Relations Among Technical, Cost and Revenue Efficiencies in Data Envelopment Analysis

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Abstract—**Efficiency is a basic issue in Data Envelopment Analysis (DEA). There are different types of efficiency in DEA, e.g. Technical Efficiency (TE), Cost Efficiency (CE) and Revenue Efficiency (RE). They may be different in terms of efficiency score values because they use different information obtained from Decision Making Units (DMUs). One of these efficiency types (TE) uses only input and output quantities, another (CE) uses input, output and input cost vectors and still another (RE) uses input, output and output benefit vectors. In real problems, not the cost of each input or the benefit of each output is usually known and accessible, and just the costs (benefits) of some inputs (outputs) of each DMU are known. In this paper, we suggest a method and models to measure the efficiency of DMUs when some input costs and some output benefits are known. The proposed measure is a generalization of TE, CE and RE. This measure is very useful to evaluate the performance of DMUs and uses more information than the other measures. We show that the CCR model to evaluate TE and the minimal cost model to evaluate CE are special cases of the proposed model. Finally we present three examples to compare the proposed measure of efficiency with the other measures of efficiency.**

Keywords: Data Envelopment Analysis, Technical efficiency, Cost efficiency, Revenue efficiency.

1 Introduction

Data Envelopment Analysis (DEA) is a mathematical method that measures the relative efficiency of a group of Decision Making Units (DMUs) with multiple inputs and outputs but with no obvious production function to aggregate the data in its entirety. Debreu (1951) provided the first measure of efficiency and Koopmans (1957) was the first to define the concept of Technical Efficiency (TE). The measurement of TE as defined by Farrell (1957) was operationalized and popularized by Charnes et al. (1978), which led to the establishment of DEA as a prominent methodological tool for assessing relative efficiency. The standard DEA method measures TE assuming Constant Returns to Scale (CRS) which was initially proposed by Charnes, Cooper and Rhodes (CCR) (1978). Some of DEA researchers worked on the theories of DEA and found some characterizations of DEA models. Fukuyama (2000), Sueyoshi and Sekitani (2007), Jahanshahloo et al. (2008) and (2009) found some properties in DEA to find returns to scale, reference set and strong defining hyperplanes. On the other hand, some of the other researchers in different fields and majors applied DEA models to evaluate commerical firms, hospitals and etc; see for example Ayaz and Alptekin (2012), Karadayi and Karsak (2014) and Yang (2009). In evaluating a DMU sometimes the optimal solution(s) of the traditional models is far from the expected results. To solve this shortcoming, the weight restrictions have been added to the traditional models regarding the management decision. Podinovski (2004) and (2007) and Podinovski and Thanassoulis (2007) presented some theories of the weight restriction methods.

Cost Efficiency (CE) evaluates the ability of a DMU to produce the current output at minimal cost, given its input prices. The concept of CE can be traced back to Farrell (1957), who originated many of the ideas underlying efficiency assessment. CE can be interpreted as an achievable measure of potential cost reduction given the outputs produced and current input prices at each DMU. The Farrell concept was developed by Färe et al. (1985), who formulated a Linear Programming (LP) model for CE assessment. This LP model requires input and output quantity data as well as input prices at each DMU. Jahanshahloo et al. (2008) simplified a version of the cost efficiency model proposed by Camanho and Dyson (2005) which they showed that the measure of CE can be obtained with the inclusion of weight restrictions in DEA models.

In Charnes et al.'s (1978) model, each member of a set of DMUs must be evaluated relative to its peers. This evaluation is generally assumed to be based on a set of quantitative output and input factors. In many real world settings, however, it is essential to take into account the presence of qualitative factors when rendering a decision

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(Advance online publication: 14 November 2015)

on the performance of a DMU. But providing a more precise, quantitative measure reflecting such a factor is generally beyond the realm of reality. In some situations, such factors can be legitimately 'quantified', but very often such quantification may be superficially forced as a modeling convenience.

In this paper we propose a model for evaluating the measure of efficiency when some input prices and some output benefits are available for all DMUs. We notate this proposed efficiency measure by PE, and we show that the multiplier form model, the minimal cost model and the maximal revenue model for evaluating TE, CE and Revenue Efficiency (RE), respectively, are special cases of our proposed model. This means TE (by using CCR model), CE and RE are special measures of PE. This method is completely different from the existing weight restriction methods. The structure of this paper is as follows: Section 2 describes the DEA models used for the estimation of TE, Farrell-Debreu and Tone CE and RE. Some definitions and models in are extracted from Cooper et al. (2007). In Section 3 we propose a model for evaluating PE. Also, in this section we prove that the minimal cost model for evaluating CE is a special case of the proposed model. Section 4 contains three examples to apply the proposed model and compare PE with the other measures of efficiency. Conclusions and suggestions for future research are presented in Section 5.

2 Background

2.1 Technical Efficiency

Relative efficiency is defined as the ratio of the total weighted output to the total weighted input. Suppose that we have n DMUs with activity vectors $(\mathbf{x}_j^t, \mathbf{y}_j^t)$; $j = 1, 2, \ldots, n$, where $\mathbf{x}_j$ and $\mathbf{y}_j$ are nonnegative and nonzero column vectors in R^m and R^s, respectively. All DMUs (DMU_j $(j = 1, \ldots, n)$) use the same number, m, of inputs (x_{ij} $(i = 1, \ldots, m)$) to produce the same number, s, of outputs (y_{rj} $(r = 1, \ldots, s)$). Note that the input and output vectors of all DMUs are the same in type but different in quantity.
Let

s be the number of outputs;

m be the number of inputs;

n be the number of DMUs whose performance must be evaluated;

y_{rj} be the value (≥ 0) of output r ($r = 1, 2, \ldots, s$) for DMU$_j$ ($j = 1, 2, \ldots, n$);

x_{ij} be the value (≥ 0) of input i ($i = 1, 2, \ldots, m$) for DMU$_j$ ($j = 1, 2, \ldots, n$);

u_{ro} be the weight (≥ 0) attached to output r ($r = 1, 2, \ldots, s$) by DMU$_o$ ($o \in \{1, 2, \ldots, n\}$);

v_{io} be the weight (≥ 0) attached to input i ($i = 1, 2, \ldots, m$) by DMU$_o$ ($o \in \{1, 2, \ldots, n\}$);

θ_o^* be the (relative) efficiency of DMU$_o$ ($o \in \{1, 2, \ldots, n\}$);

ε be a very small non-Archimedean number, smaller than any positive real number ($0 < \varepsilon \ll 1$).

The fractional model to evaluate the relative efficiency of DMU_o ($o \in J = \{1, 2, \ldots, n\}$) is as follows:

$$\begin{aligned} \theta_o^* = \max \quad & \frac{\sum_{r=1}^{s} u_{ro} y_{ro} / \sum_{i=1}^{m} v_{io} x_{io}}{\max\{\sum_{r=1}^{s} u_{ro} y_{rj} / \sum_{i=1}^{m} v_{io} x_{ij} :\ j \in J\}} \\ s.t. \quad & u_{ro} \geq 0;\ r = 1, 2, \ldots, s \\ & v_{io} \geq 0;\ i = 1, 2, \ldots, m. \end{aligned} \tag{1}$$

The measure of relative efficiency of DMU_o is θ_o^*; note that $0 \leq \theta_o^* \leq 1$ (and $\theta_o^* = 0$ if and only if $y_{ro} = 0$ $(r = 1, 2, \ldots, s)$). The result of the DEA is the determination of the hyperplanes that define an envelope surface or Pareto frontier. DMUs that lie on the envelope surface are deemed efficient, whilst those that do not are deemed inefficient. By Charnes-Cooper transformations, the fractional model is transformed to the following LP:

$$\begin{array}{lll} \max & \sum_{r=1}^{s} u_{ro} y_{ro} & \\ s.t. & \sum_{i=1}^{m} v_{io} x_{io} = 1 & \\ & \sum_{r=1}^{s} u_{ro} y_{rj} - \sum_{i=1}^{m} v_{io} x_{ij} \leq 0; & j = 1, 2, \ldots, n \\ & u_{ro} \geq 0; & r = 1, 2, \ldots, s \\ & v_{io} \geq 0; & i = 1, 2, \ldots, m. \end{array} \tag{2}$$

Model (2) is called the "input-oriented" multiplier form to evaluate the relative efficiency of DMU_o. It is assumed that the production function exhibits constant returns to scale. The dual of (2) is

$$\begin{array}{lll} \min & \theta & \\ s.t. & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{io}; & i = 1, 2, \ldots, m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro}; & r = 1, 2, \ldots, s \\ & \lambda_j \geq 0; & j = 1, 2, \ldots, n \\ & s_i^- \geq 0; & i = 1, 2, \ldots, m \\ & s_r^+ \geq 0; & r = 1, 2, \ldots, s \\ & \theta \text{ is unrestricted.} & \end{array} \tag{3}$$

Model (3) is called the "input-oriented" envelopment form to evaluate the relative efficiency of DMU_o. We know that the optimal value of Models (2) and (3) are equal. In the above model s_i^- $(i = 1, 2 \ldots, m)$ are the input excesses and s_r^+ $(r = 1, 2 \ldots, s)$ are the output shortfalls.

Definition 1 *DMU_o is called technical efficient if the optimal value of Problem (3) equals 1. Otherwise, it is called technical inefficient.*

Technical efficiency is also referred to as radial efficiency. Technical efficient DMUs are classified in two types; strong efficient DMUs and weak efficient DMUs. In evaluating DMU$_o$, which is a technical efficient DMU, by (3) if all slacks s_i^- and s_r^+ $(i = 1, ..., m,\ r = 1, ..., s)$ equal zero in all possible optimal solutions then DMU$_o$ is called a strong efficient DMU otherwise, it is called a weak efficient DMU.

If a DMU proves to be inefficient, a combination of other efficient units can produce either greater outputs for the same composite of inputs or produce the same outputs for a smaller composite of inputs. Similarly, the "output-oriented" envelopment form and the "output-oriented" multiplier form are as follows:

$$\begin{array}{lll} \phi_o^* = \max & \phi & \\ s.t. & \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = x_{io}; & \forall i \\ & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = \phi y_{ro}; & \forall r \\ & \lambda_j \geq 0; & \forall j \\ & s_i^- \geq 0; & \forall i \\ & s_r^+ \geq 0; & \forall r \\ & \phi \ is\ unrestricted. & \end{array} \tag{4}$$

$$\begin{array}{lll} \min & \sum_{r=1}^s v_i x_{io} & \\ s.t. & \sum_{r=1}^s u_r y_{ro} = 1 & \\ & \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0; & \forall j \\ & u_r \geq 0; & \forall r \\ & v_i \geq 0; & \forall i. \end{array} \tag{5}$$

We know that $\theta_o^* = \frac{1}{\phi_o^*}$, where ϕ_o^* is the optimal value of (4).

2.2 Cost Efficiency

Cost efficiency evaluates the ability of a DMU to produce the current outputs at minimal cost, given its input prices. Looking beyond TE, Farrell (1957) also proposed a measure of cost efficiency, which assumes that prices are fixed and known and maybe different among DMUs. Suppose that there exist n DMUs as defined in 2.1, and $c_{ij} \geq 0$ is the cost of the ith input of DMU_j which may vary from one DMU to another. The minimal cost model to produce at least the current output of DMU_o $(\mathbf{y}_o)$ yields the optimal value of the following LP $(\sum_{i=1}^m c_{io} x_i^*)$.

$$\begin{array}{llll} CE_o = \frac{1}{\sum_{i=1}^m c_{io} x_{io}} & \min & \sum_{i=1}^m c_{io} x_i & \\ & s.t. & \sum_{j=1}^n \lambda_j x_{ij} \leq x_i; & \forall i \\ & & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro}; & \forall r \\ & & \lambda_j \geq 0; & \forall j \\ & & x_i \geq 0; & \forall i. \end{array} \tag{6}$$

Because the cost of inputs are nonnegative for each DMU, there exists an optimal solution such as $(\lambda^{*^t}, \mathbf{x}^{*^t})$ such that all constraints $(\sum_{j=1}^n \lambda_j x_{ij} \leq x_i)$ are binding. This means the optimal value of (6) equals the optimal value of the following LP:

$$\begin{array}{llll} CE_o = \frac{1}{\sum_{i=1}^m c_{io} x_{io}} & \min & \sum_{i=1}^m c_{io} x_i & \\ & s.t. & \sum_{j=1}^n \lambda_j x_{ij} = x_i; & \forall i \\ & & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro}; & \forall r \\ & & \lambda_j \geq 0; & \forall j \\ & & x_i \geq 0; & \forall i. \end{array} \tag{7}$$

CE_o is the cost efficiency of DMU_o and it is clear that $CE_o \leq 1$. Also this cost efficiency measure is named Farrell-Debreu cost efficiency measure.

Tone(2002) found an unacceptable property of the traditional Farrell-Debreu cost efficiency measure when the unit prices of input are not identical among DMUs. He suggested a new approach to measure the cost efficiency which we call Tone Cost Efficiency (TCE). TCE is obtained from the following model:

$$\begin{array}{llll} TCE_o = \frac{1}{\sum_{i=1}^m \overline{x}_{io}} & \min & \sum_{i=1}^m \overline{x}_i & \\ & s.t. & \sum_{j=1}^n \lambda_j \overline{x}_{ij} \leq \overline{x}_i; & \forall i \\ & & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro}; & \forall r \\ & & \lambda_j \geq 0; & \forall j \\ & & \overline{x}_i \geq 0; & \forall i. \end{array} \tag{8}$$

Where $\overline{x}_{ij} = c_{ij} x_{ij}$ and $\overline{x}_i$ are variables $(i = 1, ..., m,\ j = 1, ..., n)$.

The difference between Farrell-Debreu cost model and Tone model is: in the Farrell-Debreu model, the unit cost of DMU$_o$ is fixed at c_o and then the optimal input vector x^* that produces the output vector y_o is found, while, in Tone cost model, the optimal input vector $\overline{x}^*$ is searched which can produce y_o.

2.3 Revenue Efficiency

Revenue efficiency evaluates the ability of a DMU to produce outputs at maximal revenue when it consumes the current inputs. Suppose that there exist n DMUs as defined in 2.1, and $b_{rj} \geq 0$ is the benefit of the rth output of DMU_j. The maximal revenue model to consume at most the current input of DMU_o $(\mathbf{x}_o)$ gives the optimal value of the following LP $(\sum_{r=1}^s b_{ro} y_r^*)$.

$$\begin{array}{lll} \max & \sum_{r=1}^s b_{ro} y_r & \\ s.t. & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{io}; & \forall i \\ & \sum_{j=1}^n \lambda_j y_{rj} \geq y_r; & \forall r \\ & \lambda_j \geq 0; & \forall j \\ & y_r \geq 0; & \forall r. \end{array} \tag{9}$$

Based on an optimal solution $\mathbf{y}^{*^t} = (y_1^*, y_2^*, \ldots, y_s^*)$ of this model, the revenue efficiency of DMU_o is defined as:

$$RE_o = \frac{\sum_{r=1}^s b_{ro} y_{ro}}{\sum_{r=1}^s b_{ro} y_r^*} \tag{10}$$

According to (10) we conclude that $RE_o \leq 1$.

In Cooper et al. (2006) 's a similar discussion to Tone cost model, which has been presented in 2.2, has been presented for the revenue efficiency.

3 The Proposed Efficiency Measure

In this section, we propose a model to evaluate efficiency when some (not necessarily all) elements of the input cost vector and some (not necessarily all) elements of the output benefit vector are available. We consider the fractional multiplier form model (1) to obtain the proposed model.

Suppose that there exist n DMUs, namely $DMU_1, DMU_2, \ldots, DMU_n$ where DMU_j; $j = 1, 2, \ldots, n$ has the input vector $\mathbf{x}_j$ with m elements and the output vector $\mathbf{y}_j$ with s elements.
We form the virtual input and output by weights (v_i) and (u_r) (yet unknown) as follows:

$$virtual\ input = v_1x_{1o} + v_2x_{2o} + \ldots + v_mx_{mo},$$

$$virtual\ output = u_1y_{1o} + u_2y_{2o} + \ldots + u_sy_{so}.$$

Then, we try to determine the weights, using linear programming so as to maximize the ratio

$$\frac{virtual\ output}{virtual\ input}.$$

The optimal weights may (and generally will) vary from one DMU to another. Thus, the weights in DEA are derived from the data rather than being fixed in advance. A best set of weights is assigned to each DMU with values that may vary from one DMU to another.

Let $J = \{1, 2, \ldots, n\}$. To evaluate the efficiency measure of DMU_o ($o \in \{1, 2, \ldots, n\}$), we solve the following fractional programming model to obtain values for the input weights (v_i; $i = 1, 2, \ldots, m$) and the output weights ($u_r; r = 1, 2, \ldots, s$) as variables:

$$\begin{array}{lll} \max & \theta = \dfrac{\frac{u_1y_{1o}+u_2y_{2o}+\ldots+u_sy_{so}}{v_1x_{1o}+v_2x_{2o}+\ldots+v_mx_{mo}}}{\max\{\frac{u_1y_{1j}+u_2y_{2j}+\ldots+u_sy_{sj}}{v_1x_{1j}+v_2x_{2j}+\ldots+v_mx_{mj}} : j \in J\}} & \\ s.t. & u_r \geq 0;\ r = 1, 2, \ldots, s & (11) \\ & v_i \geq 0;\ i = 1, 2, \ldots, m. & \end{array}$$

The objective is to obtain weights (v_i) and (u_r) that maximize the ratio θ of DMU_o as the DMU under assessment. According to the objective function, the optimal objective value θ^* is at most 1. Mathematically, the nonnegativity constraints are not sufficient for the fractional terms in the objective function to have a definite value. We do not consider this assumption in explicit mathematical form at this time. Instead, we put this in managerial terms by assuming that all outputs and inputs have some nonzero values and this is to be reflected in the weights u_r and v_i being assigned some positive values.

According to the above discussion, Model (11) obtains the best weights for inputs and outputs to maximize the objective function. On the other hands in real problems we may have the real value of some input prices and some output benefits, while Model (11) ignores this fact and so the efficiency obtained from this model is unacceptable because the optimal weights given by Model (11) may be different from the real weights (the market prices) that are available. Also, the optimal weights of (11) are the same for all DMUs in evaluating the DMU under assessment. For example, in assessing DMU_o, if $(v_1^*, v_2^*, \ldots, v_m^*, u_1^*, u_2^*, \ldots, u_s^*)$ is an optimal weight vector of (11), v_i^* is the weight (shadow price) of the ith input of all DMUs while in real problems the ith inputs may be different in price in each DMU.

Suppose that there exist n DMUs $(DMU_1, DMU_2, \ldots, DMU_n)$ with activity vectors $\mathbf{D}_1, \mathbf{D}_2, \ldots, \mathbf{D}_n$ where $\mathbf{D}_j = (\mathbf{x}_j^t, \mathbf{y}_j^t)$; $\mathbf{x}_j^t = (x_{1j}, x_{2j}, \ldots, x_{mj})$ and $\mathbf{y}_j^t = (y_{1j}, y_{2j}, \ldots, y_{sj})$. Furthermore, some input prices and some output benefits are available. Without loss of generality, suppose that the input prices of the first p ($p \leq m$) elements of each input vector and the output benefits of the first q ($q \leq s$) elements of each output vector are available. Let $\overline{\mathbf{v}}_j^t = (\overline{v}_{1j}, \overline{v}_{2j}, \ldots, \overline{v}_{pj})$, $\overline{\mathbf{u}}_j^t = (\overline{u}_{1j}, \overline{u}_{2j}, \ldots, \overline{u}_{qj})$, $\mathbf{v}^t = (v_{p+1}, v_{p+2}, \ldots, v_m)$ and $\mathbf{u}^t = (u_{q+1}, u_{q+2}\ldots, u_s)$ where $\overline{v}_{ij}; i = 1, 2, \ldots, p$, is the ith input cost of DMU_j and $\overline{u}_{rj}; r = 1, 2, \ldots, q$, is the rth output benefit of DMU_j. Note that input cost vectors $(\overline{v}_j^t)$ and output benefit vectors $(\overline{u}_j^t)$ may be different for each DMU.

To obtain the proposed efficiency of DMU_o consider the following model:

$$\begin{array}{ll} \theta_1 = \max & \dfrac{\frac{\overline{u}_{1o}y_{1o}+\ldots+\overline{u}_{qo}y_{qo}+u_{q+1}y_{q+1o}+\ldots+u_sy_{so}}{\overline{v}_{1o}x_{1o}+\ldots+\overline{v}_{po}x_{po}+v_{p+1}x_{p+1o}+\ldots+v_mx_{mo}}}{\max\{\frac{\overline{u}_{1j}y_{1j}+\ldots+\overline{u}_{qj}y_{qj}+u_{q+1}y_{q+1j}+\ldots+u_sy_{sj}}{\overline{v}_{1j}x_{1j}+\ldots+\overline{v}_{pj}x_{pj}+v_{p+1}x_{p+1j}+\ldots+v_mx_{mj}} :\ j \in J\}} \\ s.t. & u_r \geq 0;\ r = q+1, q+2, \ldots, s \\ & v_i \geq 0;\ i = p+1, p+2, \ldots, m, \end{array} \quad (12)$$

where $\mathbf{u}_r$ ($r = q+1, \ldots, s$) and $\mathbf{v}_i$ ($i = p+1, \ldots, m$) are variables and $\overline{u}_{rj}$, $\overline{v}_{ij}$ ($r = 1, 2, \ldots, q$, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, n$) are constants. We assume that $\overline{v}_{1j}x_{1j} + \overline{v}_{2j}x_{2j} + \ldots + \overline{v}_{pj}x_{pj} + v_{p+1}x_{p+1j} + v_{p+2}x_{p+2j} + \ldots + v_mx_{mj} \neq 0$ for all j. Let $\overline{\mathbf{x}}_j^t = (x_{1j}, x_{2j}, \ldots, x_{pj})$, $\widetilde{\mathbf{x}}_j^t = (x_{p+1j}, x_{p+2j}, \ldots, x_{mj})$, $\overline{\mathbf{y}}_j^t = (y_{1j}, y_{2j}, \ldots, y_{qj})$ and $\widetilde{\mathbf{y}}_j^t = (y_{q+1j}, y_{q+2j}, \ldots, y_{sj})$.
Any two vectors $\overline{\mathbf{v}}_j^t$, $\overline{\mathbf{x}}_j$ can be multiplied. The result of this multiplication is a real number called the inner product of the two vectors, which is defined as:

$$\overline{\mathbf{v}}_j^t\overline{\mathbf{x}}_j = \overline{v}_{1j}x_{1j} + \overline{v}_{2j}x_{2j} + \ldots + \overline{v}_{pj}x_{pj}.$$

Let $\overline{k}_j = \overline{\mathbf{v}}_j^t\overline{\mathbf{x}}_j$, $\overline{w}_j = \overline{\mathbf{u}}_j^t\overline{\mathbf{y}}_j$ and $z = \dfrac{1}{\max\{\frac{\overline{w}_j+\mathbf{u}^t\widetilde{\mathbf{y}}_j}{\overline{k}_j+\mathbf{v}^t\widetilde{\mathbf{x}}_j} :\ j \in J\}}$.

Model (12) is transformed to the following model:

$$\begin{array}{ll} \max & \frac{\overline{w}_o z+\mathbf{u}^t\widetilde{\mathbf{y}}_o z}{\overline{k}_o+\mathbf{v}^t\widetilde{\mathbf{x}}_o} \\ s.t. & \frac{\overline{w}_j z+\mathbf{u}^t\widetilde{\mathbf{y}}_j z}{\overline{k}_j+\mathbf{v}^t\widetilde{\mathbf{x}}_j} \le 1; \quad j\in J \\ & \mathbf{u}\ge \mathbf{0} \\ & \mathbf{v}\ge \mathbf{0} \\ & z\ge \varepsilon. \end{array} \tag{13}$$

To transform Model (13) to a linear programming problem, let $t=\frac{1}{\overline{k}_o+\mathbf{v}^t\widetilde{\mathbf{x}}_o}$ and use the following variables changes:

$\mathbf{u}^t tz \longrightarrow \mathbf{u}^t$
$\mathbf{v}^t t \longrightarrow \mathbf{v}^t$
$zt \longrightarrow z.$

Note that variables z and t are greater than zero. Model (13) is transformed as follows:

$$\begin{array}{rl} \theta_2=\max & \overline{w}_o z+\mathbf{u}^t\widetilde{\mathbf{y}}_o \\ s.t. & \overline{k}_o t+\mathbf{v}^t\widetilde{\mathbf{x}}_o=1, \\ & \overline{w}_j z+\mathbf{u}^t\widetilde{\mathbf{y}}_j-(\overline{k}_j t+\mathbf{v}^t\widetilde{\mathbf{x}}_j)\le 0; \quad j\in J \\ & t\ge\varepsilon, \quad \mathbf{u}\ge\mathbf{0} \\ & z\ge\varepsilon, \quad \mathbf{v}\ge\mathbf{0} \end{array} \tag{14}$$

where ε is a non-archimedean number. The above model is called the "Input-oriented multiplier proposed model."

Remark: The proposed method is completely different from the existing weight restriction methods. In the DEA literature some weight restrictions have been added to some DEA models. In evaluating a DMU by the DEA models (without any weight restriction) the corresponding weights to an input/output for all DMUs are the same. For example to assess DMU$_o$, the first input (output) of all observed DMUs have the same unknown weights and, in our notation it has been presented by v_1 (u_1). While the proposed method assigns the actual market price as a weight to each input and output which their market prices are available, and assigns the unknown weights to each input and output as variables which their market prices are not available.

Theorem 1 *The optimal values of (12) and (14) are equal.*

Proof: Suppose that $(\mathbf{u}^{*^t},\mathbf{v}^{*^t})$ is an optimal solution of (12). Let $t^*=\frac{1}{\overline{k}_o+\mathbf{v}^{*^t}\widetilde{\mathbf{x}}_o}$ and $z^*=t^*/\max\{\frac{\overline{w}_j+\mathbf{u}^{*^t}\widetilde{\mathbf{y}}_j}{\overline{k}_j+\mathbf{v}^{*^t}\widetilde{\mathbf{x}}_j}:j\in J\}$ then $(\mathbf{u}^t,\mathbf{v}^t,t,z)=(z^*\mathbf{u}^{*^t},t^*\mathbf{v}^{*^t},t^*,z^*)$ is a feasible solution of (14), so

$$\theta_2\ge\overline{w}_o z^*+\mathbf{u}^{*^t}z^*\widetilde{\mathbf{y}}_o=\theta_1. \tag{15}$$

Also, suppose that $(\mathbf{u}^{*^t},\mathbf{v}^{*^t},t^*,z^*)$ is an optimal solution of (14), then $(\mathbf{u}^t,\mathbf{v}^t)=(\frac{\mathbf{u}^{*^t}}{z^*},\frac{\mathbf{v}^{*^t}}{t^*})$ is a feasible solution of (12). Moreover, because $(\mathbf{u}^{*^t},\mathbf{v}^{*^t},t^*,z^*)$ is a feasible solution of (14), $\overline{k}_o t^*+\mathbf{v}^{*^t}\widetilde{\mathbf{x}}_o=1$ and $\overline{w}_j z^*+\mathbf{u}^{*^t}\widetilde{\mathbf{y}}_j-(\overline{k}_j t^*+\mathbf{v}^{*^t}\widetilde{\mathbf{x}}_j)\le 0$ for all $j\in J$. Therefore $\max\{\frac{\overline{w}_j z^*+\mathbf{u}^{*^t}\widetilde{\mathbf{y}}_j}{\overline{k}_j t^*+\mathbf{v}^{*^t}\widetilde{\mathbf{x}}_j}:j\in J\}\le 1$. So

$\theta_1\ge[(\overline{w}_o+\frac{\mathbf{u}^{*^t}}{z^*}\widetilde{\mathbf{y}}_o)/(\overline{k}_o+\frac{\mathbf{v}^{*^t}}{t^*}\widetilde{\mathbf{x}}_o)]/\max\{(\overline{w}_j+\frac{\mathbf{u}^{*^t}}{z^*}\widetilde{\mathbf{y}}_j)/(\overline{k}_j+\frac{\mathbf{v}^{*^t}}{t^*}\widetilde{\mathbf{x}}_j):j\in J\}=[(\overline{w}_o z^*+\mathbf{u}^{*^t}\widetilde{\mathbf{y}}_o)/(\overline{k}_o t^*+\mathbf{v}^{*^t}\widetilde{\mathbf{x}}_o)]/\max\{(\overline{w}_j z^*+\mathbf{u}^{*^t}\widetilde{\mathbf{y}}_j)/(\overline{k}_j t^*+\mathbf{v}^{*^t}\widetilde{\mathbf{x}}_j):j\in J\}\ge\overline{w}_o z^*+\mathbf{u}^{*^t}\widetilde{\mathbf{y}}_o=\theta_2.$

From (15) and the above relation, we have $\theta_1=\theta_2$ and the proof is completed.

The proof of this theorem shows that if $(\mathbf{u}^{*^t},\mathbf{v}^{*^t},t^*,z^*)$ is an optimal solution of (14), then $(\frac{\mathbf{u}^{*^t}}{z^*},\frac{\mathbf{v}^{*^t}}{t^*})$ is an optimal solution of (12), and one of the best weight vectors for the inputs and outputs of DMU_o, which their market prices are not available, is $(\frac{\mathbf{u}^{*^t}}{z^*},\frac{\mathbf{v}^{*^t}}{t^*})$.

The dual of (14) is called the "input-oriented envelopment proposed model" and is shown below:

$$\begin{array}{lll} \min & \theta-\varepsilon(s^++s^-) & \\ s.t. & \sum_{j=1}^n\lambda_j x_{ij}\le\theta x_{io}; & i=p+1,\ldots,m \\ & \sum_{j=1}^n\lambda_j\overline{k}_j+s^-=\theta\overline{k}_o & \\ & \sum_{j=1}^n\lambda_j y_{rj}\ge y_{ro}; & r=q+1,\ldots,s \\ & \sum_{j=1}^n\lambda_j\overline{w}_j-s^+=\overline{w}_o & \\ & \lambda_j\ge 0; & j\in J \\ & s^+\ge 0 & \\ & s^-\ge 0 & \\ & \theta \textit{ is unrestricted.} & \end{array} \tag{16}$$

Because of the non-Archimedean number ε, we solve the following problems to find the optimal solution of (16).

$$\begin{array}{lll} \min & \theta & \\ s.t. & \sum_{j=1}^n\lambda_j x_{ij}\le\theta x_{io}; & i=p+1,\ldots,m \\ & \sum_{j=1}^n\lambda_j\overline{k}_j\le\theta\overline{k}_o & \\ & \sum_{j=1}^n\lambda_j y_{rj}\ge y_{ro}; & r=q+1,\ldots,s \\ & \sum_{j=1}^n\lambda_j\overline{w}_j\ge\overline{w}_o & \\ & \lambda_j\ge 0; & j\in J \\ & \theta \textit{ is unrestricted.} & \end{array} \tag{17}$$

Using our knowledge of θ^*, the optimal value of (17), we solve the following LP using λ, s^+ and s^-as variables:

$$\begin{array}{lll} \max & s^++s^- & \\ s.t. & \sum_{j=1}^n\lambda_j x_{ij}\le\theta^* x_{io}; & i=p+1,\ldots,m \\ & \sum_{j=1}^n\lambda_j\overline{k}_j+s^-=\theta^*\overline{k}_o & \\ & \sum_{j=1}^n\lambda_j y_{rj}\ge y_{ro}; & r=q+1,\ldots,s \\ & \sum_{j=1}^n\lambda_j\overline{w}_j-s^+=\overline{w}_o & \\ & \lambda_j\ge 0; & j\in J \\ & s^+\ge 0 & \\ & s^-\ge 0 & \end{array} \tag{18}$$

It is easy to show that θ^* is the optimal value of (17) and $(\lambda^{*^t}, s^{+*}, s^{-*})$ is an optimal solution of (18) if and only if $(\lambda^{*^t}, \theta^*, s^{+*}, s^{-*})$ is an optimal solution of (16).

Suppose that $(\lambda^{*^t}, \theta^*, s^{+*}, s^{-*})$ is an optimal solution of (16). Consider the following virtual DMU

$$(\sum_{j=1}^{n} \lambda_j^* \mathbf{x}_j^t, \sum_{j=1}^{n} \lambda_j^* \mathbf{y}_j^t)$$

1. For the first p inputs whose weights (market prices) are known, the total cost $(\sum_{j=1}^{n} \lambda_j^* \overline{k}_j)$ is contracted with respect to $\overline{k}_o$, and
2. for the last $m-p$ inputs, whose weights are unknown, the virtual inputs are contracted with respect to $\widetilde{\mathbf{x}}_o$. These mean that the first p inputs and the last $m-p$ inputs of the virtual DMU are contracted with respect to in cost and quantity, respectively, by a contraction factor θ^*.

$(\overline{\lambda}^t, \overline{\theta}, \overline{s}^+, \overline{s}^-)$, in which $\overline{\lambda}_o = 1$, $\overline{\lambda}_j = 0$ ($j \in J$ and $j \neq o$), $\overline{\theta} = 1$, $\overline{s}^+ = 0$ and $\overline{s}^- = 0$, is a feasible solution of (16). Also if $\sum_{i=p+1}^{m} x_{io} + \overline{k}_o \neq 0$, then the optimal value of (16) is bounded. By the duality theorem the dual of (16), i.e. (14), is feasible.

Definition 2 *DMU_o is efficient if and only if the optimal value of (17) equals one and the optimal value of (18) equals zero otherwise, it is inefficient.*

In a similar way we can show that the "output-oriented multiplier proposed model" and the "output-oriented envelopment proposed model" are as follows:

$$\begin{array}{lll} \min & \overline{k}_o t + \mathbf{v}^t \widetilde{\mathbf{x}}_o & \\ s.t. & \overline{w}_o z + \mathbf{u}^t \widetilde{\mathbf{y}}_o = 1 & \\ & \overline{w}_j z + \mathbf{u}^t \widetilde{\mathbf{y}}_j - (\overline{k}_j t + \mathbf{v}^t \widetilde{\mathbf{x}}_j) \leq 0; \quad j \in J & \\ & t \geq \varepsilon, \quad \mathbf{u} \geq \mathbf{0} & \\ & z \geq \varepsilon, \quad \mathbf{v} \geq \mathbf{0} & \end{array} \tag{19}$$

$$\begin{array}{lll} \max & \phi + \varepsilon(s^+ + s^-) & \\ s.t. & \sum_{j=1}^{n} \lambda_j \widetilde{\mathbf{x}}_j \leq \widetilde{\mathbf{x}}_o & \\ & \sum_{j=1}^{n} \lambda_j \overline{k}_j + s^- = \overline{k}_o & \\ & \sum_{j=1}^{n} \lambda_j \widetilde{\mathbf{y}}_j \geq \phi \widetilde{\mathbf{y}}_o & \\ & \sum_{j=1}^{n} \lambda_j \overline{w}_j - s^+ = \phi \overline{w}_o & \\ & \lambda_j \geq 0; & j \in J \\ & s^+ \geq 0 & \\ & s^- \geq 0 & \\ & \phi \textit{ is unrestricted.} & \end{array} \tag{20}$$

Theorem 2 *$\widetilde{\phi} = \frac{1}{\theta^*}$, where '$*$' and '$\sim$' indicate the optimality of (16) and (20), respectively.*

Proof: Suppose that $(\lambda^{*^t}, \theta^*, s^{+*}, s^{-*})$ and $(\widetilde{\lambda}^t, \widetilde{\phi}, \widetilde{s^+}, \widetilde{s^-})$ are the optimal solutions of (16) and (20), respectively. $(\lambda^t, \phi, s^+, s^-) = (\frac{\lambda^{*^t}}{\theta^*}, \frac{1}{\theta^*}, \frac{s^{+*}}{\theta^*}, \frac{s^{-*}}{\theta^*})$ is a feasible solution of (21), therefore $\widetilde{\phi} + \varepsilon(\widetilde{s^+} + \widetilde{s^-}) \geq \frac{1}{\theta^*} + \frac{\varepsilon}{\theta^*}(s^{+*} + s^{-*})$. So

$$1 - \widetilde{\phi}\theta^* \leq \varepsilon[\theta^*(\widetilde{s^+} + \widetilde{s^-}) - (s^{+*} + s^{-*})]. \tag{21}$$

Also $(\lambda^t, \theta, s^+, s^-) = (\frac{\widetilde{\lambda}^t}{\widetilde{\phi}}, \frac{1}{\widetilde{\phi}}, \frac{\widetilde{s^+}}{\widetilde{\phi}}, \frac{\widetilde{s^-}}{\widetilde{\phi}})$ is a feasible solution of (16), so $\theta^* - \varepsilon(s^{+*} + s^{-*}) \leq \frac{1}{\widetilde{\phi}} - \frac{\varepsilon}{\widetilde{\phi}}(\widetilde{s^+} + \widetilde{s^-})$. It implies

$$1 - \widetilde{\phi}\theta^* \geq \varepsilon[(\widetilde{s^+} + \widetilde{s^-}) - \widetilde{\phi}(s^{+*} + s^{-*})]. \tag{22}$$

Since ε is a non-Archimedean number, then (21) and (22) show that $\widetilde{\phi}\theta^* = 1$.

Consider Model (14). If all input prices and output benefits are unknown or we do not use input cost and output benefit information in Model (14), then parameters $\overline{w}_j$ and $\overline{k}_j$ for all j and variables z and t are omitted. Therefore, what remains of Model (14) is the "input-oriented multiplier CCR model" and its optimal value is TE.

Here we show that the minimal cost model (7) for evaluating cost efficiency is a special case of our proposed model; that is, the optimal value of our proposed model when all input prices are available and all output benefits are unknown is the cost efficiency measure of DMU_o.

Consider the proposed model to evaluate DMU_o when all input prices are available and all output benefits are unknown. In evaluating the measure of PE for DMU_o, we assume that the input cost vectors of all DMUs are the same and equal to $\mathbf{c}^o$, the input cost vector of DMU_o. In this case, the proposed model is converted to:

$$\begin{array}{ll} \max & \mathbf{u}^t \mathbf{y}_o \\ s.t. & \mathbf{c}^{o^t} \mathbf{x}_o t = 1 \\ & \mathbf{u}^t \mathbf{y}_j - \mathbf{c}^{o^t} \mathbf{x}_j t \leq 0; \quad j = 1, 2, \ldots, n \\ & \mathbf{u} \geq \mathbf{0} \\ & t \geq \varepsilon. \end{array} \tag{23}$$

In the above model $\mathbf{c}^{o^t} \mathbf{x}_o > 0$ and $t = \frac{1}{\mathbf{c}^{o^t} \mathbf{x}_o}$. By setting the value of t in the other constraints, the modified model is:

$$\begin{array}{ll} \max & \mathbf{u}^t \mathbf{y}_o \\ s.t. & \mathbf{u}^t \mathbf{y}_j \leq \frac{\mathbf{c}^{o^t} \mathbf{x}_j}{\mathbf{c}^{o^t} \mathbf{x}_o}; \quad j = 1, 2, \ldots, n \\ & \mathbf{u} \geq \mathbf{0} \end{array} \tag{24}$$

The dual of the above model is:

$$\begin{array}{lll} \min & \frac{\mathbf{c}^{o^t}}{\mathbf{c}^{o^t} \mathbf{x}_o} \sum_{j=1}^{n} \lambda_j \mathbf{x}_j & \\ s.t. & \sum_{j=1}^{n} \lambda_j \mathbf{y}_j \geq \mathbf{y}_o & \\ & \lambda_j \geq 0; & j = 1, \ldots, n. \end{array} \tag{25}$$

Let $\mathbf{x} = \sum_{j=1}^{n} \lambda_j \mathbf{x}_j$, so the above model is equal to:

$$\begin{array}{lll} \min & \frac{\mathbf{c}^{o^t} \mathbf{x}}{\mathbf{c}^{o^t} \mathbf{x}_o} & \\ s.t. & \sum_{j=1}^{n} \lambda_j \mathbf{x}_j = \mathbf{x} & \\ & \sum_{j=1}^{n} \lambda_j \mathbf{y}_j \geq \mathbf{y}_o & \\ & \lambda_j \geq 0; & j = 1, \ldots, n. \end{array} \tag{26}$$

where λ_j $(j = 1, \ldots, n)$ and $\mathbf{x}$ are variables.

The preceding model is the "Cost Model" to evaluate DMU_o, and its optimal value is the cost efficiency measure of DMU_o. A similar process can be used to prove that PE is equal to TCE. In this case $\overline{k}_j = c^j x_j$ and, $v^t \widetilde{x}_j$ and $\overline{w}_j z$ will be removed from (14).

Furthermore, the model for measuring the revenue efficiency is a special case of the model for evaluating PE when all output benefits are available, and by using the "output-oriented multiplier proposed model" the proof is similar to the above.

4 Examples

Example 1
Kim et al. (1999) examined 33 telephone offices, each having 3 inputs and 5 outputs, in Korea. We select 3 inputs and 3 outputs from among their data for evaluating the DMUs. All inputs and outputs are quantitative. We use the following factors:
Inputs

(x_1) : *Manpower.* the number of regular employees.

(x_2) : *operating costs.* Various costs except for interest cost (million dollars).

(x_3) : *number of telephone lines.*
Outputs

(y_1) : *local revenues.* The total revenues of local telephone services in each office (million dollars).

(y_2) : *long distance revenues.* The total revenues of long-distance telephone services in each office (million dollars).

(y_3) : *international revenues.* The total revenues of international telephone services in each office (million dollars). Table 1 displays the data.

In this example, outputs 1, 2 and 3 are quantitative and denote the benefits attained from consuming inputs. So, in evaluating DMUs with our proposed model we can use 1,000,000 as weights for outputs 1, 2 and 3.

The DMUs in Table 1 have been evaluated using the proposed model (Model (14)). Also, we use the maximal revenue model (Model (9)) and Equation (10) to find the revenue efficiency of DMUs and apply CRS model (Model(3)) to find the technical efficiency, the results of which are presented in Table 2.

We can see that in this example PE and RE of all DMUs are equal because all output benefits are available and equal. This is an example to show that the revenue efficiency is a special case of PE. In this example, 6 DMUs are CCR-efficient but the number of the proposed efficient DMUs is 4. Also, it can be seen that PE measures are not greater than CCR efficiency measures.
Note that this result is not true generally (see Example 3).

Table 1
Data for Example 1.

DMU	x_1	x_2	x_3	y_1	y_2	y_3
1	239	7.03	158	47.99	16.67	34.04
2	261	3.94	163	37.47	14.11	19.97
3	170	2.10	90	20.70	6.80	12.64
4	290	4.54	201	41.82	11.07	6.27
5	200	3.99	140	33.44	9.81	6.49
6	283	4.65	214	42.43	11.34	5.16
7	286	6.54	197	47.03	14.62	13.04
8	375	6.22	314	55.48	16.39	7.31
9	301	4.82	257	49.20	16.15	6.33
10	333	6.87	235	47.12	13.86	6.51
11	346	6.46	244	49.43	15.88	8.87
12	175	2.06	112	20.43	4.95	1.67
13	217	4.11	131	29.41	11.39	4.38
14	441	7.71	214	61.20	25.59	33.01
15	204	3.64	163	32.27	9.57	3.65
16	216	3.24	154	32.81	11.46	9.02
17	347	5.65	301	59.01	17.82	8.19
18	288	4.66	212	42.27	14.52	7.33
19	185	3.37	178	32.95	9.46	2.91
20	242	5.12	270	65.06	24.57	20.72
21	234	2.52	126	31.55	8.55	7.27
22	204	4.24	174	32.47	11.15	2.95
23	356	7.95	299	66.04	22.25	14.91
24	292	4.52	236	49.97	14.77	6.35
25	141	5.21	63	21.48	9.76	16.26
26	220	6.09	179	47.94	17.25	22.09
27	298	3.44	225	42.35	11.14	4.25
28	261	4.30	213	41.70	11.13	4.68
29	216	3.86	156	31.57	11.89	10.48
30	171	2.45	150	24.09	9.08	2.60
31	123	1.72	61	11.97	4.78	2.95
32	89	0.88	42	6.40	3.18	1.48
33	109	1.35	57	10.57	3.43	2.00

Table 2
Results of Example 1.

DMU no	Technical Efficiency	Revenue Efficiency	Proposed Efficiency
1	1.000	1.000	1.000
2	0.990	0.965	0.965
3	1.000	0.996	0.996
4	0.822	0.668	0.668
5	0.896	0.722	0.722
6	0.792	0.636	0.636
7	0.860	0.714	0.714
8	0.721	0.606	0.606
9	0.803	0.690	0.690
10	0.748	0.577	0.577
11	0.773	0.640	0.640
12	0.780	0.609	0.609
13	0.821	0.666	0.666
14	1.000	1.000	1.000
15	0.788	0.636	0.636
16	0.857	0.810	0.810
17	0.822	0.698	0.698
18	0.794	0.695	0.695
19	0.769	0.624	0.624
20	1.000	1.000	1.000
21	1.000	0.900	0.900
22	0.731	0.587	0.587
23	0.848	0.728	0.728
24	0.872	0.734	0.734
25	1.000	1.000	1.000
26	0.975	0.928	0.928
27	0.969	0.779	0.779
28	0.795	0.776	0.776
29	0.787	0.752	0.752
30	0.774	0.677	0.677
31	0.739	0.658	0.658
32	0.803	0.618	0.618
33	0.720	0.623	0.623

Example 2

Table 3 exhibits four DMUs with two inputs and two outputs, along with the unit cost for each input. The results of evaluating TE, CE and PE are also presented in Table 4.

Table 3
Data of Example 2.

DMU	x_1	x_2	y_1	y_2	c_1	c_2
A	2	3	5	8	2	2
B	1	5	2	6	2	4
C	3	8	4	8	3	3
D	2	7	1	2	4	2

Table 4
Results of Example 2.

DMU	TE	CE	TCE	PE
A	1.000	1.000	1.000	1.000
B	1.000	0.545	0.341	0.341
C	0.571	0.455	0.303	0.303
D	0.214	0.159	0.114	0.114

According to the results we can see that PE is not equal to CE for DMUs B, C and D because in evaluating PE for each DMU the model uses the corresponding costs for each DMU, but in evaluating the cost efficiency of DMU_o, the minimal cost model actually uses the input costs of DMU_o for all other DMUs. This is while the input costs are practically different in evaluating other DMUs.
In 2.2. we said that Tone cost model has solved the shortcoming of Farrell-Debreu cost model. Regarding Table 4 we see that the results of Farrell-Debreu cost efficiency (CE) are different from TCE. Moreover, we see that the results of PE and TCE are the same. It shows that PE solves the traditional Farrell-Debreu shortcoming too. In general the results of TCE and PE, when we use the inputs and outputs quantities as well as all input prices, are the same.

Example 3.

In this example, we evaluate 12 DMUs with two inputs and two outputs. The unit cost for input 1 and the unit benefit for output 1 are available. The costs and the benefits are different for each DMU. The data are summarized in Table 5.

Table 5
Data for Example 3.

DMUs	x_1	x_2	y_1	y_2	c_1	b_1
A	20	151	100	90	500	550
B	19	131	150	50	350	400
C	25	160	160	55	450	480
D	27	168	180	72	600	600
E	22	158	94	66	300	400
F	55	255	230	90	450	430
G	33	235	220	88	500	540
H	31	206	152	80	450	420
I	30	244	190	100	380	350
J	50	268	250	100	410	410
K	53	306	260	147	440	540
L	38	284	250	120	400	295

Using the data in Table 5, we cannot evaluate cost efficiency and revenue efficiency because not all input costs or output benefits are available. However, we can obtain PE for each DMU. The results of evaluating TE and PE are summarized in Table 6.

Table 6
Results of Example 3.

DMU	A	B	C	D	E	F
PE	1.000	1.000	0.894	1.000	1.000	0.719
TE	1.000	1.000	0.883	1.000	0.763	0.835
DMUs	G	H	I	J	K	L
PE	0.942	0.732	0.936	0.765	0.949	0.847
TE	0.902	0.796	0.960	0.871	0.955	0.958

Table 6 shows that there does not exist any relation between TE and PE. For example the PE of DMUs C, E and G are greater than their TE, while the PE of F, H, I, J, K and L are less than their TE.

5 Conclusion

There are different types of efficiency measures such as technical efficiency, cost efficiency and revenue efficiency. These measures are different in terms of efficiency score values because each uses different information obtained from decision making units, e.g. technical efficiency only uses the quantities of inputs and outputs; cost efficiency uses the quantities of inputs, outputs and all input prices; and revenue efficiency uses the quantities of inputs, outputs and all output benefits. But in real problems, some input prices (not necessarily all) and some output benefits (not necessarily all) are known. The conventional techniques for measuring efficiency do not use some input prices and some output benefits together. In this paper we proposed a linear programming model to estimate the measure of efficiency when only some input prices and some output benefits are known. This measure is nearer to real world conditions than other measures (TE, CE and RE), because we use real weights in PE while the weights obtained from multiplier forms (used for evaluating TE, CE and RE) are possibly different from real weights. The weights obtained from the CCR multiplier form are the best weights to maximize TE. The measures of TE, CE and RE are special cases of PE, and in this paper we showed that TE and CE measures and the corresponding models are special cases of PE measure and the proposed model, respectively. A shortcoming of the CE measure that our proposed model overcomes is the fact that in evaluating the CE of DMU_o it uses the input cost vector of DMU_o, i.e. c^o, for all other DMUs (see Tone (2002)). Our proposed model uses their corresponding input cost vectors for other DMUs. A similar argument can be put forward for RE. PE measure can be used in real worlds problems and it can yield some methods to rank DMUs. The envelopment proposed model to evaluate PE has $n+3$ variables and $m+s+2-(p+q)$ restrictions (except the nonnegativity constraints) while the envelopment form of the CCR model has $n+1$ variables and $m+s$ restrictions (except the nonnegativity constraints). Since our proposed model has fewer constraints (when $p+q>2$) compared to the envelopment form of the CCR model, it seems to be able to evaluate PE better than the envelopment form of the CCR model in terms of complexity. Using this measure for practical problems and finding some methods to rank DMUs and applying this measure (PE) to other fields of DEA, e.g. ranking, Productivity, returns to scale and etc. are directions for future research.

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