The Impact of Utility Functions on The Equilibrium Equity Premium In A Production Economy With Jump Diffusion

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Abstract*—In this paper, we study the impact of the utility functions on the risk averse investor's equilibrium equity premium in an economy with jump diffusion under an arbitrary jump size. In other words, we provide answers to the investor's questions like how much compensation for having taken up some risk is fair enough from an investment whose initial wealth at time $t = 0$ is invested in a production economy with jump diffusion under an arbitrary jump size if one is to consume exponentially or quadratically from an accumulating wealth among other utility functions. We considered the power, negative exponential, square root and quadratic utility functions. In these four risk averse utility functions considered, the deterministic time preference function $y(t)$ affects the optimal consumption of the investor but it has no effect on the diffusive and rare-events premia thereby not affecting the equilibrium equity premium. However, the total value of an investor's wealth affects the optimal consumption but has no effect on the equilibrium equity premium of the power and square root utility functions. In case of the quadratic and negative exponential utility functions, both the optimal consumption and the equity premium are affected by the investor's total wealth value. Specifically, the negativity or positivity in the jump size does not matter on the value of the equity premium of the negative exponential utility function as this only depends on the investor's wealth value.

***Index Terms*—utility function,equilibrium, equity risk premium, jump diffusion.**

I. Introduction

THE bottom line of any market is fairness. A market where everybody should have a chance to make or lose money. However, maintaining a fair market is not easy and some people who put in very little or nothing get away with quite huge sums of profit at the expense of others.

A price in the market change, from time to time. Thus, time is money and money has value. This time value of money poses significant questions as to whether one should keep more of one's wealth in the form of money (bond) or in terms of capital goods (stocks). In addition, there is the problem of finding a portfolio that will realize the future claim.

In search of answers to market fairness, researchers continually strive to find conditions that can make the market benefit those who invest reasonably in it. That is to say, making it arbitrage free.

Option pricing has been one among the significant branches in finance. It was first modeled under the Binomial discrete model. The Binomial option pricing model is based on the assumptions that the stock price will only take two possible values in a period and that trading will be at discrete points in time. However, these assumptions do not seem to capture the real stock price dynamics and this prompted the formulation of the well celebrated Black-Scholes model. The Black-Scholes model seems to capture the real price dynamics to some extent in that, as the number of periods increases, the stock price fluctuates over a large numbers of possible values and trading is nearly continuous. The model calibrates the Binomial parameters so that it converges to a continous time model in the limit. This is according to [1]. Despite the introduction of this continuous model, current research [[2], [3], [4], [5] and [6]] seems to suggest that this model is still not effective in capturing the real stock price process. This is because, by virtue of continuity, when the price of an asset changes from say 20 pula to 23 pula, it is as good as saying it has changed through all the prices in between (continuity is a smooth curve). It is accepted [[7] and [8]] that although price changes continuously, effectively it jumps continuously in time and hence the need to formulate a model involving a jump process and a continuous process. This new process called the jump-diffusion model and was first introduced by [9]. This jump-diffusion model seems to depend heavily on the distribution of the jump sizes and in Merton's model, the jump sizes are log-normally distributed. His solutions also relied upon a product of log-normal variates being log-normally distributed. [10], on the other hand, based his model on the double exponential distribution of jump amplitudes. His derivation stresses the significance of the memoryless property of the exponential distribution. To obtain an option formula, both [9] and [10] relied on particular properties of the distributions the jumps where following, but the model by [11] argued that, no special properties were needed, and obtained an option formula for any jump distribution while working in the Fourier space.

From the brief discussions of the models above, we have concluded that establishing an equilibrium model to explain the expected returns can either be approached by a consumption-based asset pricing model which is built on an exchange economy or a production-based asset pricing model built on a production economy. The consumption based was started by [12], and continued [13] but both researchers did not include jumps in their models. More recent studies [[14], [15], [16], and [17]] that are reviewed by [18] have included the impact of jumps. The major conclusion from these studies is that consumption-based approach maximizes the investor's expected utility by choosing an optimal level of consumption at every period, while the production based does the same,

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but additional, it leaves the rest in the production to grow for future consumption. This approach was initiated by [19] and extended to consider various situations by many studies [[20], [21], [22], [3], [23], [24], and [25]]. Studies by [26] and [27] extended the study by [19] to include jumps. The partial equilibrium models only includes derivative pricing and not asset pricing which has been studied by [28], and [29].

A. THE MODEL

As in [6], suppose in only one production process, we want to invest X_t units of a particular good from time t to the time $(t+dt)$, we introduce our model by adding together the drift term, a Brownian motion and a compensated compound Poisson process which results in a Jump Diffusion process;

$$\frac{dX_t}{X_{t^-}} = \mu dt + \delta dB_t + (e^x - 1)dN_t - \lambda E(e^x - 1)dt.$$

We subtract the expected value from the drift so that the process becomes more volatile and hence a martingale because its future is unexpected. The parameters μ, δ and λ are taken as constants with μ being the expected rate of return of the investment and hence the continuous part of X_t is log-normally distributed. X_{t^-} is the value of X_t before the jump occurs. In our model, the jump size x is first taken as arbitrary so the impact of the distribution on the rare event premium can be studied and then as a random number. This random number x and stochastic processes B_t and N_t are independent. This follows directly from the definition of Brownian motion as being a continuous process and the Poisson being discrete which we obviously know that continuous processes and discrete are independent. λ is the frequency of the Poisson process and $E(dN_t) = \lambda dt$ is the probability that N_t will jump once in the given period. E is just the expectation and $(e^x - 1)$ is the value of dN_t if a jump occurs. If the jump does not occur, $dN_t = 0$. This dN_t models the sudden changes as a result of rare events happening and dB_t models small continuous changes generated by the noise whose volatility is a constant δ.

We can clearly see that the compensated compound Poisson process $(e^x - 1)dN_t - \lambda E(e^x - 1)dt$ has the mean of zero ensuring that the expected return will be μ. It has the mean of zero because

$$E[(e^x - 1)dN_t - \lambda E(e^x - 1)dt] = 0$$

because $E(dN_t) = \lambda dt$ and $E(\lambda dt) = \lambda dt$.

If we apply Itô Lemma with Jumps we have,

$$\frac{dX_t}{X_{t^-}} = \mu dt + \delta dB_t + (e^x - 1)dN_t - \lambda E(e^x - 1)dt$$

$$\frac{dX_t}{X_{t^-}} = [\mu - \lambda E(e^x - 1)]dt + \delta dB_t + (e^x - 1)dN_t$$

$$dX_t = [\mu - \lambda E(e^x - 1)]X_t dt + \delta X_t dB_t + (e^x - 1)X_t dN_t$$

Now take diffusion part

$$d^* X_t = [\mu - \lambda E(e^x - 1)]X_t dt + \delta X_t dB_t$$

so that

$$dX_t = d^* X_t + (e^x - 1)X_t dN_t$$

then

$$dc = d^* c + [c(X_t e^x, t) - c(X_t, t)]dN_t$$

'that is if the jump occurs' where

$$d^* c = c_t dt + c_{X_t} d^* X_t + \frac{1}{2} c_{X_t X_t} (d^* X_t)^2.$$

Thus

$$dc = c_t dt + c_{X_t} d^* X_t + \frac{1}{2} c_{X_t X_t} (d^* X_t)^2 + [c(X_t e^x, t) - c(X_t, t)]dN_t.$$

Now

$$c(X_t, t) = \ln X_t$$

$$c_t = 0, \quad c_{X_t} = \frac{1}{X_t}, \quad X_t c_{X_t} = 1$$

$$c_{X_t X_t} = \frac{-1}{X_t^2}, \quad X_t^2 c_{X_t X_t} = -1$$

Also

$$\begin{aligned} c(X_t e^x, t) - c(X_t, t) &= \ln X_t e^x - \ln X_t \\ &= \ln \frac{X_t e^x}{X_t} \\ &= \ln e^x \\ &= x. \end{aligned}$$

Therefore dc becomes

$$d \ln X_t = \{[\mu - \lambda E(e^x - 1)] - \frac{1}{2}\delta^2\}dt + \delta dB_t + x dN_t.$$

By integration we have

$$\ln \frac{X_T}{X_t} \equiv Y_\tau = [\mu - \lambda E(e^x - 1) - \frac{1}{2}\delta^2]\tau + \delta B_t + \sum_{i=1}^{N_\tau} x_i$$

$$for \quad \tau = T - t$$

where Y_τ is the continuously compounded investment return over the period from time t to T and τ is the investment period.

Suppose also that, at the risk-free rate ρ, the money market account $X_0(t)$ is such that

$$dX_0(t) = \rho(t) X_0(t) dt$$

whose total supply is assumed to be zero. Consider here that ρ is risk-free because it is the rate for the non risky asset (money account).

Now consider the power utility function

$$U(r) = \frac{r^\beta}{\beta}, 0 < \beta < 1$$

which is the family with a constant relative risk aversion(CRRA). We study the general equilibrium of one investor in the production economy who wishes to maximize his expected utility function(reward function)

$$\max E_t \int_t^T y(t) U(r_t) dt,$$

where $U(r_t)$ is the utility(happiness) function which models the attitude of an investor towards the risk, $y(t)$ is time preference function , $0 \leq t \leq T$,and r_t is the consumption rate at time t.

We know that happiness (Utility) depends on consumption. The more consumption is, the more the utility. We notice here that since $0 < \beta < 1$,

$$\frac{dU}{dr} = U' = \frac{\beta r^{\beta-1}}{\beta} = r^{\beta-1} > 0$$

and

$$\frac{d^2U}{dr^2} = U'' = (\beta-1)r^{\beta-1-1} = (\beta-1)r^{\beta-2} < 0$$

which implies that utility is a concave function arising from the fact that $U'' < 0$. That is to say it increases gradualy as the consumption rate increases.

So if V_t is the total value of someone's investment in this production economy at any time t then

$$V_t = V_0(t) + V_1(t)$$

where $V_0(t)$is the value of the money market account and $V_1(t)$ is the value of the investment in the stock market. Thus $V_0(t) = (1-\omega)V_t$ and $V_1(t) = \omega V_t$ where ω is the wealth ratio or simply $(\omega, 1-\omega)$ is the portfolio. This means that this investor invests ω units of assets in the risky investment and $(1-\omega)$ units in the non risky investment. Notice, here that we set $V_0(t)$ and $V_1(t)$ in such a way that one is attracted to invest more in the stock market and less in the money market. This is so because, people don't usually like risks. Hence we make the riskless alternative unfair by giving less than its expected value. If however $U(r_t)$ was linear, then this investor would be indifferent between a risky alternative and the sure thing with an expected value same as the observed value(the sure thing). In this case, his graph of consumption rate against utility will just be a straight line $\alpha r_t = U(r_t)$ for some constant α.

The investor's optimal control problem then is to maximize his expected utility function

$$\max E_t \int_t^T y(t)U(r_t)dt,$$

subject to

$$\frac{dV_t}{V_t} = [\rho + \omega\phi - \omega\lambda E(e^x - 1) - \frac{r_t}{V_t}]dt$$
$$+\omega\delta dB_t + \omega(e^x - 1)dN_t$$

where $\phi \equiv \mu - \rho$ is the equity premium which shows the significant judgements one is making as to how much risk he or she sees in this economy (market) and what price one attaches to that risk. [9], observes that if the consumer has reasonable preferences then it is possible to use utility functions to describe these preferences. The general equilibrium occurs when $\omega = 1$. The wealth ratio ω and rate of consumption r_t are control variables.

B. RESULTS AND DISCUSSION

1) Preposition: An investor's equilibrium equity premium with CRRA power utility function $U(r_t) = \frac{r_t^\beta}{\beta}, 0 < \beta < 1$, in the production economy with jump diffusion is given by

$$\phi = -(\beta-1)\delta^2 + \lambda E[(e^x - 1)(1 - (e^x)^{\beta-1})].$$

where $\phi_\delta = -(\beta-1)\delta^2$ is the diffusive risk premium and $\phi_N = \lambda E[(e^x - 1)(1 - (e^x)^{\beta-1})]$ is the rare-event premium.

Refer to Appendix A for the proof.

It is not possible to remove expectation in the rare event premium and simulate graphs because the jump size is taking to be arbitrary in this paper. This is to allow us study the effect of the jump size on the rare-event premium.

We easily notice here that if we let the jump size $x = 0$, then $(e^x - 1) = 0$ and hence regardless of what values β, $0 < \beta < 1$ assumes, the rare-event premium is zero. This result is consistent with the real life phenomena of investment because the investor cannot take a jump risk and thereby benefit from it, if there is no jump in the process.

Now, since ρ is the risk free rate for the bond, then

$$\rho = \mu - \phi$$
$$\rho = \mu + (\beta-1)\delta^2 - \lambda E[(e^x - 1)(1 - (e^x)^{\beta-1})].$$

Here, μ cannot be zero because the process must evolve with respect to time but if we assume the process is not volatile that is $\delta = 0$, and there is no jump, then the risk free rate for the bond is equal to μ. This result is consistent to that obtained in [6].

The deterministic time preference function $y(t)$ affects the optimal consumption of this investor but it has no effect on the diffusive and rare-events premia. This is because it appears in the optimal consumption equation for $r(t)$ but it does not appear anywhere in the equation for ϕ. We also note here that, since μ, δ, and λ are constants, ρ is also a constant. In addition, $\phi_\delta = -(\beta-1)\delta^2$ is the price of the diffusive risk and $\phi_N = \lambda E[(e^x - 1)(1 - (e^x)^{\beta-1})]$ is the price of the jump risk. Infact, since $0 < \beta < 1$, we easily observe that the diffusive risk premium is always positive. On the other hand, for positive jump size x, $(e^x - 1) > 0$ and $(1 - (e^x)^{\beta-1}) > 0$ which implies $(e^x - 1)(1 - (e^x)^{\beta-1}) > 0$ and hence we have a positive jump risk premium. In the case where the jump size x, is negative, we have $(e^x - 1) < 0$ and $(1 - (e^x)^{\beta-1}) < 0$ meaning $(e^x - 1)(1 - (e^x)^{\beta-1}) > 0$ and the expectation of its value against a distribution of x is also positive. Thus the rare-event premium is always positive reqardless of the sign in the jump size.

2) Preposition: Under the CARA negative exponential utility function $U(r_t) = -e^{-\alpha r_t}, \alpha > 0$, the investor's equilibrium equity premium in the production economy with jump diffusion is given by

$$\phi = \delta^2\alpha V_t + \lambda E[(e^x - 1)(1 - e^{\alpha V_t(1-e^x)})]$$

where $\phi_\delta = \delta^2\alpha V_t$ is the diffusive risk premium and

$$\phi_N = \lambda E[(e^x - 1)(1 - e^{\alpha V_t(1-e^x)})]$$

is the rare-event premium.

Refer to Appendix B for the proof.

The deterministic time preference function $y(t)$ affects the optimal consumption of this investor but it has no effect on the diffusive and rare-events premia. This is because it appears in the optimal consumption equation for $r(t)$ but it does not appear anywhere in the equation for ϕ. However, the total wealth of the investor affects both the diffusive and rare-events premia. If the value of the wealth of this investor is positive (meaning he or she has no debts), then the diffusive risk premium $\phi_\delta = \delta^2 \alpha V_t$ is always positive and the rare-event premium will now depend on the jump size x. If the jump size is also positive, then $(e^x - 1) > 0$ and $(1-e^{\alpha V_t(1-e^x)}) > 0$, hence $(e^x-1)(1-e^{\alpha V_t(1-e^x)}) > 0$ and thus we have a positive rare event premium. If the jump size is negative, $(e^x - 1) < 0$ and $(1 - e^{\alpha V_t(1-e^x)}) < 0$, implying that $(e^x - 1)(1 - e^{\alpha V_t(1-e^x)}) > 0$ and thus we have a positive rare event premium because in either case, the expectation of a positive quantity is always positive.

On the other hand if the investor has debts that is the value of his wealth is negative, the diffusive risk premium is also always negative. The rare-event premium is negative for positive jump size x and is also negative for negative jump size and thus it is always negative as long as the investor's total value of wealth is negative regardless of the positivity or negativity in the jump size x.

We therefore suggest that, under the exponential utility function, one has to always make sure he or she avoids debts otherwise, there is no way the investor can gain once the wealth value enters negative.

3) Preposition: In the production economy with jump diffusion, the investor's equilibrium equity premium with square root utility function $U(r_t) = \sqrt{r_t}, r_t > 0$, is given by

$$\phi = \frac{1}{2}\delta^2 + \lambda E[(e^x - 1)(1 - e^{-\frac{1}{2}x})].$$

where $\phi_\delta = \frac{1}{2}\delta^2$ is the diffusive risk premium and

$$\phi_N = \lambda E[(e^x - 1)(1 - e^{-\frac{1}{2}x})$$

is the rare-event premium.

Refer to Appendix C for the proof.

If the jump size $x = 0$, the rare-event premium is zero and the investor benefits nothing in terms of taking the jump risk. The diffusive risk premium is always positive except for $\delta = 0$ and the rare-event premium depends only on the jump size. If the jump size is positive, $e^x - 1 > 0$ and $1 > e^{-\frac{1}{2}x}$, hence the premium is positive. If the jump size is negative then $1 < e^{-\frac{1}{2}x}$, and $e^x - 1 < 0$, and again the premium is positive. Thus the equilibrium equity premium is always positive regardless of the sign in the jump size.

This equilibrium equity premium is neither affected by the wealth value nor the time preference. We therefore encourage investors to strongly consider consuming with the square root utility function if they are to invest in the production economy with jump diffusion as their premium is not affected by the wealth value, sign in the jump size and time preference of their investments.

4) Preposition: An investor's equilibrium equity premium with quadratic utility function $U(r_t) = r_t - ar_t^2, a > 0$ in the production economy with jump diffusion is given by

$$\phi = \frac{2aV_t\delta^2}{1 - 2aV_t} + \lambda E[(e^x - 1)(1 - \frac{(1 - 2aV_t e^x)}{1 - 2aV_t})].$$

where $\phi_\delta = \frac{2aV_t\delta^2}{1-2aV_t}$ is the diffusive risk premium and

$$\phi_N = \lambda E[(e^x - 1)(1 - \frac{(1 - 2aV_t e^x)}{1 - 2aV_t})]$$

is the rare-event premium.

See Appendix D for the proof.

It can easily be proved that, if the investor's wealth at any time t is zero, that is $V_t = 0$, the equilibrium equity premium is also zero. This means that the investor must avoid debt if some compensation is to be realised by taking any risk in this investment. On the other hand, in the case of debt, $2aV_t\delta^2 < 0$, and $1 - 2aV_t > 0$ and therefore $\frac{2aV_t\delta^2}{1-2aV_t} < 0$, implying that the diffusive risk premium is negative. Thus the diffusive risk premium is negative when the wealth value is negative.

The other interesting result here is that, if the investor has no debt and as one's value of wealth increases, the diffusive risk premium again becomes negative but the investor's equilibrium equity premium is compensated by taking the jump risk because the rare-event premium depends on the jump size and the wealth value. If both are positive, the rare-event premium is positive. If both are negative, the rare-event premium is again positive. We therefore advise investors consuming quadratically in an economy as this to invest more in the risky assets as either way the compensation realised by taking the jump risk is always positive.

II. Conclusion

In conclusion, a risk averse investor who wishes to invest in the production economy with jumps under an arbitrary jump size would like to know how much compensation one can expect from an investment in stocks and bond so he can determine which proportions to invest in them. Of course, this will depend on the investor's attitude towards risk taking. The more one becomes risk averse (risk fearing), the less one invests in stocks (risky assets) and vice-versa. We considered four risk averse utility functions namely the power, exponential, square root and quadratic utility functions and although we could not provide graphs because of the jump size being arbitrary hence making it not possible to find its expectation in the rare event premium, we were able to compute for each one of them analytical formulas for the equity premium required for an investment.

Appendix A

The optimality condition is by the Hamilton-Jacobi-Bellman (HJB) equation

$$E_t[dJ + yU(r_t)dt] = 0.$$

We apply the Itô Lemma with Jumps as follows:

$$\frac{dV_t}{V_t} = [\rho + \omega\phi - \omega\lambda E(e^x - 1) - \frac{r_t}{V_t}]dt$$
$$+\omega\delta dB_t + \omega(e^x - 1)dN_t$$

multiplying throughout by V_t we have

$$dV_t = [\rho + \omega\phi - \omega\lambda E(e^x - 1) - \frac{r_t}{V_t}]V_t dt$$

$$+\omega\delta V_t dB_t + \omega(e^x - 1)V_t dN_t.$$

Let d^*V_t be the diffusion part so that

$$d^*V_t = [\rho + \omega\phi - \omega\lambda E(e^x - 1) - \frac{r_t}{V_t}]V_t dt$$
$$+\omega\delta V_t dB_t$$

then

$$d^*J = J_t dt + J_{V_t} d^*V_t + \frac{1}{2} J_{V_t V_t}(d^*V_t)^2$$

so that

$$dJ = d^*J + [J(V_t(1 + \omega(e^x - 1)), t) - J(V_t, t)]dN_t$$

$$= J_t dt + J_{V_t}[\rho + \omega\phi - \omega\lambda E(e^x - 1) - \frac{r_t}{V_t}]V_t dt$$
$$+J_{V_t}\omega\delta V_t dB_t + \frac{1}{2} J_{V_t V_t}[\omega^2\delta^2 V_t^2 dt]$$
$$+[J(V_t(1 + \omega(e^x - 1)), t) - J(V_t, t)]dN_t.$$

Here, J_{V_t} is the partial derivative of J with respect to V_t and so is the case with all subscripts of J. Now

$$E_t[dJ + yU(r_t)dt] = 0$$

hence

$$= \max_{(r_t,\omega)}\{J_t dt + J_{V_t}[\rho + \omega\phi - \omega\lambda E(e^x - 1)]V_t dt$$
$$-J_{V_t} r_t dt + J_{V_t}\omega\delta V_t E(dB_t) + \frac{1}{2} J_{V_t V_t}[\omega^2\delta^2 V_t^2 dt]$$
$$+E[J(V_t(1 + \omega(e^x - 1)), t) - J(V_t, t)]E(dN_t)\}$$
$$+yU(r_t)dt = 0$$

and since $E(dB_t) = 0$ and $E(dN_t) = \lambda dt$ we have the result

$$= \max_{(r_t,\omega)}\{J_t dt + J_{V_t}[\rho + \omega\phi - \omega\lambda E(e^x - 1)]V_t dt$$
$$-J_{V_t} r_t dt + \frac{1}{2} J_{V_t V_t}[\omega^2\delta^2 V_t^2 dt]$$
$$+E[J(V_t(1 + \omega(e^x - 1)), t) - J(V_t, t)]\lambda dt\}$$
$$+yU(r_t)dt = 0$$

multiplying by $\frac{1}{dt}$ on both sides of the equation yields

$$\max_{(r_t,\omega)}\{J_t + J_{V_t}[\rho + \omega\phi - \omega\lambda E(e^x - 1)]V_t - J_{V_t} r_t$$
$$+\frac{1}{2} J_{V_t V_t}[\omega^2\delta^2 V_t^2] + \lambda E[J(V_t(1 + \omega(e^x - 1)), t)]$$
$$-\lambda J(V_t, t)\} + yU(r_t) = 0. \tag{1}$$

Taking partial derivatives of this Bellman equation with respect to r_t and ω, we obtain equations

$$-J_{V_t} + yU(r_t) = 0, \tag{2}$$

$$[\phi - \lambda E(e^x - 1)]V_t J_{V_t} + J_{V_t V_t}\omega\delta^2 V_t^2$$
$$+\lambda E[J_{V_t}(V_t(1 + \omega(e^x - 1)), t)V_t(e^x - 1)] = 0. \tag{3}$$

Solving for ϕ in the second equation gives

$$\phi V_t J_{V_t} = \lambda V_t J_{V_t} E(e^x - 1) - \omega\delta^2 V^2 J_{V_t V_t}$$
$$-\lambda E[J_{V_t}(V_t(1 + \omega(e^x - 1), t)V_t(e^x - 1)]$$

Taking $\omega = 1$ and dividing by $V_t J_{V_t}$ we have

$$\phi = \lambda E(e^x - 1) - \frac{J_{V_t V_t}}{J_{V_t}}\delta^2 V_t - \lambda\frac{1}{J_{V_t}} E[J_{V_t}(V_t e^x, t)(e^x - 1)]. \tag{4}$$

Substituting this ϕ into the Bellman equation we get

$$J_t + \rho V_t J_{V_t} - \frac{1}{2}\delta^2 V_t^2 J_{V_t V_t} - J_{V_t} r_t + \lambda E[J(V_t e^x, t)]$$
$$-\lambda V_t E[J_{V_t}(V_t e^x, t)(e^x - 1)] - \lambda J(V_t, t) + yU(r_t) = 0 \tag{5}$$

Taking

$$U(r_t) = \frac{r_t^\beta}{\beta}, 0 < \beta < 1,$$

we solve for $J(V_t, t)$ based on the assumption that

$$J(V_t, t) = Q(t)\frac{V_t^\beta}{\beta}. \tag{6}$$

To solve for the optimal consumption, we use first order conditiion (2) and proceed as follows:

$$yU'(r_t) = J_{V_t}$$

which implies

$$y[r_t^{\beta-1}] = Q(t)V_t^{\beta-1}$$
$$r_t = (\frac{Q(t)V_t^{\beta-1}}{y})^{\frac{1}{\beta-1}}$$

and therefore

$$r_t = (\frac{Q(t)}{y})^{\frac{1}{\beta-1}} V_t \tag{7}$$

is the optimal consumption we require.

Also, substituting the functions $J(V_t, t) = Q(t)\frac{V_t^\beta}{\beta}$, $J_{V_t} = Q(t)V_t^{\beta-1}$ and $J_{V_t V_t} = (\beta - 1)Q(t)V_t^{\beta-2}$ into the integro-partial differential equation (5) gives

$$\frac{Q(t)V_t^\beta}{\beta} + \rho Q(t)V_t^\beta - \frac{1}{2}\delta^2(\beta - 1)Q(t)V_t^\beta$$
$$-\lambda V_t^\beta Q(t)E[e^{x(\beta-1)}(e^x - 1)] + \lambda Q(t)V_t^{\beta-1}E[e^{x(\beta-1)}]$$
$$-\lambda Q(t)[\frac{V_t^\beta}{\beta}]$$
$$-Q(t)V_t^\beta[\frac{Q(t)}{y}]^{\frac{1}{\beta-1}} + \frac{y}{\beta}V_t^\beta[(\frac{Q(t)}{y})^{\frac{\beta}{1-\beta}}] = 0$$

Differentiating with respect to V_t and dividing by $V_t^{\beta-1}$ we get

$$Q(t) + \rho Q(t)\beta - \frac{1}{2}\delta^2\beta(\beta - 1)Q(t)$$
$$-\lambda\beta Q(t)E[e^{x(\beta-1)}(e^x - 1)]$$
$$+\lambda Q(t)(\beta - 1)V_t^{-1}E[e^{x(\beta-1)}] - \lambda Q(t)$$
$$-Q(t)\beta[\frac{Q(t)}{y}]^{\frac{1}{\beta-1}} + y[(\frac{Q(t)}{y})^{\frac{\beta}{1-\beta}}] = 0.$$

Thus,

$$Q(t) + aQ(t) + y[(\frac{Q(t)}{y})^{\frac{\beta}{1-\beta}}] = 0.$$

and

$$Q(T) = 0$$

as terminal condition where

$$a = \rho\beta - \frac{1}{2}\delta^2\beta(\beta-1) - \lambda\beta E[e^{x(\beta-1)}(e^x-1)]$$

$$+\lambda(\beta-1)V_t^{-1}E[e^{x(\beta-1)}] - \lambda - \beta[\frac{Q(t)}{y}]^{\frac{1}{\beta-1}}.$$

Substituting (6) into (4) gives us a formula for the equity premium

$$\phi = -(\beta-1)\delta^2 + \lambda E[(e^x-1)(1-(e^x)^{\beta-1})].$$

APPENDIX B

Suppose now that we took the exponential utility function

$$U(r_t) = -e^{-\alpha r_t}, \alpha > 0,$$

from the Constant Absolute Risk Aversion (CARA) family then

$$U'(r_t) = \alpha e^{-\alpha r_t} > 0,$$

$$U''(r_t) = -\alpha^2 e^{-\alpha r_t} < 0,$$

which implies the utility function is concave. Now the coefficient of absolute risk aversion is

$$\frac{-U''(r_t)}{U'(r_t)} = \frac{\alpha^2 e^{-\alpha r_t}}{\alpha e^{-\alpha r_t}} = \alpha > 0$$

To solve for $J(V_t, t)$, we guess that

$$J(V_t, t) = -Q_t e^{-\alpha V_t},$$

then we solve for optimal consumption using first order condition (2) as follows

$$yU'(r_t) = J_{V_t}$$

which implies

$$y(\alpha e^{-\alpha r_t}) = \alpha Q_t e^{-\alpha V_t}$$

and hence

$$-\alpha r_t = \ln[\frac{Q_t e^{-\alpha V_t}}{y}]$$

and therefore we have our optimal consumption for this investor as

$$r_t = \frac{-1}{\alpha}\ln[\frac{Q_t e^{-\alpha V_t}}{y}], \alpha > 0.$$

Substituting the functions

$$J(V_t, t) = -Q_t e^{-\alpha V_t},$$

$$J_{V_t}(V_t, t) = \alpha Q_t e^{-\alpha V_t}$$

and

$$J_{V_t V_t}(V_t, t) = -\alpha^2 Q_t e^{-\alpha V_t}$$

into the integro-partial differential equation (5) we have

$$-Q_t e^{-\alpha V_t} + \rho\alpha Q_t V_t e^{-\alpha V_t} + \frac{1}{2}\delta^2\alpha^2 Q_t V_t^2 e^{-\alpha V_t}$$

$$-\lambda\alpha Q_t E[V_t e^{-\alpha V_t e^x}(e^x-1)] + \lambda\alpha Q_t E[e^{-\alpha V_t e^x}]$$

$$-\lambda\alpha Q_t e^{-\alpha V_t} - \alpha Q_t e^{-\alpha V_t} r_t - y e^{\ln[\frac{Q_t e^{-\alpha V_t}}{y}]} = 0.$$

Substituting for r_t gives

$$-Q_t e^{-\alpha V_t} + \rho\alpha Q_t V_t e^{-\alpha V_t} + \frac{1}{2}\delta^2\alpha^2 Q_t V_t^2 e^{-\alpha V_t}$$

$$-\lambda\alpha Q_t E[V_t e^{-\alpha V_t e^x}(e^x-1)] + \lambda\alpha Q_t E[e^{-\alpha V_t e^x}]$$

$$-\lambda\alpha Q_t e^{-\alpha V_t} + Q_t e^{-\alpha V_t}\ln[\frac{Q_t e^{-\alpha V_t}}{y}]$$

$$-y e^{\ln[\frac{Q_t e^{-\alpha V_t}}{y}]} = 0.$$

Differentiating with respect to V_t and substituting into (4) gives the equity premium

$$\phi = \delta^2\alpha V_t + \lambda E[(e^x-1)(1-e^{\alpha V_t(1-e^x)})].$$

APPENDIX C

Consider the square root utility function

$$U(r_t) = \sqrt{r_t}, r_t > 0,$$

then

$$U'(r_t) = \frac{1}{2}(r_t)^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{r_t}} > 0$$

and

$$U''(r_t) = -\frac{1}{4}(r_t)^{-\frac{1}{2}-1} = -\frac{1}{4}(r_t)^{-\frac{3}{2}} = -\frac{1}{4}\times\frac{1}{\sqrt{r_t^3}}$$

$$= -\frac{1}{4\sqrt{r_t^3}} < 0$$

implying that it is a concave function. The coefficient of aversion is

$$RRA = -\frac{U'' r_t}{U'}$$

$$= \frac{\frac{r_t}{4\sqrt{r_t^3}}}{\frac{1}{2\sqrt{r_t}}} = \frac{r_t}{4\sqrt{r_t^3}} \times 2\sqrt{r_t}$$

$$= \frac{2r_t\sqrt{r_t}}{4r_t\sqrt{r_t}}$$

$$= \frac{1}{2} > 0.$$

This can easily be seen that RRA is $\frac{1}{2}$ by virtue of square root since CRRA family are of form $U(c) = c^\beta$ for some $RRA = \beta > 0$.

To solve for $J(V_t, t)$, we conjecture that

$$J(V_t, t) = Q_t\sqrt{V_t} = Q_t V_t^{\frac{1}{2}}$$

$$J_{V_t}(V_t, t) = \frac{1}{2}Q_t V_t^{\frac{1}{2}-1} = \frac{1}{2}Q_t V_t^{-\frac{1}{2}} = \frac{Q_t}{2\sqrt{V_t}}$$

$$J_{V_t V_t}(V_t, t) = -\frac{1}{4}Q_t V^{-\frac{1}{2}-1} = -\frac{1}{4}Q_t V_t^{-\frac{3}{2}}$$

$$= -\frac{Q_t}{4\sqrt{V_t^3}} = -\frac{Q_t}{4V_t\sqrt{V_t}}$$

then we solve for optimal consumption

$$yU'(r_t) = J_{V_t}$$
$$y(\frac{1}{2\sqrt{r_t}}) = \frac{Q_t}{2\sqrt{r_t}}$$
$$(\frac{y}{2\sqrt{r_t}}) = \frac{Q_t}{2\sqrt{r_t}}$$
$$2y\sqrt{V_t} = 2Q_t\sqrt{r_t}$$
$$\frac{2y\sqrt{V_t}}{2Q_t} = \sqrt{r_t}$$

which implies

$$\frac{y\sqrt{V_t}}{Q_t} = (r_t)^{\frac{1}{2}}$$

and therefore

$$r_t = (\frac{y\sqrt{V_t}}{Q_t})^2 = \frac{y^2V_t}{Q_t^2}$$

is our optimal consumption. We can clearly see that it is affected by the time preference function $y(t)$ and also by (V_t), the total wealth at time t.

Substituting J_t, J_{V_t} and $J_{V_tV_t}$ into the integro-partial differential equation we get

$$Q_tV_t^{\frac{1}{2}} + \rho V_t\frac{Q_t}{2V_t^{\frac{1}{2}}} - \frac{1}{2}\delta^2V_t^2(-\frac{Q_t}{4V_tV_t^{\frac{1}{2}}})$$
$$-\lambda V_tE[\frac{Q_t}{2(V_te^x)^{\frac{1}{2}}}(e^x-1)] + \lambda E[Q_t(V_te^x)^{\frac{1}{2}}]$$
$$-\lambda Q_tV_t^{\frac{1}{2}} - \frac{Q_t}{2V_t^{\frac{1}{2}}}r_t + y(r_t)^{\frac{1}{2}} = 0$$

Substituting for the optimal consumption r_t we get

$$Q_tV_t^{\frac{1}{2}} + \frac{1}{2}\rho Q_tV_t^{\frac{1}{2}} + \frac{1}{8}\delta^2Q_tV_t^{\frac{1}{2}}$$
$$-\frac{1}{2}\lambda Q_tV_t^{\frac{1}{2}}E[e^{-\frac{1}{2}x}(e^x-1)] + \lambda Q_tV_t^{\frac{1}{2}}E[e^{\frac{1}{2}x}]$$
$$-\lambda Q_tV_t^{\frac{1}{2}} - \frac{1}{2Q_t}y^2V_t^{\frac{1}{2}} + \frac{y^2V_t^{\frac{1}{2}}}{Q_t} = 0.$$

Differentiating with respect to V_t, dividing through out by $V_t^{-\frac{1}{2}}$ and substituting into (4) gives the equilibrium equity premium as

$$\phi = \frac{\delta^2}{2} + \lambda E[(e^x-1)(1-e^{-\frac{1}{2}x})].$$

APPENDIX D

Suppose now that this investor consumed quadratically from the investment, that is

$$U(r_t) = r_t - ar_t^2, a > 0,$$

then

$$U'(r_t) = 1 - 2ar_t > 0,$$
$$U''(r_t) = -2a < 0$$

implying $U(r_t)$ is a concave utility function by virtue of $U''(r_t)$ being negative. The

$$RRA = -\frac{U''r_t}{U'}$$
$$= \frac{2ar_t}{1-2ar_t} > 0$$

because $1 - 2ar_t > 0$ and $2ar_t > 0$.

We solve for $J(V_t,t)$ by conjecturing that

$$J(V_t,t) = Q_t(V_t - aV_t^2)$$
$$= Q_tV_t - aQ_tV_t^2$$

so that

$$J_{V_t}(V_t,t) = Q_t - 2aQ_tV_t$$

and

$$J_{V_tV_t}(V_t,t) = -2aQ_t.$$

We then proceed to solve for the optimal consumption by using the first order condition (2) as follows

$$yU'(r_t) - J_{V_t} = 0,$$

that is

$$yU'(r_t) = J_{V_t}.$$

Now, substituting $U'(r_t)$ and J_{V_t} gives

$$y(1-2ar_t) = Q_t - 2aQ_tV_t$$

so that

$$y - 2ar_ty = Q_t - 2aQ_tV_t$$
$$-2ar_ty = Q_t - 2aQ_tV_t - y$$

which implies

$$2ar_ty = -Q_t + 2aQ_tV_t + y$$

and therefore we have our optimal consumption result as

$$r_t = \frac{y - Q_t + 2aQ_tV_t}{2ay}$$

which is affected by both the time preference function $y(t)$ and V_t the total wealth at any time t.

Substituting into the integro-partial differencial equation (5) gives

$$Q_tV_t - aQ_tV_t^2 + \rho V_tQ_t - 2a\rho Q_tV_t^2 + \delta^2V_t^2aQ_t$$
$$-\lambda V_tE[(Q_t - 2aQ_tV_te^x)(e^x-1)]$$
$$+\lambda E[Q_tV_te^x - aQ_tV_t^2e^{2x}] - \lambda Q_tV_t$$
$$+\lambda aQ_tV_t^2 - Q_tr_t + 2aQ_tr_tV_t + yr_t - yar_t^2 = 0.$$

Substituting for r_t and simplifying gives

$$Q_tV_t - aQ_tV_t^2 + \rho V_tQ_t - 2a\rho Q_tV_t^2 + \delta^2V_t^2aQ_t$$
$$-\lambda V_tQ_tE[(e^x-1)] + 2a\lambda V_t^2Q_tE[e^x(e^x-1)]$$
$$+\lambda Q_tV_tE[e^x] - \lambda Q_taV_t^2E[e^{2x}] - \lambda Q_tV_t + \lambda aQ_tV_t^2$$
$$-\frac{Q_t}{2a} + \frac{Q_t^2}{2ay} - \frac{Q_t^2V_t}{y} + Q_tV_t - \frac{Q_t^2V_t}{y}$$
$$+\frac{2aQ_t^2V_t^2}{y} + \frac{y}{2a} - \frac{Q_t}{2a} + Q_tV_t$$
$$-\frac{(y-Q_t)^2}{4ay} - \frac{(y-Q_t)Q_tV_t}{y} - \frac{aQ_t^2V_t^2}{y} = 0.$$

Differentiating with respect to V_t and adding together common terms gives

$$Q_t - 2aQ_tV_t + \rho Q_t - 4a\rho Q_tV_t + 2\delta^2V_taQ_t$$

$$-\lambda Q_t E[(e^x - 1)] + 4a\lambda V_t Q_t E[e^x(e^x - 1)]$$
$$+\lambda Q_t E[e^x] - 2\lambda Q_t a V_t E[e^{2x}] - \lambda Q_t + 2\lambda a Q_t V_t$$
$$-\frac{2Q_t^2}{y} + 2Q_t - \frac{(y - Q_t)Q_t}{y} + \frac{2aQ_t^2 V_t}{y} = 0$$

and

$$Q(T) = 0$$

as terminal conditions.

So substituting into (4) we have the desired equilibrium equity premium formula

$$\phi = \frac{2aV_t\delta^2}{1 - 2aV_t} + \lambda E[(e^x - 1)(1 - \frac{(1 - 2aV_t e^x)}{1 - 2aV_t})].$$

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