Existence of Stepanov-like Square-mean Pseudo Almost Automorphic Solutions to Nonautonomous Stochastic Functional Evolution Equations

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Abstract—**We introduce the concept of bi-square-mean almost automorphic functions and Stepanov-like square-mean pseudo almost automorphic functions for stochastic processes. Using the results, we also study the existence and uniqueness theorems for Stepanov-like square-mean pseudo almost automorphic mild solutions to a class of nonautonomous stochastic functional evolution equations in a real separable Hilbert space. Finally, an application involving a stochastic parabolic system is considered.**

Index Terms—**square-mean pseudo almost automorphic, stepanov-like pseudo almost automorphic, nonautonomous stochastic evolution equations, exponential dichotomy.**

I. Introduction

The concept of pseudo almost automorphic functions is a natural generalization of almost automorphic functions, and the concept of almost automorphic functions was first created by Bochner in [1]. Since then, those functions have been widely studied and developed. For more on those functions we refer the reader to [2], [3], [4]. The existence of pseudo-almost automorphic solutions is among the most attractive topics in qualitative theory of differential equations because of their significance and applications in physics, mechanics and mathematical biology. In recent years, much attention has been paid to the existence of pseudo almost automorphic solutions on different kinds of differential equations in Banach spaces; see [5], [6], [7], [8], [9] and the references therein. On the other hand, N'Guérékata and Pankov [10] introduced the concept of Stepanov-like almost automorphy functions, which is another generalization of almost automorphic functions. Such a notion was, subsequently, utilized to study the existence of weak Stepanov-like almost automorphic solutions to some parabolic evolution equations. Very recently, Diagana in [11] introduced the notion of Stepanov-like pseudo almost automorphy as a natural generalization of the concept of pseudo almost automorphy as well as the one of Stepanov-like almost automorphy. He also studied the existence and uniqueness of Stepanov-like pseudo almost automorphic solutions to semilinear differential equations in a Banach space. Further, we refer the reader to [12], [13], [14], [15], [16], [17] and references therein for more contributions concerning the Stepanov-like almost automorphy function theory.

In many cases, deterministic models often fluctuate due to noise, which is random or at least appears to be so. Therefore, we must move from deterministic problems to stochastic ones. Recently, the existence results for square-mean almost automorphy solutions to some stochastic differential equations in Hilbert spaces have been studied in many publications. For example, Fu et al. [18] introduced a new concept of a square-mean almost automorphic stochastic process. The paper deal with the existence and uniqueness of square-mean almost automorphic mild solutions for stochastic differential equations in Hilbert spaces, which further generalized the almost automorphic theory from the deterministic version to the stochastic one. The papers [19], [20], [21] investigated the same issue for some stochastic differential equations. The authors in [22] also studied the existence and uniqueness of a Stepanov-like almost automorphic mild solution to a class of nonlinear stochastic differential equations in a real separable Hilbert space. Chen and Lin [23] introduced the concept of square-mean pseudo almost automorphy for a stochastic process and obtained the existence, uniqueness and global stability of square-mean pseudo almost automorphic solutions for a class of stochastic evolution equations. For the case of fractional stochastic differential equations; see [24]. Yan et al. [25] introduced the concept of Stepanov-like square-mean pseudo almost automorphic functions, and discussed the existence and uniqueness of Stepanov-like square-mean pseudo almost automorphic mild solutions to a neutral stochastic functional differential equation. They results are more general and complicated than the almost periodic mild solutions or pseudo almost periodic mild solutions to some stochastic differential equations. One can refer to Bezandry and Diagana [26], [27], [28], in which the authors made extensive use of the almost periodicity to study the existence and uniqueness of almost periodic mild solutions to the class of semilinear stochastic differential equations. We would like to point out that other interesting results on almost periodicity and square-mean almost periodic mild solution to stochastic evolution equations have been obtained in [29], [30], [31], [32], [33], [34], [35].

In this paper, we investigate the existence and uniqueness of Stepanov-like square-mean pseudo almost automorphic to the following nonautonomous stochastic functional evolution equations:

$$\begin{aligned} dx(t) &= A(t)x(t)dt + h(t, x(t-r))dt \\ &\quad + f(t, x(t-r))dW(t), \quad t \in R, \end{aligned} \tag{1}$$

where $A(t) : D((A(t)) \subseteq L^2(P,H) \to L^2(P,H)$ is a family of densely defined closed linear operators satisfying the so-called "Acquistapace-Terrani " conditions, $W(t)$ is a two-sided standard one-dimensional Brownian motion de-

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fined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$, where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \leq t\}$. $r \geq 0$ is a fixed constant and h, f are appropriate functions to be specified later.

We introduce the notion of bi-square-mean almost automorphic and Stepanov-like square-mean pseudo almost automorphic for stochastic processes, which, in turn generalizes all the above-mentioned concepts, in particular, the notion of square-mean pseudo almost automorphy and Stepanov-like square-mean almost automorphic. Using the new concepts, the theory of evolution family and exponential dichotomy, we study the existence and uniqueness of Stepanov-like square-mean pseudo almost automorphic mild solutions to the partial stochastic evolution equations of the form Eq. (1). To the best of our knowledge, there is no work reported on the interesting problem, which in fact is the main motivation of the present paper.

The paper is organized as follows. In Section 2, we recall briefly some basic notations and definitions, lemmas related with evolution system and Stepanov-like square-mean pseudo almost automorphic functions. Section 3 is devoted to the existence and uniqueness of the Stepanov-like square-mean pseudo almost automorphic solutions for the problem (1). Finally in Section 4, for illustration, we propose to study the existence and uniqueness of Stepanov-like square-mean pseudo almost automorphic solutions for the model arising in physical systems.

II. Preliminaries

In this section, we introduce some basic definitions, notations and lemmas which are used throughout this paper.

Throughout the paper, we assume that $(H, \|\cdot\|)$ is assumed to be a real and separable Hilbert space. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. The notation $L^2(P, H)$ stands for the space of all H-valued random variables x such that $E \parallel x \parallel^2 = \int_\Omega \parallel x \parallel^2 dP < \infty$, which is a Banach space with the norm $\parallel x \parallel_2 = (\int_\Omega \parallel x \parallel^2 dP)^{\frac{1}{2}}$. It is routine to check that $L^2(P, H)$ is a Hilbert space equipped with the norm $\|\cdot\|$. We let $L(K, H)$ be the space of all linear bounded operators from K into H, equipped with the usual operator norm $\parallel \cdot \parallel_{L(K,H)}$; in particular, this is simply denoted by $L(H)$ when $K = H$. $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$, where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \leq t\}$.

A. Square-mean pseudo almost automorphy

Let $C(R, L^2(P, H)), BC(R, L^2(P, H))$ stand for the collection of all continuous functions from R into $L^2(P, H)$, the Banach space of all bounded continuous functions from R into $L^2(P, H)$, equipped with the sup norm $\parallel \cdot \parallel_\infty$, respectively. Similarly, $C(R \times L^2(P, H), L^2(P, H))$ and $BC(R \times L^2(P, H), L^2(P, H))$ stand, respectively, for the class of all jointly continuous functions from $R \times L^2(P, H)$ into $L^2(P, H)$ and the collection of all jointly bounded continuous functions from $R \times L^2(P, H)$ into $L^2(P, H)$.

Definition 1 ([23]). A stochastic process $x(t) : R \to L^2(P, H)$ is said to be stochastically bounded if there exists $\widehat{M} > 0$ such that $E \parallel x(t) \parallel \leq \widehat{M}$ for all $t \in R$.

Definition 2 ([23]). A stochastic process $x : R \to L^2(P, H)$ is said to be stochastically continuous if

$$\lim_{t \to s} E \parallel x(t) - x(s) \parallel^2 = 0.$$

Denote by $BC(R, L^2(P, H))$ the collection of all the stochastically bounded and continuous processes. Then several properties of the space $BC(R, L^2(P, H))$ are listed as follows.

Remark 1 ([23]). $BC(R, L^2(P, H))$ is a linear space.

Remark 2 ([23]). $BC(R, L^2(P, H))$ is a Banach space with the norm

$$\parallel x \parallel_\infty := \sup_{t \in R} (E \parallel x(t) \parallel^2)^{\frac{1}{2}},$$

for $E \parallel x(t) \parallel^2 = (\int_\Omega \parallel x(t) \parallel^2 dP)^{\frac{1}{2}}$.

Definition 3 ([18]). A stochastically continuous stochastic process $x : R \to L^2(P, H)$ is said to be square-mean almost automorphic if for every sequence of real numbers $(s'_n)_{n \in N}$ there is a subsequence $(s_n)_{n \in N}$ and a stochastic process $y : R \to L^2(P, H)$ such that

$$\lim_{n \to \infty} E \parallel x(t + s_n) - y(t) \parallel^2 = 0,$$

$$\lim_{n \to \infty} E \parallel y(t - s_n) - x(t) \parallel^2 = 0$$

holds for each $t \in R$. This limit means that

$$\lim_{m \to \infty} \lim_{n \to \infty} E \parallel x(t + s_n - s_m) - x(t) \parallel^2 = 0$$

for each $t \in R$. Denote the set of all such stochastically continuous processes by $AA(L^2(P, H))$.

Remark 3 ([18]). If $x(t) \in AA(R, L^2(P, H))$, then $x(t)$ is bounded, that is, $\parallel x \parallel_\infty < \infty$.

Refer to Lemma 2.3 of [18] for the detailed proof of Remark 3.

Lemma 1 ([2]). Let $f, g : R \to L^2(P, H))$ are square-mean almost automorphic and λ is any scalar. Then the following holds true:

(1) $f + g, \lambda f, f_\tau(t) = f(t + \tau), \hat{f}(t) := f(-t)$ are square-mean almost automorphic.
(2) The range R_f of f is precompact, so f is bounded.
(3) If $\{f_n\}$ is a sequence of almost automorphic functions and $f_n \to f$ uniformly on R, then f is almost automorphic.

Definition 4 A continuous function $f(t, s) : R \times R \to L^2(P, H)$ is called bi-square-mean almost automorphic if for every sequence of real numbers $\{s'_n\}_{n \in N}$ there exists a subsequence $\{s_n\}_{n \in N}$ and a stochastic process $\tilde{f}(t, s) : R \times R \to L^2(P, H)$ such that

$$\lim_{n \to +\infty} E \parallel f(t + s_n, s + s_n) - \tilde{f}(t, s) \parallel^2 = 0$$

is well defined in $t, s \in R$, and

$$\lim_{n \to +\infty} E \parallel \tilde{f}(t - s_n, s - s_n) - f(t, s) \parallel^2 = 0$$

for each $t, s \in R$. Denote the set of all such stochastically continuous processes by $bAA(R \times R, L^2(P, H))$.

In other words, a function $f(t, s) : R \times R \to L^2(P, H)$ is said to be bi-square-mean almost automorphic if for any sequence of real numbers $\{s'_n\}_{n \in N}$ there exists a subsequence $\{s_n\}_{n \in N}$ such that

$$\lim_{m \to +\infty} \lim_{n \to \infty} E \parallel f(t + s_n - s_m, s + s_n - s_m) - f(t, s) \parallel^2 = 0$$

for each $t, s \in R$.

We set

$$PAP_0(L^2(P,H)) = \left\{ f \in BC(R, L^2(P,H)) : \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E \parallel x(t) \parallel^2 dt = 0 \right\}.$$

Definition 5 ([23]). A stochastically continuous process $f(t) : R \to L^2(P,H)$ is said to be square-mean pseudo almost automorphic if it can be decomposed as $f = g + \varphi$, where $g \in AAL^2(P,H)$ and $\varphi \in PAP_0(L^2(P,H))$. Denote the set of all such stochastically continuous processes by $PAA(L^2(P,H))$.

Remark 4 ([23]). $PAA(L^2(P,H))$ is a linear closed subspace of $BC(R, L^2(P,H))$.

Remark 5 ([23]). $PAA(L^2(P,H))$ is a Banach space with the norm $\parallel \cdot \parallel_\infty$.

Definition 6 ([18]). A function $f : R \times L^2(P,H) \to L^2(P,H)$, $(t,x) \to f(t,x)$, which is jointly continuous, is said to be square-mean almost automorphic in $t \in R$ for each $x \in L^2(P,H)$ if for every sequence of real numbers $\{s'_n\}_{n\in N}$ there exists a subsequence $\{s_n\}_{n\in N}$ and a stochastic process $\tilde{f} : R \times L^2(P,H) \to L^2(P,H)$ such that

$$\lim_{n\to\infty} E \parallel f(t+s_n,x) - \tilde{f}(t,x) \parallel^2 = 0,$$

$$\lim_{n\to\infty} E \parallel \tilde{f}(t-s_n,x) - f(t,x) \parallel^2 = 0$$

for each $t \in R$ and each $x \in L^2(P,H)$. Denote the set of all such stochastically continuous processes by $AA(R \times L^2(P,H))$.

Lemma 2 ([18]). Let $f : R \times L^2(P,H) \to L^2(P,H)$, $(t,x) \to f(t,x)$ be square-mean almost automorphic in $t \in R$ for each $x \in L^2(P,H)$, and assume that f satisfies a Lipschitz condition in the following sense:

$$E \parallel f(t,\phi) - f(t,\psi) \parallel^2 \leq \tilde{M} E \parallel \phi - \psi \parallel^2$$

for all $\phi, \psi \in L^2(P,H)$ and for each $t \in R$, where $\tilde{M} > 0$ is independent of t. Then for any square-mean almost automorphic process $x : R \to L^2(P,H)$, the stochastic process $F : R \to L^2(P,H)$ given by $F(\cdot) = f(\cdot, x(\cdot))$ is square-mean almost automorphic.

Denote

$$AA_0(R \times L^2(P,H)) = \left\{ f \in BC(R \times L^2(P,H), L^2(P,H)) : \lim_{T\to\infty} \int_{-T}^{T} E \parallel f(t,x) \parallel^2 dt = 0 \right\}.$$

Definition 7 ([23]). A function $f(t,x) : R \times L^2(P,H) \to L^2(P,H)$, which is jointly continuous, is said to be square-mean pseudo almost automorphic in t for any $x \in L^2(P,H)$ if it can be decomposed as $f = g + \varphi$, where $g \in AA(R \times L^2(P,H))$ and $\varphi \in AA_0(R \times L^2(P,H))$. Denote the set of all such stochastically continuous processes by $PAA(L^2(R \times L^2(P,H))$.

B. Stepanov-like square-mean almost automorphy

Definition 8 ([32]). The Bochner transform $x^b(t,s), t \in R, s \in [0,1]$, of a stochastic process $x : R \to L^2(P,H)$ is defined by

$$x^b(t,s) := x(t+s).$$

Remark 6 ([32]). A stochastic process $\psi(t,s), t \in R, s \in [0,1]$, is the Bochner transform of a certain stochastic process f,

$$\psi(t,s) = x^b(t,s),$$

if and only if

$$\psi(t+\tau, s-\tau) = \psi(s,t)$$

for all $t \in R, s \in [0,1]$ and $\tau \in [s-1, s]$.

Definition 9 ([10]). The Bochner transform $F^b(t,s,u), t \in R, s \in [0,1], u \in L^2(P,H)$, of a function $F : R \times L^2(P,H) \to L^2(P,H)$ is defined by

$$F^b(t,s,u) := F(t+s,u)$$

for each $u \in L^2(P,H)$.

Definition 10 ([32]). The space $BS^2(L^2(P,H))$ of all Stepanov bounded stochastic processes consists of all measurable stochastic processes $x : R \to L^2(P,H)$ such that

$$x^b = L^\infty(R; L^2(0,1; L^2(P,H))).$$

This is a Banach space with the norm

$$\parallel x \parallel_{S^2} = \parallel x^b \parallel_{L^\infty(R;L^2)} = \sup_{t\in R} \left(\int_t^{t+1} E \parallel x(\tau) \parallel^2 d\tau \right)^{\frac{1}{2}}.$$

Definition 11 ([19]). A stochastic process $x \in BS^2$ $(L^2(P,H))$ is called Stepanov-like square-mean almost automorphic (or S^2-almost automorphic) if

$$x^b \in AA(R; L^2(0,1; L^2(P,H))).$$

In other words, a stochastic process

$$x \in L^2_{loc}(R, L^2(P,H))$$

is said to be Stepanov-like almost automorphic if its Bochner transform

$$x^b : R \to L^2(0,1; L^2(P,H))$$

is square-mean almost automorphic in the sense that for every sequence of real numbers $\{s'_n\}_{n\in N}$, there exist a subsequence $\{s_n\}_{n\in N}$, and a stochastic process

$$y \in L^2_{log}(R, L^2(P,H))$$

such that

$$\int_t^{t+1} E \parallel x(s+s_n) - y(s) \parallel^2 ds \to 0,$$

$$\int_t^{t+1} E \parallel y(s-s_n) - x(s) \parallel^2 ds \to 0$$

as $n \to \infty$ pointwise on R. Denote the set of all such stochastically continuous processes by $AS^2(L^2(P,H))$.

Remark 7 ([19]). It is clear that, if $x : R \to L^2(P,H)$ is a square-mean almost automorphic stochastic process, then x is S^2-almost automorphic, that is, $AA(L^2(P,H)) \subset AS^2(L^2(P,H))$.

C. Stepanov-like square-mean pseudo almost automorphy

Definition 12. A stochastic process $f \in BS^2(R, L^2(P,H))$ is said to be Stepanov-like square-mean pseudo almost automorphic (or S^2-pseudo almost automorphic) if it can be decomposed as $f = h + \varphi$, where $h^b \in AA(L^2(0,1;L^2(P,H)))$ and $\varphi^b \in PAP_0(L^2(0,1;L^2(P,H)))$. Denote the set of all such stochastically continuous processes by $PAA^2(L^2(P,H))$.

In other words, a stochastic process

$$f \in L^2_{loc}(R, L^2(P,H))$$

is said to be Stepanov-like square-mean pseudo almost automorphic if its Bochner transform

$$f^b : R \to L^2(0,1;L^2(P,H))$$

is square-mean pseudo almost automorphic in the sense that there exist two functions $h, \varphi : R \to L^2(P,H)$ such that $f = h + \varphi$, where $h^b \in AA(L^2(0,1;L^2(P,H)))$ and $\varphi^b \in PAP_0(L^2(0,1;L^2(P,H)))$.

Obviously, the following inclusions hold:

$$AP(L^2(P,H)) \subset AA(L^2(P,H)) \subset PAA(L^2(P,H)) \\ \subset PAA^2(L^2(P,H)),$$

where $AP(L^2(P,H))$ stands for the collection of all $L^2(P,H)$-valued almost periodic functions.

Definition 13. A stochastic process $f \in BS^2(R \times L^2(P,H),\ L^2(P,H))$ is said to be Stepanov-like square-mean pseudo almost automorphic (or S^2-pseudo almost automorphic) if it can be decomposed as $f = h + \varphi$, where $h^b \in AA(R \times L^2(0,1;L^2(P,H)))$ and $\varphi^b \in PAP_0(R \times L^2(0,1;L^2(P,H)))$ Denote the set of all such stochastically continuous processes by $PAA^2(R \times L^2(P,H))$.

Lemma 3. Assume $f \in PAA^2(R \times L^2(P,H))$. Suppose that $f(t,u)$ is Lipschitz in $u \in L^2(P,H)$ uniformly in $t \in R$, in the sense that there exists $L > 0$ such that

$$\| f(t,u) - f(t,v) \| \leq L \| u - v \|$$

for all $t \in R, u, v \in L^2(P,H)$. If $\phi(\cdot) \in PAA^2(L^2(P,H))$ then $f(\cdot, \phi(\cdot)) \in PAA^2(L^2(P,H))$.

Lemma 3 can be proved by using Definition 11, Definition 12 and Lemmas 2. One may refer to Theorem 3.5 in [11] for more details about the proof of Lemma 3.

D. Evolution family and exponential dichotomy

We also need the following concepts concerning evolution family and exponential dichotomy. For more details, we refer the reader to [36].

Definition 14. A set $U(t,s) : t \geq s, t, s \in R$ of bounded linear operators on $L^2(P,H)$ is called an evolution family if

(a) $U(s,s) = I, U(t,s) = U(t,\tau)U(\tau,s)$ for $t \geq \tau \geq s$ and $t, \tau, s \in R$;
(b) $\{(\tau,\sigma) \in R^2 : \tau \geq \sigma\} \ni (t,s) \mapsto U(t,s)$ is strongly continuous.

Definition 15. An evolution family U is called hyperbolic (or has exponential dichotomy) if there are projections $P(t), t \in R$, uniformly bounded and strongly continuous in t, and constants $M, \delta > 0$ such that

(a) $U(t,s)P(s) = P(t)U(t,s)$ for all $t \geq s$;
(b) the restriction
$$U_Q(t,s) : Q(s)L^2(P,H) \to Q(t)L^2(P,H)$$
is invertible for all $t \geq s$ (and we set $U_Q(s,t) = U_Q(t,s)^{-1}$);
(c) $\| U(t,s)P(s) \| \leq Me^{-\delta(t-s)}$ and $\| U_Q(s,t)Q(t) \| \leq Me^{-\delta(t-s)}$ for all $t \geq s$.

Here and below $Q := I - P$.

Remark 8. Exponential dichotomy is a classical concept in the study of the long-term behavior of evolution equations, see [37], [38], [39]. If $P(t) = I$ for $t \in R$, then $(U(t,s))_{t \geq s}$ is exponentially stable, see [36], [37], [38], [39], for example.

Definition 16. If U is a hyperbolic evolution family, then

$$\Gamma(t,s) := \begin{cases} U(t,s)P(s), & t \geq s, t, s \in R, \\ -U_Q(t,s)Q(s), & t < s, t, s \in R \end{cases}$$

is called Green's function corresponding to U and $P(\cdot)$.

III. Existence Results

In this section, we prove that there is a unique mild solution for the problem (1). For that, we make the following hypotheses:

(H1) There exist constants $\lambda_0 > 0, \theta \in (\frac{\pi}{2}, \pi)$, $K_0, K_1 \geq 0$, and $\alpha_1, \alpha_2 \in (0,1]$ with $\alpha_1 + \alpha_2 > 1$ such that
$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0),\ \| R(\lambda, A(t) - \lambda_0) \| \leq \frac{K_1}{1 + |\lambda|},$$
and
$$\| (A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) \\ -R(\lambda_0, A(s))] \| \leq K_0 |t-s|^{\alpha_1} |\lambda|^{-\alpha_2}$$
for $t, s \in R, \lambda \in \Sigma_\theta := \{\lambda \in \mathbf{C} \setminus \{0\} : |\arg \lambda| \leq \theta\}$.
(H2) The evolution family $U(t,s)$ generated by $A(t)$ has an exponential dichotomy with constants $M, \delta > 0$, dichotomy projections $P(t), t \in R$, and Green's function Γ.
(H3) $\Gamma(t,s)x \in bAA(R \times R, L^2(P,H))$ uniformly for all x in any bounded subset of $L^2(P,H)$.
(H4) The functions h, f are Lipschitz with to the second argument uniformly in the first argument in the sense that: there exist $L_h, L_f > 0$ such that
$$E \| h(t,x) - h(t,y) \|^2 \leq L_h E \| x - y \|^2,$$
and
$$E \| f(t,x) - f(t,y) \|^2 \leq L_f E \| x - y \|^2$$
for all $t \in R$ and each $x, y \in L^2(P,H)$.
(H5) The functions $h, f \in PAA^2(R \times L^2(P,H)) \cap C(R \times L^2(P,H), L^2(P,H))$.

Remark 9. Assumption (H1) is usually called "Acquistapace-Terreni" conditions, which was firstly introduced in [40] and widely used to investigate nonautonomous evolution equations in [5], [36], [37]. If (H1) holds, t hen there exists a unique evolution family $\{U(t,s), t \geq s > -\infty\}$ on $L^2(P,H)$.

In this section, we investigate the existence of a Stepanov-like square-mean pseudo almost automorphic mild solution for the problem (1). To do this, we first consider the existence of square-mean Stepanov-like pseudo almost automorphic mild solutions to the linear stochastic differential equation

$$dx(t) = A(t)x(t)dt + h(t)dt + f(t)dW(t), \quad t \in R, \quad (2)$$

where $A(t) : D((A(t)) \subseteq L^2(P, H) \to L^2(P, H)$ is a family of densely defined closed linear operators satisfying the so-called "Acquistapace-Terrani" conditions. $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$, where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \leq t\}$. $h, f \in PAA^2(L^2(P, H)) \cap C(R, L^2(P, H))$.

Definition 17. An $\mathcal{F}_t$-progressively measurable stochastic process $x : R \to L^2(P, H)$ is called a mild solution of the system (3.1) if $x(t)$ satisfies

$$\begin{aligned} x(t) = U(t,s)x(s) + \int_s^t U(t,\tau)h(\tau)d\tau \\ + \int_s^t U(t,\tau)f(\tau)dW(\tau) \quad (3) \end{aligned}$$

for all $t \geq s$ and all $s \in R$.

Now we are ready to state the first main result.

Theorem 1. *Suppose that* (H1), (H2) *and* (H3) *hold. If* $h, f \in PAA^2(L^2(P, H)) \cap C(R, L^2(P, H))$*, then there exists a unique solution* $x \in PAA(L^2(P, H))$ *of equation* (2) *such that*

$$\begin{aligned} x(t) = &\int_{-\infty}^t U(t,\tau)P(\tau)h(\tau)d\tau \\ &+ \int_t^{+\infty} U_Q(t,\tau)Q(\tau)f(\tau)dW(\tau), \quad t \in R. \quad (4) \end{aligned}$$

Proof. Firstly, we show that Eq. (2) admits a unique bounded solution given by Eq. (4), which is similar to the proofs of Theorem 4.28 in [39]. See also [5]. From exponential dichotomy of $U(t,s)_{t\geq s}$, we deduce

$$\begin{aligned} x(t) = &\int_{-\infty}^t U(t,\tau)P(\tau)h(\tau)d\tau \\ &- \int_t^{+\infty} U_Q(t,\tau)Q(\tau)h(\tau)d\tau \\ &+ \int_{-\infty}^t U(t,\tau)P(\tau)f(\tau)dW(\tau) \\ &- \int_t^{+\infty} U_Q(t,\tau)Q(\tau)f(\tau)dW(\tau) \quad (5) \end{aligned}$$

is well defined for each $t \in R$.

To prove that x satisfies equation (2) for all $t \geq s$, all $s \in R$, we let

$$\begin{aligned} x(s) = &\int_{-\infty}^s U(t,\tau)P(\tau)h(\tau)d\tau \\ &- \int_s^{+\infty} U_Q(t,\tau)Q(\tau)h(\tau)d\tau \\ &+ \int_{-\infty}^s U(t,\tau)P(\tau)f(\tau)dW(\tau) \\ &- \int_s^{+\infty} U_Q(t,\tau)Q(\tau)f(\tau)dW(\tau), \quad s \in R. \quad (6) \end{aligned}$$

Multiply both sides of (6) by $U(t,s)$ for all $t \geq s$, then

$$\begin{aligned} U(t,s)x(s) & \\ = &\int_{-\infty}^s U(t,\tau)P(\tau)h(\tau)d\tau \\ &- \int_s^{+\infty} U_Q(t,\tau)Q(\tau)h(\tau)d\tau \\ &+ \int_{-\infty}^s U(t,\tau)P(\tau)f(\tau)dW(\tau) \\ &- \int_s^{+\infty} U_Q(t,\tau)Q(\tau)f(\tau)dW(\tau) \\ = &\int_{-\infty}^t U(t,\tau)P(\tau)h(\tau)d\tau \\ &- \int_s^t U(t,\tau)P(\tau)h(\tau)d\tau \\ &- \int_t^{+\infty} U_Q(t,\tau)Q(\tau)h(\tau)d\tau \\ &- \int_s^t U_Q(t,\tau)Q(\tau)h(\tau)d\tau \\ &+ \int_{-\infty}^t U(t,\tau)P(\tau)f(\tau)dW(\tau) \\ &- \int_s^t U(t,\tau)P(\tau)f(\tau)dW(\tau) \\ &- \int_t^{+\infty} U_Q(t,\tau)Q(\tau)f(\tau)dW(\tau) \\ &- \int_s^t U_Q(t,\tau)Q(\tau)f(\tau)dW(\tau) \\ = &\, x(t) - \int_s^t U(t,\tau)h(\tau)d\tau \\ &- \int_s^t U(t,\tau)f(\tau)dW(\tau). \end{aligned}$$

Hence x is a mild solution to Eq. (2).

To prove the uniqueness, let $y \in PAA(L^2(P, H))$ satisfy equation (4). Then, exponential dichotomy of $U(t,s)_{t\geq s}$ imply that

$$\begin{aligned} P(t)y(t) = &\int_{-\infty}^t U(t,\tau)P(\tau)h(\tau)d\tau \\ &+ \int_{-\infty}^t U(t,\tau)P(\tau)f(\tau)dW(\tau), \quad t \in R, \end{aligned}$$

$$\begin{aligned} Q(t)y(t) = &\int_{+\infty}^t U_Q(t,\tau)Q(\tau)h(\tau)d\tau \\ &+ \int_{+\infty}^t U_Q(t,\tau)Q(\tau)f(\tau)dW(\tau), \quad t \in R. \end{aligned}$$

Thus,

$$\begin{aligned} y(t) = P(t)y(t) + Q(t)y(t) = &\int_{-\infty}^t U(t,\tau)P(\tau)h(\tau)d\tau \\ &+ \int_{+\infty}^t U_Q(t,\tau)Q(\tau)h(\tau)d\tau \\ &+ \int_{-\infty}^t U(t,\tau)P(\tau)f(\tau)dW(\tau) \\ &+ \int_{+\infty}^t U_Q(t,\tau)Q(\tau)f(\tau)dW(\tau) = x(t). \end{aligned}$$

Now, we will prove that $x \in PAA(L^2(P,H))$. Since

$$h, f \in PAA^2(L^2(P,H)) \cap C(R, L^2(P,H),$$

write

$$h = h_1 + h_2, f = f_1 + f_2,$$

where

$$h_1^b, f_1^b \in AA(L^2(0,1; L^2(P,H))) \\ \cap C(R, L^2(0,1; L^2(P,H)))$$

and

$$h_2^b, f_2^b \in PAP_0(L^2(0,1; L^2(P,H))) \\ \cap C(R, L^2(0,1; L^2(P,H))),$$

then $x(t)$ can be decomposed as

$$\begin{aligned} x(t) = & \int_{-\infty}^t U(t,\tau)P(\tau)h_1(\tau)d\tau \\ & + \int_{-\infty}^t U(t,\tau)P(\tau)h_2(\tau)d\tau \\ & + \int_{+\infty}^t U_Q(t,\tau)Q(\tau)h_1(\tau)d\tau \\ & + \int_{+\infty}^t U_Q(t,\tau)Q(\tau)h_2(\tau)d\tau \\ & + \int_{-\infty}^t U(t,\tau)P(\tau)f_1(\tau)dW(\tau) \\ & + \int_{-\infty}^t U(t,\tau)P(\tau)f_2(\tau)dW(\tau) \\ & + \int_{+\infty}^t U_Q(t,\tau)Q(\tau)f_1(\tau)dW(\tau) \\ & + \int_{+\infty}^t U_Q(t,\tau).Q(\tau)f_2(\tau)dW(\tau). \end{aligned}$$

Set

$$\begin{aligned} H_1(t) = & \int_{-\infty}^t U(t,\tau)P(\tau)h_1(\tau)d\tau \\ & + \int_{+\infty}^t U_Q(t,\tau)Q(\tau)h_1(\tau)d\tau, \end{aligned}$$

$$\begin{aligned} H_2(t) = & \int_{-\infty}^t U(t,\tau)P(\tau)h_2(\tau)d\tau \\ & + \int_{+\infty}^t U_Q(t,\tau)Q(\tau)h_2(\tau)d\tau, \end{aligned}$$

and

$$\begin{aligned} F_1(t) = & \int_{-\infty}^t U(t,\tau)P(\tau)f_1(\tau)dW(\tau) \\ & + \int_{+\infty}^t U_Q(t,\tau)Q(\tau)f_1(\tau)dW(\tau), \end{aligned}$$

$$\begin{aligned} F_2(t) = & \int_{-\infty}^t U(t,\tau)P(\tau)f_2(\tau)dW(\tau) \\ & + \int_{+\infty}^t U_Q(t,\tau)Q(\tau)f_2(\tau)dW(\tau). \end{aligned}$$

Next we show that $H_1, F_1 \in AA(L^2(P,H))$ and $H_2, F_2 \in PAP_0(L^2(P,H))$.

To prove that $H_1 \in AA(L^2(P,H))$, we consider

$$\begin{aligned} & H_{1,k}(t) \\ & = \int_{t-k}^{t-k+1} U(t,\tau)P(\tau)h_1(\tau)d\tau \\ & \quad + \int_{t+k}^{t+k-1} U_Q(t,\tau)Q(\tau)h_1(\tau)d\tau \\ & = \int_{k-1}^{k} U(t,t-\tau)P(t-\tau)h_1(t-\tau)d\tau \\ & \quad + \int_{k}^{k-1} U_Q(t,t+\tau)Q(t+\tau)h_1(t+\tau)d\tau \end{aligned}$$

for each $t \in R$ and $k = 1, 2, 3, \ldots$. Then, using exponential dichotomy of $U(t,s)_{t \geq s}$ and Hölder's inequality, it follows that

$$\begin{aligned} & E \parallel H_{1,k}(t) \parallel^2 \\ & \leq 2E \left\| \int_{t-k}^{t-k+1} U(t,\tau)P(\tau)h_1(\tau)d\tau \right\|^2 \\ & \quad +2E \left\| \int_{t+k}^{t+k-1} U_Q(t,\tau)Q(\tau)h_1(\tau)d\tau \right\|^2 \\ & \leq 2E \left(\int_{t-k}^{t-k+1} \parallel U(t,\tau)P(\tau) \parallel \parallel h_1(\tau) \parallel d\tau \right)^2 \\ & \quad +2E \left(\int_{t+k}^{t+k-1} \parallel U_Q(t,\tau)Q(\tau) \parallel \parallel h_1(\tau) \parallel d\tau \right)^2 \\ & \leq 2M^2 E \left(\int_{t-k}^{t-k+1} e^{-\delta(t-\tau)} \parallel h_1(\tau) \parallel d\tau \right)^2 \\ & \quad +2M^2 E \left(\int_{t+k}^{t+k-1} e^{\delta(t-\tau)} \parallel h_1(\tau) \parallel d\tau \right)^2 \\ & \leq 2M^2 \left(\int_{t-k}^{t-k+1} e^{-2\delta(t-\tau)} d\tau \right) \\ & \quad \times \left(\int_{t-k}^{t-k+1} E \parallel h_1(\tau) \parallel^2 d\tau \right) \\ & \quad +2M^2 \left(\int_{t+k}^{t+k-1} e^{2\delta(t-\tau)} d\tau \right) \\ & \quad \times \left(\int_{t+k}^{t+k-1} E \parallel h_1(\tau) \parallel^2 d\tau \right) \\ & \leq 2M^2 \left(\int_{k-1}^{k} e^{-2\delta\tau} d\tau \right) \parallel h_1 \parallel_{S^2}^2 \\ & \quad +2M^2 \left(\int_{-k}^{-k+1} e^{2\delta\tau} d\tau \right) \parallel h_1 \parallel_{S^2}^2 \\ & \leq \frac{2M^2}{\delta} e^{-2\delta k}(e^{2\delta} - 1) \parallel h_1 \parallel_{S^2}^2 . \end{aligned}$$

Since $\frac{2M^2}{\delta}(e^{2\delta} - 1) \parallel h_1 \parallel_{S^2}^2 \sum_{k=1}^{\infty} e^{-2\delta k} < \infty$, we deduce from the well-known Weierstrass test that the series $\sum_{k=1}^{\infty} H_{1,k}(t)$ is uniformly convergent on R. Furthermore,

$$\begin{aligned} H_1(t) = & \int_{-\infty}^t U(t,\tau)P(\tau)h_1(\tau)d\tau \\ & + \int_{+\infty}^t U_Q(t,\tau)Q(\tau)h_1(\tau)d\tau = \sum_{k=1}^{\infty} H_{1,k}(t). \end{aligned}$$

Let us take a sequence $(s'_n)_{n \in N}$ and show that there exists a subsequence $(s_n)_{n \in N}$ of $(s'_n)_{n \in N}$ such that

$$\lim_{m \to \infty} \lim_{n \to \infty} E \parallel H_{1,k}(t + s_n - s_m) - H_{1,k}(t) \parallel^2 = 0$$

for each $t \in R$. Let $\varepsilon > 0, N_\varepsilon > 0$. By $h_1^b \in AA(L^2(0,1;L^2(P,H)))$ and (H3), there exists a subsequence $(s_n)_{n\in N}$ of $(s'_n)_{n\in N}$ such that, for each $t \in R$,

$$\int_t^{t+1} E \parallel h_1(s+s_n-s_m)-h_1(s) \parallel^2 ds < \varepsilon, \quad (7)$$

$$\begin{aligned}&\parallel U(t+s_n-s_m, t+s_n-s_m-\tau)\\ &\quad \times P(t+s_n-s_m-\tau)\\ &\quad -U(t,t-\tau)P(t-\tau) \parallel^2 < \varepsilon,\end{aligned} \quad (8)$$

$$\begin{aligned}&\parallel U_Q(t+s_n-s_m, t+s_n-s_m+\tau)\\ &\quad Q(t+s_n-s_m+\tau)\\ &\quad -U_Q(t,t+\tau)Q(t+\tau) \parallel^2 < \varepsilon\end{aligned} \quad (9)$$

for all $n,m \geq N_\varepsilon$. On the other hand, using the inequality (7)-(9), exponential dichotomy of $U(t,s)_{t\geq s}$ and Hölder's inequality, we obtain that

$$\begin{aligned}
&E \parallel H_{1,k}(t+s_n-s_m) - H_{1,k}(t) \parallel^2\\
&\leq 2E\Big\| \int_{k-1}^{k} [U(t+s_n-s_m, t+s_n-s_m-\tau)\\
&\quad \times P(t+s_n-s_m-\tau)h_1(t+s_n-s_m-\tau)\\
&\quad -U(t,t-\tau)P(t-\tau)h_1(t-\tau)]d\tau\Big\|^2\\
&\quad +2E\Big\| \int_{k}^{k-1} [U_Q(t+s_n-s_m, t+s_n-s_m+\tau)\\
&\quad \times Q(t+s_n-s_m+\tau)h_1(t+s_n-s_m+\tau)\\
&\quad -U_Q(t,t+\tau)Q(t+\tau)h_1(t+\tau)]d\tau\Big\|^2\\
&\leq 4M^2E\Big(\int_{k-1}^{k} e^{-\delta\tau} \parallel h_1(t+s_n-s_m-\tau)\\
&\quad -h_1(t-\tau) \parallel d\tau\Big)^2\\
&\quad +4E\Big(\int_{k-1}^{k} \parallel U(t+s_n-s_m, t+s_n-s_m-\tau)\\
&\quad \times P(t+s_n-s_m-\tau)\\
&\quad -U(t,t-\tau)P(t-\tau) \parallel\parallel h_1(t-\tau) \parallel d\tau\Big)^2\\
&\quad +4M^2E\Big(\int_{k-1}^{k} e^{-\delta\tau} \parallel h_1(t+s_n-s_m+\tau)\\
&\quad -h_1(t+\tau) \parallel d\tau\Big)^2\\
&\quad +4E\Big(\int_{k-1}^{k} \parallel U_Q(t+s_n-s_m, t+s_n-s_m+\tau)\\
&\quad \times Q(t+s_n-s_m+\tau)\\
&\quad -U_Q(t,t+\tau)Q(t+\tau) \parallel\parallel h_1(t+\tau) \parallel d\tau\Big)^2\\
&\leq 4M^2\Big(\int_{k-1}^{k} e^{-2\delta\tau}d\tau\Big)\Big(\int_{k-1}^{k} E \parallel h_1(t+s_n-s_m\\
&\quad -\tau)-h_1(t-\tau) \parallel^2 d\tau\Big)\\
&\quad +4\Big(\int_{k-1}^{k} \parallel U(t+s_n-s_m, t+s_n-s_m-\tau)\\
&\quad \times P(t+s_n-s_m-\tau)\\
&\quad -U(t,t-\tau)P(t-\tau) \parallel^2 d\tau\Big)\\
&\quad \times\Big(\int_{k-1}^{k} E \parallel h_1(t-\tau) \parallel^2 d\tau\Big)\\
&\quad +4M^2\Big(\int_{k-1}^{k} e^{-2\delta\tau}d\tau\Big)\Big(\int_{k-1}^{k} E \parallel h_1(t+s_n-s_m\\
&\quad +\tau)-h_1(t+\tau) \parallel^2 d\tau\Big)\\
&\quad +4\Big(\int_{k-1}^{k} \parallel U_Q(t+s_n-s_m, t+s_n-s_m+\tau)\\
&\quad \times Q(t+s_n-s_m+\tau)\\
&\quad -U_Q(t,t+\tau)Q(t+\tau) \parallel^2 d\tau\Big)\\
&\quad \times\Big(\int_{k-1}^{k} E \parallel h_1(t+\tau) \parallel^2 d\tau\Big)\\
&\leq \frac{2M^2}{\delta}e^{-2\delta k}(e^{2\delta}-1)\Big(\int_{t-k}^{t-k+1} E \parallel h_1(s+s_n-s_m)\\
&\quad -h_1(s) \parallel^2 ds\Big)\\
&\quad +4\Big(\int_{k-1}^{k} \parallel U(t+s_n-s_m, t+s_n-s_m-\tau)\\
&\quad \times P(t+s_n-s_m-\tau)\\
&\quad -U(t,t-\tau)P(t-\tau) \parallel^2 d\tau\Big) \parallel h_1 \parallel_{S^2}^2\\
&\quad +\frac{2M^2}{\delta}e^{-2\delta k}(e^{2\delta}-1)\Big(\int_{t+k-1}^{t+k} E \parallel h_1(s+s_n-s_m)\\
&\quad -h_1(s) \parallel^2 ds\Big)\\
&\quad +4\Big(\int_{k-1}^{k} \parallel U_Q(t+s_n-s_m, t+s_n-s_m+\tau)\\
&\quad \times Q(t+s_n-s_m+\tau)\\
&\quad -U_Q(t,t+\tau)Q(t+\tau) \parallel^2 d\tau\Big) \parallel h_1 \parallel_{S^2}^2\\
&< 4\Big[\frac{M^2}{\delta}e^{-2\delta k}(e^{2\delta}-1)+2 \parallel h_1 \parallel_{S^2}^2\Big]\varepsilon.
\end{aligned}$$

Thus, we immediately obtain that

$$\lim_{m\to\infty}\lim_{n\to\infty} E \parallel H_{1,k}(t+s_n-s_m) - H_{1,k}(t) \parallel^2 = 0$$

for each $t \in R$. Therefore, we get that $H_{1,k} \in AA(L^2(P,H))$. Applying Lemma 1, we deduce that the uniform limit

$$H_1(t) = \sum_{k=1}^{\infty} H_{1,k}(t) \in AA(L^2(P,H)).$$

Next, we will prove that $H_2 \in PAP_0(L^2(P,H))$. It is obvious that $H_2 \in BC(R, L^2(P,H)$, the left task is to show that

$$\lim_{r\to\infty} \frac{1}{2T}\int_{-T}^{T} E \parallel H_2(t) \parallel^2 dt = 0.$$

For this, we consider

$$\begin{aligned}&H_{2,k}(t)\\ &\quad = \int_{t-k}^{t-k+1} U(t,\tau)P(\tau)h_2(\tau)d\tau\end{aligned}$$

$$+\int_{t+k}^{t+k-1} U_Q(t,\tau)Q(\tau)h_2(\tau)d\tau$$
$$=\int_{k-1}^{k} U(t,t-\tau)P(t-\tau)h_2(t-\tau)d\tau$$
$$+\int_{k}^{k-1} U_Q(t,t+\tau)Q(t+\tau)h_2(t+\tau)d\tau$$

for each $t\in R$ and $k=1,2,3,\ldots$. Then, using exponential dichotomy of $U(t,s)_{t\geq s}$ and Hölder's inequality, it follows that

$$\begin{aligned}
&E\parallel H_{2,k}(t)\parallel^2\\
&\leq 2E\Big\|\int_{t-k}^{t-k+1} U(t,\tau)P(\tau)h_2(\tau)d\tau\Big\|^2\\
&\quad+2E\Big\|\int_{t+k}^{t+k-1} U_Q(t,\tau)Q(\tau)h_2(\tau)d\tau\Big\|^2\\
&\leq 2M^2E\Big(\int_{t-k}^{t-k+1} e^{-\delta(t-\tau)}\parallel h_2(\tau)\parallel d\tau\Big)^2\\
&\quad+2M^2E\Big(\int_{t+k}^{t+k-1} e^{\delta(t-\tau)}\parallel h_2(\tau)\parallel d\tau\Big)^2\\
&\leq 2M^2\Big(\int_{t-k}^{t-k+1} e^{-2\delta(t-\tau)}d\tau\Big)\\
&\quad\times\Big(\int_{t-k}^{t-k+1} E\parallel h_2(\tau)\parallel^2 d\tau\Big)\\
&\quad+2M^2\Big(\int_{t+k}^{t+k-1} e^{2\delta(t-\tau)}d\tau\Big)\\
&\quad\times\Big(\int_{t+k}^{t+k-1} E\parallel h_2(\tau)\parallel^2 d\tau\Big)\\
&\leq 2M^2\Big(\int_{k-1}^{k} e^{-2\delta\tau}d\tau\Big)\Big(\int_{t-k}^{t-k+1} E\parallel h_2(\tau)\parallel^2 d\tau\Big)\\
&\quad+2M^2\Big(\int_{-k}^{-k+1} e^{2\delta\tau}d\tau\Big)\\
&\quad\times\Big(\int_{t+k}^{t+k-1} E\parallel h_2(\tau)\parallel^2 d\tau\Big)\\
&\leq \frac{M^2}{\delta}e^{-2\delta k}(e^{2\delta}-1)\Big[\Big(\int_{t-k}^{t-k+1} E\parallel h_2(\tau)\parallel^2 d\tau\Big)\\
&\quad+\Big(\int_{t+k}^{t+k-1} E\parallel h_2(\tau)\parallel^2 d\tau\Big)\Big].
\end{aligned}$$

Since $h_2^b\in PAP_0(L^2(0,1;L^2(P,H)))$, the above inequality leads to $H_2\in PAP_0(L^2(P,H))$. The above inequality leads also to

$$E\parallel H_{2,k}(t)\parallel^2\leq \frac{2M^2}{\delta}e^{-2\delta k}(e^{2\delta}-1)\parallel h_2\parallel_{S^2}^2 .$$

By using $\frac{2M^2}{\delta}(e^{2\delta}-1)\sum_{k=1}^{\infty}\parallel h_2\parallel_{S^2}^2\ e^{-2\delta k}<\infty$, we deduce from the well-known Weierstrass test that the series $\sum_{k=1}^{\infty}H_{2,k}(t)$ is uniformly convergent on R. Furthermore,

$$H_2(t)=\int_{-\infty}^{t} U(t,\tau)P(\tau)h_2(\tau)d\tau$$
$$+\int_{+\infty}^{t} U_Q(t,\tau)Q(\tau)h_2(\tau)d\tau=\sum_{k=1}^{\infty}H_{2,k}(t).$$

Applying $H_{2,k}\in PAP_0(L^2(P,H))$ and the inequality

$$\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T}E\parallel H_2(t)\parallel^2 dt\\
&\quad\leq\frac{1}{T}\int_{-T}^{T}E\Big\|H_2(t)-\sum_{k=1}^{n}H_{2,k}(t)\Big\|^2dt\\
&\quad+\frac{n}{T}\sum_{k=1}^{n}\int_{-T}^{T}E\parallel H_{2,k}(t)\parallel^2 dt,
\end{aligned}$$

we deduce that the uniform limit $H_2(t)=\sum_{k=1}^{\infty}H_{2,k}(t)\in PAP_0(L^2(P,H))$.

To prove that $F_1\in AA(L^2(P,H))$, we consider

$$\begin{aligned}
&F_{1,k}(t)\\
&\quad=\int_{t-k}^{t-k+1} U(t,\tau)P(\tau)f_1(\tau)dW(\tau)\\
&\quad+\int_{t+k}^{t+k-1} U_Q(t,\tau)Q(\tau)f_1(\tau)dW(\tau)\\
&\quad=\int_{k-1}^{k} U(t,t-\tau)P(t-\tau)f_1(t-\tau)dW(\tau)\\
&\quad+\int_{k}^{k-1} U_Q(t,t+\tau)Q(t+\tau)f_1(t+\tau)dW(\tau)
\end{aligned}$$

for each $t\in R$ and $k=1,2,3,\ldots$. Then, by using an estimate on the Ito integral established in [41], it follows that

$$\begin{aligned}
&E\parallel F_{1,k}(t)\parallel^2\\
&\leq 2E\Big\|\int_{t-k}^{t-k+1} U(t,\tau)P(\tau)f_1(\tau)dW(\tau)\Big\|^2\\
&\quad+2E\Big\|\int_{t+k}^{t+k-1} U_Q(t,\tau)Q(\tau)f_1(\tau)dW(\tau)\Big\|^2\\
&\leq 2M^2\int_{t-k}^{t-k+1} e^{-2\delta(t-\tau)}E\parallel f_1(\tau)\parallel^2 d\tau\\
&\quad+2M^2\int_{t+k}^{t+k-1} e^{2\delta(t-\tau)}E\parallel f_1(\tau)\parallel^2 d\tau\\
&\quad\leq 2M^2\int_{k-1}^{k} e^{-2\delta\tau}E\parallel f_1(t-\tau)\parallel^2 d\tau\\
&\quad+2M^2\int_{-k}^{-k+1} e^{2\delta\tau}E\parallel f_1(t+\tau)\parallel^2 d\tau\\
&\leq 2M^2\sup_{\tau\in[k-1,k]}e^{-2\delta\tau}\int_{k-1}^{k}E\parallel f_1(t-\tau)\parallel^2 d\tau\\
&\quad+2M^2\sup_{\tau\in[-k,-k+1]}e^{2\delta\tau}\int_{-k}^{-k+1}E\parallel f_1(t+\tau)\parallel^2 d\tau\\
&\leq 2M^2e^{-2\delta(k-1)}\parallel f_1\parallel_{S^2}^2+2M^2e^{2\delta(-k+1)}\parallel f_1\parallel_{S^2}^2\\
&\leq 4M^2e^{-2\delta k}e^{2\delta}\parallel f_1\parallel_{S^2}^2 .
\end{aligned}$$

Since $4M^2e^{2\delta}\parallel f_1\parallel_{S^2}^2\sum_{k=1}^{\infty}e^{-2\delta k}<\infty$, we deduce from the well-known Weierstrass test that the series $\sum_{k=1}^{\infty}H_{1,k}(t)$ is uniformly convergent on R. Furthermore,

$$F_1(t)=\int_{-\infty}^{t} U(t,\tau)P(\tau)f_1(\tau)d\tau$$
$$+\int_{+\infty}^{t} U_Q(t,\tau)Q(\tau)f_1(\tau)d\tau=\sum_{k=1}^{\infty}F_{1,k}(t).$$

Let us take a sequence $(s'_n)_{n\in N}$ and show that there exists a subsequence $(s_n)_{n\in N}$ of $(s'_n)_{n\in N}$ such that

$$\lim_{m\to\infty}\lim_{n\to\infty} E \parallel F_{1,k}(t+s_n-s_m)-F_{1,k}(t)\parallel^2=0$$

for each $t\in R$. Let $\varepsilon>0, N_\varepsilon>0$. By $f_1^b\in AA(L^2(0,1;L^2(P,H)))$ and (H3), there exists a subsequence $(s_n)_{n\in N}$ of $(s'_n)_{n\in N}$ such that, for each $t\in R$,

$$\int_t^{t+1} E\parallel f_1(s+s_n-s_m)-f_1(s)\parallel^2 ds<\varepsilon \qquad (10)$$

for all $n,m\geq N_\varepsilon$. On the other hand, using the inequality (8)-(10), exponential dichotomy of $U(t,s)_{t\geq s}$ and the Ito integral, we obtain that

$$\begin{aligned}
&E\parallel F_{1,k}(t+s_n-s_m)-F_{1,k}(t)\parallel^2\\
&\quad\leq 2E\Bigg\|\int_{k-1}^{k}[U(t+s_n-s_m,t+s_n-s_m-\tau)\\
&\qquad\times P(t+s_n-s_m-\tau)f_1(t+s_n-s_m-\tau)\\
&\qquad-U(t,t-\tau)P(t-\tau)f_1(t-\tau)]dW(\tau)\Bigg\|^2\\
&\qquad+2E\Bigg\|\int_{k}^{k-1}[U_Q(t+s_n-s_m,t+s_n-s_m+\tau)\\
&\qquad\times Q(t+s_n-s_m+\tau)f_1(t+s_n-s_m+\tau)\\
&\qquad-U_Q(t,t+\tau)Q(t+\tau)f_1(t+\tau)]dW(\tau)\Bigg\|^2\\
&\quad\leq 4M^2\int_{k-1}^{k}e^{-2\delta\tau}E\parallel f_1(t+s_n-s_m-\tau)\\
&\qquad-f_1(t-\tau)\parallel^2 d\tau\\
&\qquad+4\int_{k-1}^{k}\parallel U(t+s_n-s_m,t+s_n-s_m-\tau)\\
&\qquad\times P(t+s_n-s_m-\tau)\\
&\qquad-U(t,t-\tau)P(t-\tau)\parallel^2 E\parallel f_1(t-\tau)\parallel^2 d\tau\\
&\qquad+4M^2\int_{k-1}^{k}e^{-\delta\tau}E\parallel f_1(t+s_n-s_m+\tau)\\
&\qquad-f_1(t+\tau)\parallel^2 d\tau\\
&\qquad+4E\int_{k-1}^{k}\parallel U_Q(t+s_n-s_m,t+s_n-s_m+\tau)\\
&\qquad\times Q(t+s_n-s_m+\tau)\\
&\qquad-U_Q(t,t+\tau)Q(t+\tau)\parallel^2 E\parallel f_1(t+\tau)\parallel^2 d\tau\\
&\quad\leq 4M^2\sup_{\tau\in[k-1,k]}e^{-2\delta\tau}\int_{k-1}^{k}E\parallel f_1(t+s_n-s_m-\tau)\\
&\qquad-f_1(t-\tau)\parallel^2 d\tau\\
&\qquad+4\varepsilon^2\int_{t-k}^{t-k+1}E\parallel f_1(\tau)\parallel^2 d\tau\\
&\qquad+4M^2\sup_{\tau\in[k-1,k]}e^{-2\delta\tau}\int_{k-1}^{k}E\parallel f_1(t+s_n-s_m\\
&\qquad+\tau)-f_1(t+\tau)\parallel^2 d\tau\\
&\qquad+4\varepsilon^2\int_{t+k}^{t+k-1}E\parallel f_1(\tau)\parallel^2 d\tau\\
&\quad\leq 4M^2e^{-2\delta k}e^{2\delta}\int_{t-k}^{t-k+1}E\parallel f_1(s+s_n-s_m)\\
&\qquad-f_1(s)\parallel^2 ds+4\varepsilon^2\parallel f_1\parallel^2_{S^2}\\
&\qquad+4M^2e^{-2\delta k}e^{2\delta}\int_{t+k-1}^{t+k}E\parallel f_1(s+s_n-s_m)\\
&\qquad-f_1(s)\parallel^2 ds+4\varepsilon^2\parallel f_1\parallel^2_{S^2}\\
&\quad<8(M^2e^{-2\delta k}e^{2\delta}+\parallel f_1\parallel^2_{S^2}\varepsilon)\varepsilon.
\end{aligned}$$

Thus, we immediately obtain that

$$\lim_{m\to\infty}\lim_{n\to\infty} E \parallel F_{1,k}(t+s_n-s_m)-F_{1,k}(t)\parallel^2=0$$

for each $t\in R$. Therefore, we get that $F_{1,k}\in AA(L^2(P,H))$. Applying Lemma 1, we deduce that the uniform limit

$$F_1(t)=\sum_{k=1}^{\infty}F_{1,k}(t)\in AA(L^2(P,H)).$$

Next, we will prove that $F_2\in PAP_0(L^2(P,H))$. It is obvious that $F_2\in BC(R,L^2(P,H)$, the left task is to show that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}E\parallel F_2(t)\parallel^2 dt=0.$$

For this, we consider

$$\begin{aligned}
&F_{2,k}(t)\\
&\quad=\int_{t-k}^{t-k+1}U(t,\tau)P(\tau)f_2(\tau)d\tau\\
&\qquad+\int_{t+k}^{t+k-1}U_Q(t,\tau)Q(\tau)f_2(\tau)d\tau\\
&\quad=\int_{k-1}^{k}U(t,t-\tau)P(t-\tau)f_2(t-\tau)d\tau\\
&\qquad+\int_{k}^{k-1}U_Q(t,t+\tau)Q(t+\tau)f_2(t+\tau)d\tau
\end{aligned}$$

for each $t\in R$ and $k=1,2,3,\ldots$. Then, by using the exponential dichotomy of $U(t,s)_{t\geq s}$ and the Ito integral, it follows that

$$\begin{aligned}
&E\parallel F_{2,k}(t)\parallel^2\\
&\quad\leq 2E\Bigg\|\int_{t-k}^{t-k+1}U(t,\tau)P(\tau)f_2(\tau)d\tau\Bigg\|^2\\
&\qquad+2E\Bigg\|\int_{t+k}^{t+k-1}U_Q(t,\tau)Q(\tau)f_2(\tau)d\tau\Bigg\|^2\\
&\quad\leq 2M^2\int_{t-k}^{t-k+1}e^{-2\delta(t-\tau)}E\parallel f_2(\tau)\parallel^2 d\tau\\
&\qquad+2M^2\int_{t+k}^{t+k-1}e^{2\delta(t-\tau)}E\parallel f_2(\tau)\parallel^2 d\tau\\
&\quad\leq 2M^2\int_{k-1}^{k}e^{-2\delta\tau}E\parallel f_2(t-\tau)\parallel^2 d\tau\\
&\qquad+2M^2\int_{-k}^{-k+1}e^{2\delta\tau}E\parallel f_2(t+\tau)\parallel^2 d\tau\\
&\quad\leq 2M^2\sup_{\tau\in[k-1,k]}e^{-2\delta\tau}\int_{k-1}^{k}E\parallel f_2(t-\tau)\parallel^2 d\tau\\
&\qquad+2M^2\sup_{\tau\in[-k,-k+1]}e^{2\delta\tau}\int_{-k}^{-k+1}E\parallel f_2(t+\tau)\parallel^2 d\tau\\
&\quad\leq 2M^2e^{-2\delta(k-1)}\parallel f_2\parallel^2_{S^2}+2M^2e^{2\delta(-k+1)}\parallel f_2\parallel^2_{S^2}\\
&\quad\leq 4M^2e^{-2\delta k}e^{2\delta}\parallel f_2\parallel^2_{S^2}.
\end{aligned}$$

Since $f_2^b\in PAP_0(L^2(0,1;L^2(P,H)))$, the above inequality leads to $F_2\in PAP_0(L^2(P,H))$. The above inequality leads also to

$$E\parallel F_{2,k}(t)\parallel^2\leq 4M^2e^{-2\delta k}e^{2\delta}\parallel f_2\parallel^2_{S^2}.$$

By using $4M^2e^{2\delta}\sum_{k=1}^{\infty} \parallel f_2 \parallel_{S^2}^2 e^{-2\delta k} < \infty$, we deduce from the well-known Weierstrass test that the series $\sum_{k=1}^{\infty} F_{2,k}(t)$ is uniformly convergent on R. Furthermore,

$$\begin{aligned} F_2(t) = & \int_{-\infty}^{t} U(t,\tau)P(\tau)f_2(\tau)d\tau \\ & + \int_{+\infty}^{t} U_Q(t,\tau)Q(\tau)f_2(\tau)d\tau = \sum_{k=1}^{\infty} F_{2,k}(t). \end{aligned}$$

Applying $F_{2,k} \in PAP_0(L^2(P,H))$ and the inequality

$$\begin{aligned} & \frac{1}{2T}\int_{-T}^{T} E \parallel F_2(t) \parallel^2 dt \\ & \quad \leq \frac{1}{T}\int_{-T}^{T} E\left\| F_2(t) - \sum_{k=1}^{n} F_{2,k}(t)\right\|^2 dt \\ & \quad + \frac{n}{T}\sum_{k=1}^{n}\int_{-T}^{T} E \parallel F_{2,k}(t) \parallel^2 dt, \end{aligned}$$

we deduce that the uniform limit $F_2(t) = \sum_{k=1}^{\infty} F_{2,k}(t) \in PAP_0(L^2(P,H))$, which ends the proof.

Next, we establish the existence and uniqueness theorem of pseudo almost automorphic mild solutions to evolution equation (1).

Definition 18. An $\mathcal{F}_t$-progressively measurable stochastic process $x : R \to L^2(P,H)$ is called a mild solution of the system (1) if $x(t)$ satisfies

$$\begin{aligned} x(t) = U(t,s)x(s) + & \int_{s}^{t} U(t,\tau)h(\tau, x(\tau - r))d\tau \\ & + \int_{s}^{t} U(t,\tau)f(\tau, x(\tau - r))dW(\tau) \end{aligned}$$

for all $t \geq s$ and all $s \in R$.

Lemma 4. If $x(\cdot) \in PAA(L^2(P,H))$, then $x(\cdot - r) \in PAA(L^2(P,H))$, where $r \geq 0$ is a fixed constant.

The proof is similar to the proof of Lemma 3.2 in [5], and we omit the details here.

Theorem 2. *Assume that* (H1)-(H5) *hold. If*

$$\frac{8M^2}{\delta^2}L_h + \frac{4M^2}{\delta}L_f < 1, \qquad (11)$$

then Eq. (1) *admits a unique pseudo almost automorphic mild solution on* R.

Proof. Using similar arguments as in the proof of Theorem 1, it is easy to see that each mild solution x to Eq. (1) is given by

$$\begin{aligned} x(t) = & \int_{-\infty}^{t} U(t,s)P(s)h(s, x(s-r))ds \\ & - \int_{t}^{+\infty} U_Q(t,s)Q(s)h(s, x(s-r))ds \\ & + \int_{-\infty}^{t} U(t,s)P(s)f(s, x(s-r))dW(s) \\ & - \int_{t}^{+\infty} U_Q(t,s)Q(s)f(s, x(s-r))dW(s), \ t \in R. \end{aligned}$$

Now consider the nonlinear operator on $BC(R, L^2(P,H))$ defined by

$$\begin{aligned} (\Psi x)(t) = & \int_{-\infty}^{t} U(t,s)P(s)h(s, x(s-r))ds \\ & - \int_{t}^{+\infty} U_Q(t,s)Q(s)h(s, x(s-r))ds \\ & + \int_{-\infty}^{t} U(t,s)P(s)f(s, x(s-r))dW(s) \\ & - \int_{t}^{+\infty} U_Q(t,s)Q(s)f(s, x(s-r))dW(s), \ t \in R. \end{aligned}$$

Let $x(\cdot) \in PAA(L^2(P,H)) \subset PAA^2(L^2(P,H))$. From Lemma 4 it is clear that $x(\cdot - s) \in PAA(L^2(P,H)) \subset PAA^2(L^2(P,H))$. Using (H4), (H5) and composition theorem on Stepanov-like pseudo almost automorphic functions, we deduce that $h(\cdot, x(\cdot - s)), f(\cdot, x(\cdot - s)) \in PAA^2(L^2(P,H))$. It is easy to check that $h(\cdot, x(\cdot - s)), f(\cdot, x(\cdot - s)) \in C(R, L^2(P,H))$. Applying Theorem 1 for $h(\cdot) = h(\cdot, x(\cdot - s)), f(\cdot) = f(\cdot, x(\cdot - s)) \in PAA^2(L^2(P,H))$, it follows that the operator Ψ maps $PAA(L^2(P,H))$ into $PAA(L^2(P,H))$.

Let $x, y \in PAA(L^2(P,H))$, then (H2), (H4) and (H5) yield that

$$\begin{aligned} & E \parallel (\Psi x)(t) - (\Psi y)(t) \parallel^2 \\ & \leq 4E\left\| \int_{-\infty}^{t} U(t,s)P(s)[h(s, x(s-r)) \right. \\ & \quad \left. - h(s, y(s-r))]ds \right\|^2 \\ & \quad + 4E\left\| \int_{t}^{+\infty} U_Q(t,s)Q(s)[h(s, x(s-r)) \right. \\ & \quad \left. - h(s, y(s-r))]ds \right\|^2 \\ & \quad + 4E\left\| \int_{-\infty}^{t} U(t,s)P(s)[f(s, x(s-r)) \right. \\ & \quad \left. - f(s, y(s-r))]dW(s) \right\|^2 \\ & \quad + 4E\left\| \int_{t}^{+\infty} U_Q(t,s)Q(s)[f(s, x(s-r)) \right. \\ & \quad \left. - f(s, y(s-r))]dW(s) \right\|^2. \end{aligned}$$

By using the Cauchy-Schwarz inequality, we first evaluate the first second term of the right-hand side

$$\begin{aligned} & 4E\left\| \int_{-\infty}^{t} U(t,s)P(s)[h(s, x(s-r)) - h(s, y(s-r))]ds \right\|^2 \\ & \quad + 4E\left\| \int_{t}^{+\infty} U_Q(t,s)Q(s)[h(s, x(s-r)) \right. \\ & \quad \left. - h(s, y(s-r))]ds \right\|^2 \\ & \leq 4M^2E\Big(\int_{-\infty}^{t} e^{-\delta(t-s)} \parallel h(s, x(s-r)) \\ & \quad - h(s, y(s-r)) \parallel ds \Big)^2 \\ & \quad + 4M^2E\Big(\int_{t}^{+\infty} e^{\delta(t-s)} \parallel h(s, x(s-r)) \end{aligned}$$

$$
\begin{aligned}
&-h(s,y(s-r)) \parallel ds\Big)^2 \\
&\le 4M^2\Big(\int_{-\infty}^t e^{-\delta(t-s)}ds\Big)\Big(\int_0^t e^{-\delta(t-s)} \\
&\quad \times E \parallel h(s,x(s-r)) - h(s,y(s-r)) \parallel^2 ds\Big) \\
&\quad +4M^2\Big(\int_t^{+\infty} e^{\delta(t-s)}ds\Big)\Big(\int_0^t e^{\delta(t-s)} \\
&\quad \times E \parallel h(s,x(s-r)) - h(s,y(s-r)) \parallel^2 ds\Big) \\
&\le 4M^2L_h\Big(\int_{-\infty}^t e^{-\delta(t-s)}ds\Big)\Big(\int_{-\infty}^t e^{-\delta(t-s)} \\
&\quad \times E \parallel x(s-r) - y(s-r) \parallel^2 ds\Big) \\
&\quad +4M^2L_h\Big(\int_t^{+\infty} e^{\delta(t-s)}ds\Big)\Big(\int_t^{+\infty} e^{\delta(t-s)} \\
&\quad \times E \parallel x(s-r) - y(s-r) \parallel^2 ds\Big) \\
&\le 4M^2L_h\Big(\int_{-\infty}^t e^{-\delta(t-s)}ds\Big)^2 \sup_{s\in R} E \parallel x(s) - y(s) \parallel^2 \\
&\quad +4M^2L_h\Big(\int_t^{+\infty} e^{\delta(t-s)}ds\Big)^2 \\
&\quad \times \sup_{s\in R} E \parallel x(s) - y(s) \parallel^2 \\
&\le \frac{8M^2}{\delta^2}L_h \parallel x-y \parallel_\infty^2 .
\end{aligned}
$$

As to the last second term, by the Ito integral, we get

$$
\begin{aligned}
&4E\Big\| \int_{-\infty}^t U(t,s)P(s)[f(s,x(s-r)) \\
&\quad -f(s,y(s-r))]dW(s)\Big\|^2 \\
&\quad +4E\Big\| \int_t^{+\infty} U_Q(t,s)Q(s)[f(s,x(s-r)) \\
&\quad -f(s,y(s-r))]dW(s)\Big\|^2 \\
&\le 4M^2\int_{-\infty}^t e^{-2\delta(t-s)}E \parallel f(s,x(s-r)) \\
&\quad -f(s,y(s-r)) \parallel^2 ds \\
&\quad +4M^2\int_t^{+\infty} e^{2\delta(t-s)}E \parallel f(s,x(s-r)) \\
&\quad -f(s,y(s-r)) \parallel^2 ds \\
&\le 4M^2L_f\int_{-\infty}^t e^{-2\delta(t-s)} \\
&\quad \times E \parallel x(s-r) - y(s-r) \parallel^2 ds \\
&\quad +4M^2L_f\int_t^{+\infty} e^{2\delta(t-s)} \\
&\quad \times E \parallel x(s-r) - y(s-r) \parallel^2 ds \\
&\le 4M^2L_f\int_{-\infty}^t e^{-2\delta(t-s)}ds \sup_{s\in R} E \parallel x(s) - y(s) \parallel^2 \\
&\quad +4M^2L_f\int_t^{+\infty} e^{2\delta(t-s)}ds \sup_{s\in R} E \parallel x(s) - y(s) \parallel^2 \\
&\le \frac{4M^2}{\delta}L_f \parallel x-y \parallel_\infty^2 .
\end{aligned}
$$

Thus, by combining the above inequality together, we obtain that, for each $t \in R$,

$$
\begin{aligned}
&E \parallel (\Psi x)(t) - (\Psi y)(t) \parallel^2 \\
&\quad \le \Big[\frac{8M^2}{\delta^2}L_h + \frac{4M^2}{\delta}L_f\Big] \parallel x-y \parallel_\infty^2 .
\end{aligned}
$$

Hence

$$\parallel \Psi x - \Psi y \parallel_\infty \le \sqrt{L_0} \parallel x-y \parallel_\infty,$$

where $L_0 = \frac{8M^2}{\delta^2}L_h + \frac{4M^2}{\delta}L_f < 1$, then the operator Ψ becomes a strict contraction. By the Banach contraction principle, we draw a conclusion that there exists a unique fixed point $x(\cdot)$ for Ψ in $PAA(L^2(P,H))$. It is clear that the fixed point is the mild solution to (1), which ends the proof.

IV. Application

Let $\Gamma \subset R^N (N \ge 1)$ be an open bounded subset with C^2 regular boundary $\partial\Gamma$ and let $H = L^2(\Gamma)$ be equipped with its natural topology $\| \cdot \|_{L^2(\Gamma)}$. We study the existence of Stepanov-like pseudo almost automorphic solutions to the following nonautonomous stochastic functional differential equations:

$$
\begin{aligned}
dz(t,x) &= a(t,x)\Delta z(t,x)dt + \mu_1(t,z(t-r,x))dt \\
&\quad +\mu_2(t,z(t-r,x))dW(t), (t,x) \in R\times\Gamma, \quad (12)\\
z(t,x) &= 0, (t,x) \in R\times\partial\Gamma, \quad (13)
\end{aligned}
$$

where $\Delta = \sum_{i=1}^N \partial^2/\partial x_i^2$ is the Laplace operator on Γ. $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$. In this system, μ_1, μ_2 are Stepanov-like pseudo almost automorphic continuous functions.

Let $A(t)$ be the linear operator given by

$$A(t)u = a(t,x)\Delta u$$

for all

$$u \in D(A(t)) = H_0^1(\Gamma) \cap H^2(\Gamma),$$

where $a : R\times\Gamma \to R$, in addition of being pseudo almost automorphic satisfies the following assumptions:

(i) $\inf_{t\in R, x\in\Gamma} a(t,x) = \xi_0$, and
(ii) there exists $L_0 > 0$ and $0 < \mu_0 \le 1$ such that

$$|a(t,x) - a(s,x)| \le L_0|s-t|^{\mu_0}$$

for all $t, s \in R$ uniformly in $x \in \Gamma$.

Clearly, $A(t)$ is sectorial and invertible. Moreover it can be shown that the analytic semigroup $(e^{-sA(t)})_{s\ge0}$ associated with $-A(t)$ is exponentially stable and hence hyperbolic. It is clear that the operators $A(t)$ defined above are invertible and satisfy Acquistapace-Terreni conditions.

Let $h, f \in PAA^2(R\times L^2(P,H)) \cap C(R\times L^2(P,H), L^2(P,H))$ be defined for $x \in \Gamma$ and $t \in R$ by

$$h(t,u)(x) = \mu_1(t, u(t-r)(x)),$$

$$f(t,u)(x) = \mu_2(t, u(t-r)(x)).$$

Then, the above equation can be written in the abstract form as the system (1). Assume that there exist constants L_1, L_2 such that

$$E \parallel \mu_1(t,u) - \mu_1(t,v) \parallel_{L^2(\Gamma)}^2 \le L_1 \parallel u-v \parallel_{L^2(\Gamma)}^2,$$

$$E \parallel \mu_2(t,u) - \mu_2(t,v) \parallel^2_{L^2(\Gamma)} \leq L_2 \parallel u - v \parallel^2_{L^2(\Gamma)}$$

for all $t \in R$ and each $u, v \in L^2(P, L^2(\Gamma))$.

Consequently all assumptions (H1)-(H5) are satisfied, then by Theorem 2, we deduce the following result.

Proposition 1. Under the above assumption, if

$$\frac{4M^2}{\delta}\left[\frac{2}{\delta}L_1 + L_2\right] < 1.$$

Then, Eq. (12)-(13) has a unique pseudo almost automorphic solution on R.

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