Robust Exponential Stabilization of a Class of Stochastic Time-Delay Systems

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Abstract—**This paper is concerned with the exponential stability problem for a class of grey (uncertain) stochastic systems with time delays. To the best of the author' knowledge, up to now, there is little attention paid to the stability analysis problem of grey stochastic system except limited papers, due to the lack of fully understand or information source to the grey systems. This situation motives our present research. In this paper, by using a suitable Lyapunov-Krasovskii functional, in particular, employing decomposition technique of continuous matrix-covered sets of grey matrix, novel sufficient stability criteria are derived to guarantee the exponential stability in mean square and almost surely exponential stability for our considered systems. Finally, a numerical example is given to demonstrate the effectiveness of the proposed criteria.**

Index Terms—**Grey Stochastic Systems, Time Delays, Exponential Stability, Lyapunov-Krasovskii Functional, Decomposition Technique**

I. Introduction

As is well-known, many physical systems have variable structures subject to random abrupt changes, which may result from abrupt phenomena such as random failures of the components, sudden environmental changes and so on. Therefore, stochastic model has come to play an important role in many branches of science or industry, and the stability analysis for stochastic systems has been widely investigated in recent years [1-8]. On the other hand, time-delay is frequently encountered in many real-word control systems, and it often results in instability or poor performance. For instance, the existence of time-delays often brings about oscillation, divergence, even instability, which is the disadvantage of applications of neural networks. Hence, the study of stochastic systems with time delays has received much attention from many scholars, and a large amount of results have appeared in the literature [9-20]. For example, in [13], author proposed two sufficient conditions for the stability of dynamic systems with mulitple time-varying delays and nonlinear uncertainties, by utilizing the Lyapunov stability theory and the linear matrix inequality approach.

However, in practical applications, it is often very difficult to obtain some parameters of stochastic systems accurately, because of the lack of fully understand and information source to the systems. So, we often have to estimate the parameters of systems. As pointed out by [21] that, assume that the parameters of stochastic systems are estimated by using grey numbers, the systems can be described uncertainly and become grey (uncertain) stochastic systems, which leads to the research on the stability of grey stochastic systems, accordingly. But, till now, there have been very few papers tackling the stability problem of the stochastic systems except limited papers, such as [21-24], which is still open and remains challenging. In [21], the p-moment exponential robust stability for the grey stochastic systems with distributed delays and interval parameters was studied, and the obtained results were very significant and innovative.

Motivated by the above observations, in this paper, we deal with the exponential stability problem for a class of grey stochastic systems with time delays. First, we construct a suitable Lyapunov-Krasovskii functional. Then, by using the decomposition technique of the continuous matrix-covered sets of grey matrix, and some well-known differential formulas, the sufficient stability criteria are obtained, which will ensure the systems in the mean square and almost surely exponential stability. Finally, an example is provided to show the effectiveness of the obtained result.

Notations: The following notations will be used throughout this paper. R^n denotes the n-dimensional Euclidean space, $R^{m\times n}$ is the set of all m×n real matrices. The superscript "T" denotes matrix transposition, $\|\cdot\|$ stands for the Euclidean norm for vector or the spectral norm of matrices. For real symmetric matrices X and Y , the notation $X \geq Y$ (respectively, $X > Y$), means that $X - Y$ is positive semi-definite (respectively, positive definite). Moreover, let $\left(\Omega, F, \{F_t\}_{t\geq 0}, P\right)$ be the complete probability space with a filtration $\{F_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous), Let $\tau > 0$ and $C([-\tau,0];R^n)$ be the family of continuous functions φ from $[-\tau,0]$ to R^n , Let $L^2_{F_0}([-\tau,0];R^n)$ be the family of all F_0-measurable bounded $C([-\tau,0];R^n)$-valued random variables $\xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$

II. Preliminaries and Problem formulation

In this paper, we consider a class of grey stochastic systems with time delays as follows:

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$$\begin{cases} dx(t) = [A(\otimes)x(t) + B(\otimes)x(t-\tau)]dt \\ \qquad + g(x(t), x(t-\tau), t)dw(t), \quad t \ge 0 \\ x_0 = \xi, \ \ \xi \in L^2_{F_0}([-\tau,0];R^n), \quad -\tau \le t \le 0 \end{cases} \tag{2.1}$$

Where $A(\otimes)$ and $B(\otimes)$ are grey (uncertain) n×n matrices, and let $A(\otimes) = (\otimes^a_{ij})$, $B(\otimes) = (\otimes^b_{ij})$.

Here,

$\otimes^a_{ij}, \otimes^b_{ij}$ are said to be grey elements of $A(\otimes)$ and $B(\otimes)$. Moreover, define

$$[L_a, U_a] = \{A(\hat{\otimes}) = (a_{ij}) : \underline{a_{ij}} \le a_{ij} \le \overline{a_{ij}}, i, j = 1,2,..n\}$$

$$[L_b, U_b] = \{B(\hat{\otimes}) = (b_{ij}) : \underline{b_{ij}} \le b_{ij} \le \overline{b_{ij}}, i, j = 1,2,..n\}$$

Which are said to be the continuous matrix-covered sets of $A(\otimes)$ and $B(\otimes)$.

Here,

$A(\hat{\otimes})$ and $B(\hat{\otimes})$ are whitened (deterministic) matrices of $A(\otimes)$ and $B(\otimes)$, $[\underline{a_{ij}}, \overline{a_{ij}}]$ and $[\underline{b_{ij}}, \overline{b_{ij}}]$ are said to be the number-covered sets of $\otimes^a_{ij}, \otimes^b_{ij}$.

In addition, the following assumptions are made on grey stochastic time-delay systems:

(H1) $H : R^n \times R^n \times R_+ \to R^{n\times n}$, and satisfies the local Lipschitz condition.

(H2) Supposing there exist constants $\alpha \ge 0$, $\beta \ge 0$, for arbitrary $(x, y, t) \in H : R^n \times R^n \times R_+$, the following inequality holds:

$$Trace[g^T(x, y, t)g(x, y, t) \le \alpha|x|^2 + \beta|y|^2$$

Definition 2.1. System(2.1)is said to be exponentially stable in mean square, if for all $\xi \in L^2_{F_0}([-\tau,0];R^n)$ and whitened matrices $A(\hat{\otimes}) \in [L_a, U_a]$, $B(\hat{\otimes}) \in [L_b, U_b]$, there exist constants $r > 0$ and $C > 0$, such that

$$E|x(t;\xi)|^2 \le Ce^{-rt} \sup_{-\tau\le\theta\le0} E|\xi(\theta)|^2, t \ge 0$$

Definition 2.2. Systems(2.1)is said to be almost surely exponential stability, if for all $\xi \in L^2_{F_0}([-\tau,0];R^n)$ and whitened matrices $A(\hat{\otimes}) \in [L_a, U_a]$, $B(\hat{\otimes}) \in [L_b, U_b]$, there exists constant $\hat{r} > 0$, such that

$$\limsup_{t\to\infty} \frac{1}{t} \ln|x(t;\xi)| \le -\frac{\hat{r}}{2}, \quad a.s.$$

First, let us introduce the following lemmas, in particular, lemma 2.1, which will be important for the proof of our main results.

Lemma 2.1. [21] If $A(\otimes) = (\otimes_{ij})_{m\times n}$ is a grey matrix, $[\underline{a_{ij}}, \overline{a_{ij}}]$ is a number-covered sets of grey element $\otimes_{ij}$, then for arbitrary whitened matrix $A(\hat{\otimes}) \in [L_a, U_a]$, we have

i) $A(\hat{\otimes}) = \dfrac{U_a + L_a}{2} + \Delta A$

ii) $0 \le \Delta A \le \dfrac{U_a - L_a}{2}$

iii) $\|A(\hat{\otimes})\| \le \left\|\dfrac{U_a + L_a}{2}\right\| + \left\|\dfrac{U_a - L_a}{2}\right\|$

Where $L_a = (\underline{a_{ij}})_{m\times n}$, $U_a = (\overline{a_{ij}})_{m\times n}$, $\Delta A = (\dfrac{\overline{a_{ij}} - \underline{a_{ij}}}{2}\hat{r}_{ij})_{m\times n}$,

Here, $\hat{r}_{ij}$ is a whitened number of γ_{ij}, $\hat{r}_{ij} \in [-1,1]$, $[-1,1]$ is a number-covered sets of γ_{ij}, and γ_{ij} is said to be a unit grey number.

Noting that for arbitrary whitened number $\hat{\otimes}_{ij} \in [\underline{a_{ij}}, \overline{a_{ij}}]$, there must exist corresponding whitened number $\hat{r}_{ij} \in [-1,1]$, such that $\hat{\otimes}_{ij} = \dfrac{\overline{a_{ij}} + \underline{a_{ij}}}{2} + \dfrac{\overline{a_{ij}} - \underline{a_{ij}}}{2}\hat{r}_{ij}$. Moreover, by applying the inequality of nonnegative matrix norm, it follows that the Lemma 2.1 holds.

Lemma 2.2. [25] For any vectors $x, y \in R^n$, N is real matrix of appropriate dimensions, and constant ε > 0, the following inequality holds:

$$2x^T Ny \le \varepsilon x^T x + \varepsilon^{-1} y^T N^T Ny$$

III. Main Results and Proofs

In this section, we will investigate the stability problem of grey stochastic time-delay systems, main results are given in the following theorems, which ensure our considered systems in the mean-square exponential stability and almost surely exponential stability.

Theorem 3.1. system (2.1) is said to be exponentially robustly stable in mean square, if there exist symmetric matrices $Q > 0, R > 0$, and positive constants $\varepsilon_1, \varepsilon_2$, such that

$$\Psi = \begin{pmatrix} M & Q\dfrac{U_b + L_b}{2} \\ \dfrac{U_b^T + L_b^T}{2}Q & N \end{pmatrix} < 0$$

where,

$$M = R + Q\frac{U_a + L_a}{2} + \frac{U_a^T + L_a^T}{2}Q + (\varepsilon_1 + \varepsilon_2)Q^2 + \varepsilon_1^{-1}\left\|\frac{U_a - L_a}{2}\right\|^2 + \alpha\lambda_{\max}(Q)$$

$$N = -R + \varepsilon_2^{-1}\left\|\frac{U_b - L_b}{2}\right\|^2 + \beta\lambda_{\max}(Q)$$

Then, for all $\xi \in L^2_{F_0}([-\tau,0];R^n)$, the following inequality holds:

$$E|x(t;\xi)|^2 \le \frac{\lambda_{\max}(Q)+\tau\lambda_{\max}(R)+r\lambda_{\max}(R)\tau^2 e^{r\tau}}{\lambda_{\min}(Q)} \sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2 e^{-rt} \quad (3.1)$$

here, r is the unique positive solution of the following equation:

$$r\lambda_{\max}(Q)+\lambda_{\max}(\Psi)+r\lambda_{\max}(R)\tau e^{r\tau}=0 \quad (3.2)$$

Proof Fix $\xi\in L^2_{F_0}([-\tau,0];R^n)$ and whitened matrices $A(\hat{\otimes})\in[L_a,U_a]$, $B(\hat{\otimes})\in[L_b,U_b]$ arbitrarily, and write $x(t,\xi)=x(t)$.

First, we construct the following Lyapunov-Krasovskii functional for system (2.1)

$$V(x(t),t)=x^T(t)Qx(t)+\int_{t-\tau}^{t} x^T(\theta)Rx(\theta)d\theta \quad (3.3)$$

By $It\hat{o}^{/}s$ differential formula, the stochastic derivative of $V(x(t),t)$ along the trajectory of system (2.1) can be obtained as follows:

$$\begin{aligned} &dV(x(t),t)\\ &=LV(x(t),t)dt \\ &\quad +2x^T(t)Qg(x(t),x(t-\tau),t)dw(t)\end{aligned} \quad (3.4)$$

Here, the weak infinitesimal operator L is given as

$$\begin{aligned} &LV(x(t),t)\\ &=x^T(t)Rx(t)-x^T(t-\tau)Rx(t-\tau)\\ &\quad +2x^T(t)QA(\hat{\otimes})x(t)+2x^T(t)QB(\hat{\otimes})x(t-\tau)\\ &\quad +Trace[g^T(x(t),x(t-\tau),t)Q\,g(x(t),x(t-\tau),t)]\end{aligned} \quad (3.5)$$

For the positive constants $\varepsilon_1>0$, $\varepsilon_2>0$, it follows from Lemma 2.1 and Lemma 2.2 that

$$\begin{aligned} &2x^T(t)QA(\hat{\otimes})x(t)\\ &=x^T(t)(Q\frac{U_a+L_a}{2}+\frac{U_a^T+L_a^T}{2}Q)x(t)\\ &\quad +2x^T(t)Q\Delta Ax(t)\\ &\le x^T(t)(Q\frac{U_a+L_a}{2}+\frac{U_a^T+L_a^T}{2}Q)x(t)\\ &\quad +\varepsilon_1 x^T(t)Q^2x(t)+\varepsilon_1^{-1}\left\|\frac{U_a-L_a}{2}\right\|^2 x^T(t)x(t)\end{aligned} \quad (3.6)$$

and

$$\begin{aligned} &2x^T(t)QB(\hat{\otimes})x(t-\tau)\\ &\le x^T(t)Q\frac{U_b+L_b}{2}x(t-\tau)\\ &\quad +x^T(t-\tau)\frac{U_b^T+L_b^T}{2}Qx(t)+2x^T(t)Q\Delta Bx(t-\tau)\\ &\le x^T(t)Q\frac{U_b+L_b}{2}x(t-\tau)\\ &\quad +x^T(t-\tau)\frac{U_b^T+L_b^T}{2}Qx(t)\\ &\quad +\varepsilon_2x^T(t)Q^2x(t)\\ &\quad +\varepsilon_2^{-1}\left\|\frac{U_b-L_b}{2}\right\|^2 x^T(t-\tau)x(t-\tau)\end{aligned} \quad (3.7)$$

Furthermore, it follows from the **(H2)** of assumption that

$$\begin{aligned} &Trace[g^T(x(t),x(t-\tau),t)Q\,g(x(t),x(t-\tau),t)]\\ &\le\lambda_{\max}(Q)[\alpha x^T(t)x(t)+\beta x^T(t-\tau)x(t-\tau)]\end{aligned} \quad (3.8)$$

Substituting of (3.6) - (3.8) into (3.5), and noting the definitions of Ψ, we can get

$$\begin{aligned} &LV(x(t),t)\\ &\le[R+(Q\frac{U_a+L_a}{2}+\frac{U_a^T+L_a^T}{2}Q)\\ &\quad +\varepsilon_1Q^2+\varepsilon_1^{-1}\left\|\frac{U_a-L_a}{2}\right\|^2\\ &\quad +\varepsilon_2Q^2+\alpha\lambda_{\max}(Q)]x^T(t)x(t)\\ &\quad +x^T(t)Q\frac{U_b+L_b}{2}x(t-\tau)\\ &\quad +x^T(t-\tau)\frac{U_b^T+L_b^T}{2}Qx(t)\\ &\quad +[-R+\varepsilon_2^{-1}\left\|\frac{U_a-L_a}{2}\right\|^2+\beta\lambda_{\max}(Q)]\\ &\qquad x^T(t-\tau)x(t-\tau)\\ &=(x^T(t),x^T(t-\tau))\Psi\begin{pmatrix}x(t)\\ x(t-\tau)\end{pmatrix}\\ &\le\lambda_{\max}(\Psi)(|x(t)|^2+|x(t-\tau)|^2)\end{aligned} \quad (3.9)$$

Therefore, by (3.4) and (3.9), we can get

$$\begin{aligned} &dV(x(t),t)\\ &\le\lambda_{\max}(\Psi)(|x(t)|^2+|x(t-\tau)|^2)dt\\ &\quad +2x^T(t)Qg(x(t),x(t-\tau),t)dw(t)\end{aligned} \quad (3.10)$$

On the other hand, from (3.3), we can have

$$\begin{aligned} &V(x(t),t)\\ &\le\lambda_{\max}(Q)x^T(t)x(t)\\ &\quad +\lambda_{\max}(R)\int_{t-\tau}^{t}x^T(\theta)x(\theta)d\theta\end{aligned} \quad (3.11)$$

Using (3.10) and (3.11), we can obtain the following results:

$$\begin{aligned} &d[e^{rt}V(x(t),t)]\\ &=e^{rt}[rV(x(t),t)dt+dV(x(t),t)]\\ &\le e^{rt}[(r\lambda_{\max}(Q)|x(t)|^2+r\lambda_{\max}(H)\int_{t-\tau}^{t}|x(\theta)|^2d\theta)dt\end{aligned}$$

$$+\lambda_{\max}(\Psi)(|x(t)|^2+|x(t-\tau)|^2)dt$$
$$+2x^T(t)Qg(x(t),x(t-\tau),t)dw(t)]$$
$$=e^{rt}[(r\lambda_{\max}(Q)+\lambda_{\max}(\Psi))|x(t)|^2 dt$$
$$+r\lambda_{\max}(R)(\int_{t-\tau}^{t}|x(\theta)|^2 d\theta)dt$$
$$+\lambda_{\max}(\Psi)|x(t-\tau)|^2 dt$$
$$+2x^T(t)Qg(x(t),x(t-\tau),t)dw(t)] \tag{3.12}$$

Integrating both sides of (3.12) from 0 to t>0, and taking the mathematical expectation, we have

$$E[e^{rt}V(x(t),t)]$$
$$\le[\lambda_{\max}(Q)+\tau\lambda_{\max}(R)]\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2$$
$$+(r\lambda_{\max}(Q)+\lambda_{\max}(\Psi))\int_0^t e^{rs}E|x(s)|^2 ds$$
$$+r\lambda_{\max}(R)E\int_0^t e^{rs}\int_{t-\tau}^{t}|x(\theta)|^2 d\theta ds$$
$$+\lambda_{\max}(\Psi)\int_0^t e^{rs}E|x(s-\tau)|^2 ds \tag{3.13}$$

In addition, we can also have the following two estimates:

$$E\int_0^t e^{rs}\int_{t-\tau}^{t}|x(\theta)|^2 d\theta ds$$
$$\le\tau e^{r\tau}(\int_0^t e^{rs}E|x(s)|^2 ds+\int_{-\tau}^{0}E|\xi(\theta)|^2 d\theta) \tag{3.14}$$

and

$$\int_0^t e^{rs}E|x(s-\tau)|^2 ds$$
$$\le e^{r\tau}(\int_0^t e^{rs}E|x(s)|^2 ds+\int_{-\tau}^{0}E|\xi(\theta)|^2 d\theta) \tag{3.15}$$

Then, combining (3.13)–(3.15) together, and note that $\lambda_{\max}(\Psi)<0$, we can eventually obtain

$$E[e^{rt}V(x(t),t)]$$
$$\le[\lambda_{\max}(Q)+\tau\lambda_{\max}(R)]\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2$$
$$+(r\lambda_{\max}(Q)+\lambda_{\max}(\Psi))\int_0^t e^{rs}E|x(s)|^2 ds$$
$$+r\lambda_{\max}(R)\tau e^{r\tau}(\int_0^t e^{rs}E|x(s)|^2+ds\int_{-\tau}^{0}E|\xi(\theta)|^2 d\theta)$$
$$+\lambda_{\max}(\Psi)e^{r\tau}(\int_0^t e^{rs}E|x(s)|^2 ds+\int_{-\tau}^{0}E|\xi(\theta)|^2 d\theta)$$
$$=[\lambda_{\max}(Q)+\tau\lambda_{\max}(H)+r\lambda_{\max}(R)\tau^2 e^{r\tau}$$
$$+\lambda_{\max}(\Psi)\tau e^{r\tau}]\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2$$
$$+[r\lambda_{\max}(Q)+\lambda_{\max}(\Psi)+r\lambda_{\max}(R)\tau e^{r\tau}$$
$$+\lambda_{\max}(\Psi)e^{r\tau}]\int_0^t e^{rs}E|x(s)|^2 ds$$
$$\le[\lambda_{\max}(Q)+\tau\lambda_{\max}(H)$$
$$+r\lambda_{\max}(R)\tau^2 e^{r\tau}]\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2$$
$$+[r\lambda_{\max}(Q)+\lambda_{\max}(\Psi)$$
$$+r\lambda_{\max}(R)\tau e^{r\tau}]\int_0^t e^{rs}E|x(s)|^2 ds \tag{3.16}$$

Let $f(r)$ be the left of equation (3.2), $f(r)$ can be regarded as the function of r. Then, the derivative of $f(r)$ with respect to r is

$$f'(r)=\lambda_{\max}(Q)+\lambda_{\max}(R)\tau(e^{r\tau}+re^{r\tau}\tau)$$

Obviously, $f'(r)>0$, which shows that $f(r)$ is a rigidly increasing function.

By (3.2), we get $f(0)=\lambda_{\max}(\Psi)<0$, $f(+\infty)=+\infty$, Hence, equation (3.2) must have a uniquely positive solution r, and we can also obtain the following result:

$$E[e^{rt}V(x(t),t)]$$
$$\le[\lambda_{\max}(Q)+\tau\lambda_{\max}(R)$$
$$+r\lambda_{\max}(R)\tau^2 e^{r\tau}]\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2$$

or equivalently

$$E|x(t;\xi)|^2$$
$$\le\frac{\lambda_{\max}(Q)+\tau\lambda_{\max}(R)+r\lambda_{\max}(R)\tau^2 e^{r\tau}}{\lambda_{\min}(Q)}$$
$$\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2 e^{-rt}$$

Which implies system (2.1) is exponentially stable in the mean square. This completes the proof of this theorem.

Remark 3.1. If the grey matrices $A(\otimes)$ and $B(\otimes)$ of the system (2.1) are replaced by the known determinate matrices A and B, systems (2.1) becomes the following deterministic system.

$$\begin{cases} dx(t)=[Ax(t)+Bx(t-\tau)]dt+g(x(t),x(t-\tau),t)dw(t) \\ x_0=\xi,\ \xi\in L^2_{F_0}([-\tau,0];R^n),\ -\tau\le t\le 0 \end{cases} \tag{3.17}$$

Let $L_a=U_a=A, L_b=U_b=B$, Similar to the proof of the Theorem **3.1**, We have the following result.

Corollary 3.1. if there exist symmetric matrices $Q>0, R>0$, and positive constants $\varepsilon_1,\varepsilon_2$, such that

$$\phi=$$
$$\begin{pmatrix} R+QA+A^TQ+(\varepsilon_1+\varepsilon_2)Q^2+\alpha\lambda_{\max}(Q) & QB \\ B^TQ & -R+\beta\lambda_{\max}(Q) \end{pmatrix}$$
$$<0$$

Then, system (3.17) is exponentially stable in mean square, and r is the unique positive solution of the following equation

$$r\lambda_{\max}(Q)+\lambda_{\max}(\phi)+r\lambda_{\max}(R)\tau e^{r\tau}=0$$

Theorem 3.2. Under the conditions of Theorem 1, for all $\xi \in L^2_{F_0}([-\tau,0];R^n)$, the following inequality holds:

$$\lim_{t\to\infty}\sup\frac{1}{t}\ln|x(t;\xi)| \le -\frac{r}{2}, \quad a.s. \tag{3.18}$$

where, r is the unique positive solution of equation (3.2). In other words, system (2.1) is almost surely exponentially robustly stable.

Proof Under the conditions of Theorem 1, system (2.1) is said to be exponentially robustly stable in mean square, so we have

$$E|x(t;\xi)|^2 \le K \sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2 e^{-rt},$$

$$\xi \in L^2_{F_0}([-\tau,0];R^n), \quad t \ge 0$$

Using the similar method as in [24], by Doob's martingale inequality, Cauchy inequality, for arbitrary integer $k \ge 1$ and $\delta \in (0,r)$, we have

$$\begin{aligned} &P\left(\omega : \sup_{0\le s\le\tau}|x(k\tau+s)|^2 > e^{-(r-\delta)k\tau}\right) \\ &\le \frac{E|x(k\tau+\tau)|^2}{e^{-(r-\delta)k\tau}} \\ &\le \frac{Ke^{-r(k\tau+\tau)}\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2}{e^{-(r-\delta)k\tau}} \\ &= Ke^{-r\tau}e^{-\delta k\tau}\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2 \end{aligned} \tag{3.19}$$

By Borel-Cantelli lemma, for almost all $\omega \in \Omega$ and all but finitely many k, we can obtain

$$\sup_{0\le s\le\tau}|x(k\tau+s)|^2 \le e^{-(r-\delta)k\tau} \tag{3.20}$$

Then, there exists $k_0(\omega)$, for all $\omega \in \Omega$, including a P-null set, whenever $k \ge k_0$, such that (3.20) holds.

In other words, if $t \ge k_0\tau$, then for almost all $\omega \in \Omega$, we can get $|x(t)|^2 \le e^{-(r-\delta)t}$.

Noticing that $|x(t)|^2$ is finite on $[0,k_0\tau]$, and for almost all $\omega \in \Omega$, there is a finite $C = C(\omega)$, such that

$$|x(t)|^2 \le Ce^{-(r-\delta)t}, t \ge 0.$$

Thus, it can be concluded that

$$\lim_{t\to\infty}\sup\frac{1}{t}\ln|x(t;\xi)| \le -\frac{r-\delta}{2}, \quad a.s. \tag{3.21}$$

By (3.21) and $\delta \to 0$, we can obtain

$$\lim_{t\to\infty}\sup\frac{1}{t}\ln|x(t;\xi)| \le -\frac{r}{2}, \quad a.s.$$

Which implies system (2.1) is the almost surely exponential stability. The proof is complete.

IV. Examples

In this section, we provide a simple numerical example to demonstrate the correctness and effectiveness of the main results.

Consider the following grey stochastic time-delay systems

$$\begin{cases} dx(t) = [A(\otimes)x(t) + B(\otimes)x(t-0.5)]dt \\ \qquad + \sigma(x(t),x(t-0.5),t)dW(t) \\ x_0 = \xi, \xi \in L^2_{F_0}([-0.5,0];R^2), \quad -0.5 \le t \le 0 \end{cases} \tag{4.1}$$

where

$$L_a = \begin{bmatrix} -3.35 & 0.22 \\ 0.23 & -3.34 \end{bmatrix}$$

$$U_a = \begin{bmatrix} -3.15 & 0.32 \\ 0.31 & -3.45 \end{bmatrix}$$

$$L_b = \begin{bmatrix} -1.15 & 0.20 \\ 0.23 & -1.16 \end{bmatrix}$$

$$U_b = \begin{bmatrix} -1.12 & 0.22 \\ 0.31 & -1.09 \end{bmatrix}$$

Here,

L_a, U_a ; L_b, U_b are the lower bound and upper bound matrices of $A(\otimes)$ and $B(\otimes)$.

In addition,

$$\sigma(x(t),x(t-0.5),t) = \begin{bmatrix} \frac{1}{2}x_1(t)\sin(x_2(t-0.5)) \\ \frac{1}{2}x_2(t)\sin(x_1(t-0.5)) \end{bmatrix}$$

Clearly,

$$Trace[\sigma^T(x(t),x(t-0.5),t)\sigma(x(t),x(t-0.5),t)] \le 0.25x^2(t)$$

With the help of the programmed procedure (see[24]), we can calculate and optimize $\varepsilon_1, \varepsilon_2$ satisfing the formula (3.2)，and it is easy to obtain that $r = 1.5627$. Therefore, by Theorem 3.1, we can conclude that the system (4.1) is exponentially stable in mean square.

V. Conclusion

This paper has focused on the stability analysis problem for a class of grey stochastic systems with time delays. By choosing an appropriate Lyapunov-Krasovskii functional, especially, using decomposition technique of the continuous matrix-covered sets of grey matrix, novel sufficient stability criteria have been obtained, which ensure the grey systems in the mean-square and almost surely exponential stability. Finally, a numerical example has been provided to show the effectiveness of the proposed methods in this paper.

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REFERENCES

[1] E. K. Boukas, and Z. K. Liu. "Robust stability and stabilizability of Markov jump linear uncertain systems with mode-dependent time delays," *Journal of Optimization Theory and Applications*, vol. 109, no. 3, pp. 587-600, 2001.

[2] R. Z. Khas'minskii, *Stochastic Stability of Differential Equations*, Sijthoff and Noordhoff, 1981.

[3] H. Kushner, *Stochastic Stability and Control*, Academic Press, New York, 1967.

[4] H. J. Chu, and L. X. Gao, "Robust exponential stability and H∞ control for jumping stochastic Cohen-Grossberg neural networks with mixed delays", *Journal of Computational Information Systems*, vol. 7, no. 3, pp. 794 -806, 2011.

[5] X. Mao, *Stochastic Differential Equations and Applications, Chichester*, UK: Horwood Publishing, 1997.

[6] X. Mao. "Robustness of exponential stability of stochastic differential delay equation," *IEEE Trans.Autom. Control*, vol. 41, no. 3, pp. 442 -447, 1996.

[7] H. Yan, Q. H. Meng, X. Huang, and H. Zhang, "Robust exponential stability of stochastic time-delay systems with uncertainties and nonlinear perturbations, " *International Journal of Information Acquisition*, vol. 6, no. 1, pp. 61 - 71, 2009.

[8] W. Chen, and X. Lu, "Mean square exponential stability of uncertain stochastic delayed neural networks," *Physics Letter A*, vol. 372, no. 7, pp. 1061 - 1069, 2008.

[9] X. Mao, N. Koroleva, and A. Rodkina, "Robust Stability of Uncertain Stochastic Differential Delay Equations," *Systems & Control Letters*, vol. 35, no. 5, pp. 325-336, 1998.

[10] E. K. Boukas, and Z. K. Liu, *Deterministic and Stochastic Time-Delay Systems*. Boston, MA: Birkhauser, 2002.

[11] S. Arik, "Global robust stability analysis of neural networks with discrete time delays", *Chaos, Solitons and Fractals*, vol. 26, no. 5, pp. 1407-1414, 2005.

[12] E. K. Boukas, and Z. K. Liu, "Robust stability and stabilizability of Markov jump linear uncertain systems with mode-dependent time delays," *Journal of Optimization Theory and Applications*, vol. 109, no. 3, pp. 587-600, 2001.

[13] J. H. Park, "Robust stabilization for dynamic systems with mulitple time-varying delays and nonlinear uncertainties," *Journal of Optimization Theory and Applications*, vol. 108, no. 1, pp. 155 -174, 2001.

[14] P. Park, "A delay-dependent stability criterion for systems with uncertain time-invariant delays". *IEEE Transactions on Automatic Control*, vol. 44, no. 4, pp. 876-877, 1999.

[15] R. Rakkiyappan, and P. Balasubramaniam, "New global exponential stability results for neutral type neural networks with distributed time delays," *Neurocomputing*, vol. 171, no. 4, pp. 1039-1045, 2008.

[16] W. J. Xiong, and J. L. Liang, "Novel stability criteria for neutral systems with multiple time delays," *Chaos, Solitons & Fractals*, vol. 32, no. 5, pp. 1735-1741, 2007.

[17] Y. Glizer, V. Turetsky, and J. Shinar, "Terminal Cost Distribution in Discrete-Time Controlled System with Disturbance and Noise-Corrupted State Information," *IAENG International Journal of Applied Mathematics*, vol. 42, no. 1, pp. 52-59, 2012.

[18] L. Lin, "Stabilization of LTI Switched Systems with Input Time Delay," *Engineering Letters*, vol. 14, no. 2, pp. 117-123, 2007.

[19] T. C. Kuo, and Y. J. Huang, "Global Stabilization of Robot Control with Neural Network and Sliding Mode," *Engineering Letters*, vol. 16, no. 1, pp. 56-60, 2008.

[20] R. Samli, and S. Arik, "New results for global stability of a class of neutral-type neural systems with time delays," *Applied Mathematics and Computation*, vol. 210, no. 2, pp. 564 -570, 2009.

[21] C. H. Su, and S. F. Liu, "The p-moment exponential robust stability for stochastic systems with distributed delays and interval parameters," *Applied Mathematics and Mechanics*, vol. 30, no. 7, pp. 915-924, 2009.

[22] J. J. Li, and C. H. Su, "Mean-square Exponential Robust Stability for a Class of Grey Stochastic Systems with Distributed Delays", *Chin. Quart. J. of Math.*, vol. 25, no. 3, pp. 451-458, 2010.

[23] C. H. Su, and S. F. Liu, "Robust Stability of Grey Stochastic Nonlinear Systems with Distributed-Delays," *Mathematics in Practice and Theory*, vol. 38, no. 22, pp. 218-223, 2008.

[24] C. H. Su, and S. F. Liu, "Exponential Robust Stability of Grey Neutral Stochastic Systems with Distributed Delays," *Chinese Journal of Engineering Mathematics*, vol. 27, no. 3, pp. 403-414, 2010.

[25] A. Friedman, *Stochastic differential equations and their applications*, New York, Academic Press, 1976.