - 2\nu\Phi(-\frac{\nu}{\varrho}),$$

where

$$\nu = \beta + (y_0 - \beta)e^{-\alpha t},$$
$$\varrho^2 = \frac{1 - e^{-2\alpha t}}{2\alpha}\beta^2.$$

Then we choose $\sigma(t) = E[|Y_t|]$. Unfortunately, we cannot price the European option price analytically with the underlying asset price $S(t)$, with this deterministic volatility $\sigma(t)$. That is to say we cannot use Method 1 to accelerate Monte Carlo method for pricing the option with the Stein-Stein model.

B. Method 2

This method is proposed by Ma and Xu(2010) when they priced variance swaps by control variate Monte Carlo method. However, they just considered the first two order moments for choosing a control variate. Here, we extend it to $\forall m \in R$, and apply it to pricing European option under stochastic volatility models.

First we calculate

$$S(t) = E[S_t], \tag{28}$$
$$S^2(t) = E[S_t^2]. \tag{29}$$

Then we choose $\sigma(t)$ such that $S(t) = E[S_t]$, and $S^2(t) = E[S_t^2]$. Finally the auxiliary process $S(t)$ is obtained for the underlying asset of the control variate option.

For the Hull-White model, we can derive the m-th order of the underlying asset price S_t with the stochastic volatility σ_t.

$$\begin{aligned} &E[S_t^m] \\ &= E[s_0^m \exp\{mrt - \frac{m}{2}\int_0^t \sigma_s^2 ds + m\int_0^t \sigma_s dW_{1s}\}] \\ &= E[s_0^m e^{mrt} \exp\{-\frac{m}{2}\int_0^t Y_s ds + m\int_0^t \sqrt{Y}_s dW_{1s}\}] \\ &\approx E[s_0^m e^{mrt} \exp\{-\frac{m}{2}\int_0^t Y_s ds + m^2\int_0^t Y_s ds\}] \end{aligned} \tag{30}$$

where the first $\approx$ is obtained by $\int_0^t \sigma_s dW_{1s} \approx \int_0^t \sigma_s^2 ds$, the second one by $Y_t \approx E[Y_t]$.

We do the same to the auxiliary process $S(t)$ with non-random volatility $\sigma(t)$,

$$\begin{aligned} &E[S^m(t)] \\ &= E[s_0^m \exp\{mrt - \frac{m}{2}\int_0^t \sigma^2(s)ds + m\int_0^t \sigma(s)dW_{1s}\}] \\ &= s_0^m e^{mrt} E[\exp\{-\frac{m}{2}\int_0^t \sigma^2(s)ds + m\int_0^t \sigma(s)dW_{1s}\}] \\ &= s_0^m e^{mrt} E[\exp\{-\frac{m}{2}\int_0^t \sigma^2(s)ds + m^2\int_0^t \sigma^2(s)ds\}] \end{aligned} \tag{31}$$

Then we derive

$$\sigma(t) = Y_0^{\frac{1}{2}} \exp\{\frac{1}{2}t(\mu - \frac{1}{4}\sigma^2)\}, \tag{32}$$

by $E[S^m(t)] = E[S_t^m]$. This is the case when $m = 1$ as that by Method 1.

For the Heston model, we know that

$$\begin{aligned} E[S_t] &= s_0 \exp\{rt\}, \\ E[S_t^2] &= E[s_0^2 \exp\{2rt - \int_0^t \sigma_s^2 ds + 2\int_0^t \sigma_s dW_{1s}\}] \\ &= S0^2 e^{2rt} E[\exp\{-\int_0^t \sigma_s^2 ds + 2\int_0^t \sigma_t dW_{1s}\}] \\ &= s_0^2 e^{2rt} E[\exp\{-\int_0^t Y_s ds + 2\int_0^t \sqrt{Y_s} dW_{1s}\}] \\ &\approx s_0^2 e^{2rt} E[\exp\{-\int_0^t E[Y_s]ds + 2\int_0^t Y_s ds\}] \\ &\approx s_0^2 e^{2rt} E[\exp\{-\int_0^t E[Y_s]ds + 2\int_0^t E[Y_s]ds\}] \\ &= s_0^2 e^{2rt} E[\exp\{\int_0^t E[Y_s]\}]ds, \end{aligned}$$

and

$$\begin{aligned} E[S(t)] &= E[s_0 \exp\{rt - \frac{1}{2}\int_0^t \sigma^2(s)ds + \int_0^t \sigma(s)dW_{1s}\}] \\ &= s_0 e^{rt}, \\ E[S^2(t)] &= E[s_0^2 \exp\{2rt - \int_0^t \sigma^2(s)ds + 2\int_0^t \sigma(s)dW_{1s}\}] \\ &= s_0^2 e^{2rt} E[\exp\{-\int_0^t \sigma^2(s)ds + 2\int_0^t \sigma(s)dW_{1s}\}] \\ &= s_0^2 e^{2rt} \exp\{\int_0^t \sigma^2(s)ds\}. \end{aligned} \tag{33}$$

Then we can have

$$\sigma(t) = \sqrt{E[Y_t]} \tag{34}$$

by $E[S^2(t)] = E[S_t^2]$.

From the above analysis, we can see that the final step in Method 1 is to get non-random volatility $\sigma(t)$ by calculating $E[Y_t]$, which is the only one step in our new method.

For Stein-Stein model, we can derive the same deterministic volatility as that by Method 1, with Method 2, then we cannot apply Method 2 to price options with the Stein-Stein model.

We can see that it is difficult to derive the exact expression of the m-th order moment for the underlying asset price process even with non-random volatility. Just as mentioned by Ma and Xu(2010), we get the auxiliary underlying asset price process by some approximations. The control variate by Method 2 with the Hull-White model, or the Heston model, is the special case as that by our new method and Method 1.

IV. Numerical Experiment

In last section, the control variate constructed by Method 1 and Method 2 can also be derived by our new control variate method, which is much simpler than any one of them. Then we just give the experiments of our new control variate method to show the variance reduction efficiency in options pricing, including European put option and Asian option.

From Glasserman(2004)[6] we know that the variance reduction ratio(the variance by the ordinary Monte Carlo method to that by control variate Monte Carlo method) is used to illustrate the accelerating efficiency of our new method. The greater the variance reduction ratio is, the faster convergence rate for Monte Carlo method in options pricing is. Just as done in Ma and Xu(2010), we ignore the simulation time of the control variate, because it is neglectable comparing with that by the ordinary Monte Carlo method.

A. European option pricing

This experiment gives the variance reduction ratios which the ratios are between the variance of the European put option price by the new control variate Monte Carlo method and that by ordinary Monte Carlo method. In the following numerical experiment results, CV is the option price of the control variate chosen with our new method, MC is the price of European option with ordinary Monte Carlo method, the standard deviation of the estimator is denoted as STD1. MC+CV is the option price with new control variate Monte Carlo method, the standard deviation of the estimator is STD2. The variance reduction ratio denoted by $\hat{R}$, which is the square of the ratio of STD1 to STD2, SteinNew is the option price given by Stein and Stein(1991).

1) Hull-White model: The parameters in the model are set as follows, $r = 0.05$, $y_0 = 0.02$, $\mu = 0.02$, $K = 40$, $NSim = 10^5$, $s_0 \in [34, 50]$, $\rho \in [-1, 1]$, $m \in [-75, 75]$. In Table II, $\rho = 0$, $m = 1$; in Table III, $s_0 = 40$, $m = 1$; in Table IV, $s_0 = 40$, $\rho = 0$.

TABLE II
VARIANCE REDUCTION RATIO BY NEW CONTROL VARIATE METHOD WITH DIFFERENT s_0

s_0	CV	MC	STD1	MC+CV	STD2	$\hat{R}$
34	4.6344	4.6684	0.0122	4.6417	0.0003	2224.63
36	3.2329	3.2812	0.0109	3.2421	0.0003	1769.87
40	1.3263	1.3802	0.0076	1.3360	0.0003	709.40
44	0.4368	0.4735	0.0044	0.4436	0.0003	268.38
46	0.2326	0.2601	0.0032	0.2375	0.0002	172.30
50	0.0579	0.0704	0.0016	0.0601	0.0002	79.75

TABLE III
VARIANCE REDUCTION RATIO BY NEW CONTROL VARIATE METHOD WITH DIFFERENT ρ

ρ	CV	MC	STD1	MC+CV	STD2	$\hat{R}$
-1	1.3263	1.4431	0.0079	1.3966	0.0001	8634.39
-0.6	1.3263	1.4179	0.0078	1.3724	0.0002	1092.58
-0.1	1.3263	1.3865	0.0076	1.3421	0.0003	720.26
0	1.3263	1.3802	0.0076	1.3360	0.0003	709.40
0.1	1.3263	1.3739	0.0075	1.3300	0.0003	711.43
0.6	1.3263	1.3426	0.0073	1.2998	0.0002	1005.99
1	1.3263	1.3175	0.0072	1.2756	0.0001	5846.07

The results in Table II-IV show that our new control variate method has good variance reduction efficiency for European options pricing under the Hull-White model. The variance ratios vary as different parameters change. For European put option, the greater initial price of the stock is, the greater the variance reduction ratio is. The absolute of the relative coefficient between the stock and the stochastic volatility

TABLE IV
VARIANCE REDUCTION RATIO BY NEW CONTROL VARIATE METHOD WITH DIFFERENT m

m	CV	MC	STD1	MC+CV	STD2	$\hat{R}$
-75	1.1555	1.3786	0.0076	1.3465	0.0006	145.50
-50	1.2088	1.3786	0.0076	1.3415	0.0006	161.35
-10	1.2999	1.3786	0.0076	1.3357	0.0006	176.99
0	1.3239	1.3786	0.0076	1.3346	0.0006	178.19
1	1.3263	1.3786	0.0076	1.3345	0.0006	178.24
2	1.3287	1.3786	0.0076	1.3344	0.0006	178.28
5	1.3361	1.3786	0.0076	1.3341	0.0006	178.33
10	1.3484	1.3786	0.0076	1.3337	0.0006	178.17
75	1.5204	1.3786	0.0076	1.3311	0.0006	153.39

increase, the ratio increases. The smaller the order number m is, the greater the variance reduction ratio is.

2) Heston Model: Just as Heston(1993) and Knoch(1992), we set the parameters in the model as follows, $K = 100$, $r = 0$, $y_0 = 0.01$, $k = 2$, $\theta = 0.01$, $M = 10^5$, $M = 50$. In Table V: $\rho = 0$, $\sigma = 0.1$, $T = 0.5$; in Table VI: $s_0 = 100$, $\sigma = 0.1$, $T = 0.5$; in Table VII: $s_0 = 100$, $\rho = 0$, $T = 0.5$; in Table VIII: $s_0 = 100$, $\rho = 0$, $\sigma = 0.1$.

TABLE V
VARIANCE REDUCTION RATIO BY NEW METHOD WITH DIFFERENT INITIAL STOCK PRICES s_0

s_0	CV	MC	STD1	MC+CV	STD2	$\hat{R}$
90	10.2010	10.2206	0.0187	10.2227	0.0038	569.63
100	2.8204	2.7903	0.0126	2.8065	0.0032	252.49
110	0.3046	0.3139	0.0042	0.3227	0.0020	21.89

TABLE VI
VARIANCE REDUCTION RATIO BY NEW METHOD WITH DIFFERENT ρ

ρ	CV	MC	STD1	MC+CV	STD2	$\hat{R}$
-1	2.8204	4.0099	0.0163	4.0314	0.0024	44.98
-0.6	2.8204	3.5161	0.0149	3.5355	0.0030	24.76
-0.1	2.8204	2.9098	0.0130	2.9265	0.0032	16.71
0	2.8204	2.7903	0.0126	2.8065	0.0032	15.89
0.1	2.8204	2.6713	0.0122	2.6869	0.0031	15.25
0.6	2.8204	2.0840	0.0099	2.0968	0.0026	14.73
1	2.8204	1.6214	0.0079	1.6318	0.0015	26.48

TABLE VII
VARIANCE REDUCTION RATIO BY NEW METHOD WITH DIFFERENT σ

σ	CV	MC	STD1	MC+CV	STD2	$\hat{R}$
0.01	2.8204	2.8171	0.0125	2.8336	0.0015	68.24
0.05	2.8204	2.8111	0.0125	2.8276	0.0020	37.84
0.1	2.8204	2.7903	0.0126	2.8065	0.0032	15.89
0.15	2.8204	2.7557	0.0126	2.7715	0.0044	8.21
0.2	2.8204	2.7101	0.0127	2.7253	0.0057	5.03
0.25	2.8204	2.6559	0.0128	2.6703	0.0069	3.50

TABLE VIII
VARIANCE REDUCTION RATIO BY NEW METHOD WITH DIFFERENT T

T	CV	MC	STD1	MC+CV	STD2	$\hat{R}$
0.25	1.9945	1.9797	0.0090	1.9911	0.0019	21.68
0.5	2.8204	2.7903	0.0126	2.8065	0.0032	15.89
0.75	3.4539	3.4124	0.0153	3.4323	0.0041	15.82
1	3.9878	3.9383	0.0175	3.9615	0.0049	12.80
1.5	4.8830	4.8243	0.0211	4.8531	0.0061	11.83

TABLE IX
VARIANCE REDUCTION RATIO BY NEW METHOD WITH DIFFERENT k

k	CV	MC	STD1	MC+CV	STD2	$\hat{R}$
1	2.8204	2.7797	0.0126	2.7958	0.0035	12.88
2	2.8204	2.7903	0.0126	2.8065	0.0032	15.89
4	2.8204	2.8019	0.0125	2.8182	0.0027	21.39
8	2.8204	2.8107	0.0125	2.8270	0.0023	30.00
14	2.8204	2.8145	0.0125	2.8309	0.0020	38.42

The results in Table V-IX show that our new control variate method has good variance reduction efficiency for European options pricing with the Heston model. The variance reduction ratios vary as different parameters changes, which is the same as that under Hull-White model.

3) Stein-Stein Model: The parameters in the model are set as Stein and Stein(1989), $r = 0.095$, $s_0 = 100$, $y_0 = 0.1$, $K = 100$, $NSim = 10^5$, $N = 50$. $\rho \in [-1, 1]$; $\alpha \in [4, 20]$, $\beta \in [0.20, 0.35]$, $\sigma \in [0.15, 0.40]$.

In Table X, $\alpha = 4$, $\beta = 0.2$, $\rho = 0$, $\sigma = 0.1$, $T = 0.5$; in Table XI, $\alpha = 4$, $\beta = 0.2$, $\sigma = 0.1$, $T = 0.5$, $K = 100$; in Table XII, $\alpha = 4$, $\beta = 0.2$, $\rho = 0$, $T = 0.5\rho = 0$; in Table XIII, $\alpha = 4$, $\beta = 0.2$, $\sigma = 0.1$, $\rho = 0$, $K = 100$; in Table XIV, $\beta = 0.2$, $\sigma = 0.1\rho = 0$, $T = 0.5$, $K = 100$.

TABLE XI
VARIANCE REDUCTION RATIO BY NEW CONTROL VARIATE METHOD WITH DIFFERENT ρ

ρ	CV	MC	STD1	MC+CV	STD2	$\hat{R}$
-1	8.1417	5.2407	0.0232	5.3362	0.0035	44.43
-0.6	8.1417	6.3112	0.0272	6.4230	0.0047	33.37
-0.1	8.1417	7.7414	0.0321	7.8733	0.0055	34.18
0	8.1417	8.0401	0.0331	8.1761	0.0056	35.21
0.1	8.1417	8.3434	0.0340	8.4833	0.0056	36.53
0.6	8.1417	9.9258	0.0388	10.0859	0.0056	48.75
1	8.1417	11.2699	0.0425	11.4459	0.0049	74.72

TABLE XII
VARIANCE REDUCTION RATIO BY NEW CONTROL VARIATE METHOD WITH DIFFERENT σ

σ	CV	MC	STD1	MC+CV	STD2	$\hat{R}$
0.01	8.1417	8.0061	0.0326	8.1421	0.0006	3403.46
0.1	8.1417	8.0401	0.0331	8.1761	0.0056	35.21
0.15	8.1417	8.0844	0.0336	8.2201	0.0083	16.21
0.2	8.1417	8.1468	0.0343	8.2824	0.0111	9.58

TABLE XIII
VARIANCE REDUCTION RATIO BY NEW CONTROL VARIATE METHOD WITH DIFFERENT T

T	CV	MC	STD1	MC+CV	STD2	$\hat{R}$
1/12	2.7110	2.6765	0.0118	2.7152	0.0012	90.80
0.25	5.2292	5.1642	0.0219	5.2453	0.0033	45.54
0.5	8.1417	8.0401	0.0331	8.1761	0.0056	35.21
0.75	10.6700	10.5332	0.0423	10.7208	0.0075	32.20
1	12.9921	12.8188	0.0505	13.0568	0.0091	30.83

The results in Table X-XIII show that our new control variate method has good variance reduction for European option pricing with Stein-Stein model. The smaller the initial stock price, smaller σ is, and the smaller life time of the option is, the greater the variance reduction ratio is. The greater the absolute of the coefficient ρ is, the greater the ratio is.

TABLE XIV
VARIANCE REDUCTION RATIO BY NEW CONTROL VARIATE METHOD WITH DIFFERENT α

α	CV	MC	STD1	MC+CV	STD2	$\hat{R}$
4	8.1417	8.0401	0.0331	8.1761	0.0056	35.21
8	8.1417	8.0287	0.0329	8.1646	0.0043	58.86
14	8.1417	8.0205	0.0327	8.1563	0.0034	94.82
16	8.1417	8.0188	0.0327	8.1546	0.0032	106.82
20	8.1417	8.0162	0.0327	8.1520	0.0029	130.78
100	8.1417	8.0065	0.0326	8.1424	0.0013	608.63

B. Asian option pricing

Theorem 2. *Suppose that the stochastic volatility σ_t in (1) is replaced by a deterministic square-integrable volatility $\sigma(t)$, there is an analytic solution for the fixed-strike continuous sampling geometric average Asian (call) option,*

$$\begin{aligned} X_{1cGAO}|_{t=0} &= E[e^{-rT}(X_{1cGAO}|_{t=T})] \\ &= e^{-rT}E[(e^{\frac{1}{T}\int_0^T logS(t)dt} - K)^+] \\ &= e^{\frac{1}{2}\widehat{\sigma}^2 - rT + a}N(d_+) - Ke^{-rT}N(d_-), \end{aligned} \tag{35}$$

where

$$a = \log S_0 + \frac{1}{2}rT - \frac{1}{2T}\int_0^T[\int_0^t \sigma^2(s)ds]dt,$$

$$\widehat{\sigma}^2 = \lim_{n\to\infty}\frac{1}{n^2}\sum_{j=1}^{n}[2(n-j)+1]\int_0^{j\frac{T}{n}}\sigma^2(s)ds,$$

and $d_- = \frac{a - \log K}{\widehat{\sigma}}$, $d_+ = d_- + \widehat{\sigma}$.

For Hull-White model, the option value as the control variate is as follows

$$a = \begin{cases} \log S_0 + \frac{1}{2}rT - \frac{1}{4}\sigma_0^2 T, & \text{if } a_m = 0 \\ \log S_0 + \frac{1}{2}rT & \\ -\frac{\sigma_0^2}{2Ta_m}[\frac{1}{a_m}(e^{a_m T} - 1) - T], & \text{if } a_m \neq 0 \end{cases} \tag{36}$$

$$\widehat{\sigma}^2 = \begin{cases} \frac{1}{3}\sigma_0^2 T, & \text{if } a_m = 0 \\ \frac{2\sigma_0^2}{T^2 a_m^3}(e^{a_m T} - 1) - \frac{2\sigma_0^2}{Ta_m^2} - \frac{\sigma_0^2}{a_m}, & \text{if } a_m \neq 0 \end{cases} \tag{37}$$

where $a_m = \mu + \frac{1}{2}(m-1)\sigma^2$.

This experiment gives the standard deviation reduction ratios, which square are variance reduction ratios, when X_{1cGAO} is used as the control variate for continuous sampling Arithmetic average or Geometric average Asian option. The parameters in the model are set as follows: $T = 1$, $n = 100$, $N = 50$, $r = 0.05$, $\mu = 0.05$, $s_0 = 100$, $\sigma = 0.01$, $y_0 = \sigma_0^2 = 0.15^2$, $p = 10000$. We give the standard deviation reduction ratios when m, ρ, K vary.

The data in Table XV show that our new control variate method has good variance reduction efficiency for Asian options pricing, and X_{1cGAO} has better variance reduction ratios for V_{1cGAO} than that for V_{1cAAO}. For both options, the greater strike prices(call options), the greater variance reduction ratios. When $m = 0$, the variance reduction ratio is greater than that in any other cases. The greater the order number m is, the less the variance reduction ratio is. When $m = 1 - \frac{2\mu}{\sigma^2}$, that is the case for Method 2, which the variance ratio is the least one.

TABLE X
VARIANCE REDUCTION RATIO BY NEW CONTROL VARIATE METHOD WITH DIFFERENT K

K	CV	MC	STD1	MC+CV	STD2	$\hat{R}$	SteinNew
90	15.1179	15.0109	0.0406	15.1587	0.0062	42.95	15.16
95	11.3422	11.2373	0.0373	11.3813	0.0059	40.18	11.38
100	8.1417	8.0401	0.0331	8.1761	0.0056	35.21	8.18
105	5.5836	5.4965	0.0282	5.6187	0.0053	28.70	5.62
110	3.6583	3.5886	0.0233	3.6981	0.0050	20.07	3.69

TABLE XV
THE STANDARD DEVIATION REDUCTION RATIO BY USING X_{1cGAO} AS THE CONTROL VARIATE FOR V_{1cGAO} AND V_{1cAAO}

		m=-25	m=0	m=1	m=2	m=50	m=$1-\frac{2\mu}{\sigma^2}$
V_{1cGAO}	K=90	425.0241	422.7731	422.6536	422.5319	414.2153	174.1980
	K=100	379.4825	376.7030	376.5696	376.4345	368.1058	167.0983
	K=110	247.2158	247.3690	247.3625	247.3551	245.8597	112.1999
V_{1cAAO}	K=90	52.6871	52.8379	52.8439	52.8499	53.1350	46.1588
	K=100	46.4262	46.6143	46.6218	46.6294	46.9910	39.4808
	K=110	26.1005	26.2199	26.2247	26.2295	26.4615	22.1204

V. CONCLUSIONS

In this paper, we present a new simple control variate method for instruments pricing with stochastic volatility models. Our idea is using a deterministic volatility $\sigma(t)$ to replace the stochastic volatility σ_t by choosing the factor $Y(t)$ with the same order moment as that of the stochastic factor Y_t. Numerical experiments report that our new control variate works quite well in that the variance reduction ratio $\hat{R}$ and the ratio is obviously better than one formed by the constant volatility which $m = 1 - \frac{2\mu}{\sigma^2}$. This method is much easier in computing than that of Method 1, Method 2, and the martingale control variate method. In addition, our new control variate method has a promising wider-range application and can be extended to any other stochastic volatility models in options pricing, or other financial instruments pricing.

ACKNOWLEDGMENT

The author would like to thank the anonymous reviewers very much for their valuable suggestions on improving this paper.

REFERENCES

[1] F. Black and M. Scholes, "The Valuation of Options and Corporate Liabilities," *Journal of Political Economy*, 637-654, 1973.

[2] K. Du, G. Liu and G. Gu, "A Class of Control Variates for Pricing Asian Options under Stochastic Volatility Models," *IAENG International Journal of Applied Mathematics*, 43(2), 45-53, 2013.

[3] K. Du, G. Liu and G. Gu, "Accelerating Monte Carlo Method for Pricing Multi-asset Options under Stochastic Volatility Models," *IAENG International Journal of Applied Mathematics*, 44(2), 62-70, 2014.

[4] J.P. Fouque and C.H. Han, "A Martingale Control Variate Method for Option Pricing with Stochastic Volatility," *Probability and Statistics*, 2007.

[5] R. Geske, "The Valuation of Compound Options," *Journal of Financial Economics*, 7, 63-81, 1979.

[6] P. Glasserman, *Monte Carlo Methods in Financial Engineering*, Springer, New York, 2004.

[7] S.L. Heston, "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options," *The Review of Financial Studies*, 6, 327-343, 1993.

[8] J. Hull and A. White, "The Pricing of Options on Assets with Stochastic Volatilities," *Journal of Finance*, 42, 281-300, 1987.

[9] H. Johnson and D. Shanno, "Option pricing when volatility is changing," *Journal of Financial and Quantitative Analysis*, 22, 143-152, 1987.

[10] J.M. Ma, and C.L. Xu, "An Efficient Control Variate Method for Pricing Variance Derivatives," *Journal of Computational and Applied Mathematic*, 235, 108-119, 2010.

[11] R. Merton, "The theory of rational option pricing," *Journal of Economics and Management Science*, 4, 141-183, 1973.

[12] R. Roll, "An Analytical Valuation Formula for Unprotected American Call Options on Stocks with Known Dividends," *Journal of Financial Economics*, 5, 251-258, 1977.

[13] L.O. Scott, "Option Pricing when the Variance Change Randomly: Theory, Estimation, and an Application," *Journal of Financial and Quantitative Analysis*, 22, 419-438, 1987.

[14] E.M. Stein, and J.C. Stein, "Stock Price Distributions with Stochastic Volatility: an Analytic Approach," *Review of Finance Studies*, 4, 727-752, 1991.

[15] R.E. Whaley, "On the Valuation of American Call Options on Stocks with Known Dividends," *Journal of Financial Economics*, 9, 207-211, 1981.