Nonexistence of Global Solutions to a Nonlinear Fractional Reaction-Diffusion System

Belgacem Rebiai and Kamel Haouam

Abstract—In this paper we consider a Cauchy problem presented as a parabolic fractional reaction diffusion system with nonlinear terms of the form $|u|^p_i u_i$, $i = 1, 2$ where $p_1,q_2 \geq 0$ and $p_2,q_1 > 1$ are constants. We prove a nonexistence result which is more general than the interesting result obtained by Kirane et al. [8] which concerns the case $p_1 = q_2 = 0$. We also show that this result extends the works of Yamauchi [15] and Zheng [16] done in the classical case.

Index Terms—Fractional derivatives, nonlinear reaction diffusion system, test-function, critical exponent.

I. INTRODUCTION

The fractional calculus is a field of mathematics which is as old as the differential calculus, the concept is born from a question asked by the Hospital in 1695 to Leibniz about the meaning of $d^n y/dx^n = D^n y$ if $n = 1/2$ and what happens for $D^n$ if $n$ is not an integer number; Leibniz replied “That is a paradox from which one day useful consequences will be drawn”. In the last few years non integer derivatives and integrals have been used in different fields of science to describe many phenomena and they are nowadays playing a great role in modelling engineering problems including fluid flow, rheology, diffusive transport, networks, probability and hereditary properties of various materials. Fractional and boundary value problems for nonlinear fractional differential equations have received much attention and it becomes natural that many authors try to solve the fractional derivatives, fractional integrals and fractional differential equations, see, e.g., [1], [6], [8], [10], [11], [12], [13], [14] and references therein.

The present paper is concerned with the following Cauchy problem for the nonlinear reaction-diffusion system

\[
\begin{aligned}
D^{\alpha_1}_0u + (-\Delta)^{\beta_1/2}u &= |u|^{p_1}|u|^{q_1}, \\
D^{\alpha_2}_0v + (-\Delta)^{\beta_2/2}v &= |u|^{p_2}|v|^{q_2}, \\
u(0,x) &= u_0(x) \geq 0, \neq 0, \\
v(0,x) &= v_0(x) \geq 0, \neq 0,
\end{aligned}
\]

where $(t,x) \in R^+ \times R^N$, $p_1 \geq 0$, $q_2 \geq 0$, $p_2 > 1$, $q_1 > 1$, $0 < \alpha_i < 1$, $1 \leq \beta_i \leq 2$, $i = 1, 2$ are constants. $D^{\alpha}_0$ denotes the derivatives of order $\alpha_i$ in the sense of Caputo (see, e.g., [13]) and $(-\Delta)^{\beta_i/2}$ is the fractional power of the Laplacian $(-\Delta)$ defined by

\[
(-\Delta)^{\beta/2}u(t,x) = F^{-1}(\xi^{\beta/2}F(u)(\xi))(t,x),
\]

where $F$ is the Fourier transform and $F^{-1}$ its inverse.

In the case where $\alpha_i = 1$, $\beta_i = 2$, $i = 1, 2$, the problem (1) was treated by many authors in several contexts, see for example [2], [3], [4], [15], [16]. Escobedo and Herrero [2] proved that if $p_2q_1 > 1$, $p_1 = q_2 = 0$ and $(\gamma + 1)/(p_2q_1 - 1) \geq N/2$ with $\gamma = \max(p_2,q_1)$, then the only solution of the problem (1) is the trivial one, i.e., $u \equiv v \equiv 0$. Later in [3] Escobedo and Levine showed that if $p_1 \geq 1$ and $p_2 + q_2 \geq p_1 + q_1 > 0$, then the problem (1) behaves like the Cauchy problem for the single equation $u_t - \Delta u = u^{p_1+q_1}$, with respect to Fujita-type blowup theorems, see [5]. In [15], Yamauchi considered the problem

\[
\begin{aligned}
u_t - \Delta u &= |x|^\sigma |u|^{p_1}|u|^{q_1}, \\
v_t - \Delta v &= |x|^\sigma |p_2|^{q_2}, \\
u(0,x) &= u_0(x) \geq 0, \neq 0, \\
v(0,x) &= v_0(x) \geq 0, \neq 0,
\end{aligned}
\]

where $p_i, q_i \geq 0$, $\sigma_i \geq \max(-2,-N)$, $i = 1, 2$. He proved a nonexistence results under some conditions concerning relation between exponents $p_i, q_i, \sigma_i$ and initial data.

In the case of real order $0 < \alpha_i < 1$ and $1 \leq \beta_i \leq 2$, Kirane et al. [8] considered the following Cauchy problem

\[
\begin{aligned}
D^{\alpha_1}_0u + (-\Delta)^{\beta_1/2}u &= |u|^{p_1}, \\
D^{\alpha_2}_0v + (-\Delta)^{\beta_2/2}v &= |u|^{p_2}, \\
u(0,x) &= u_0(x) \geq 0, \\
v(0,x) &= v_0(x) \geq 0,
\end{aligned}
\]

(2)

and they proved that if

\[
q_1 > 1, p_2 > 1, q_1p_1' = q_1 + q_1', p_2p_2' = p_2 + p_2'
\]

and

\[
N \leq \max \left\{ \frac{\alpha_2}{p_2} + \alpha_1 - (1 - \frac{1}{p_2q_1}), \frac{\alpha_1}{p_1} + \alpha_2 - (1 - \frac{1}{p_2q_1}) \right\},
\]

then the problem (2) does not admit nontrivial global weak nonnegative solutions.

The aim of this paper is to prove a nonexistence result which is more general than the above interesting result cited in [8] which concerns the case $p_1 = q_2 = 0$.

II. PRELIMINARIES

In this section, we present some definitions and results concerning fractional derivatives that will be used hereafter.

Definition 1: Let $0 < \alpha < 1$ and $\phi \in L^1(0,T)$. The left-sided and right-sided Riemann-Liouville derivatives of order $\alpha$ for $\phi$ are defined, respectively, by:

\[
D_{0+}^{\alpha_1}\phi(t) = \frac{1}{\Gamma(1-\alpha_1)} \frac{d}{dt} \int_0^t (t-s)^{\alpha_1-1} \phi(s) ds,
\]

(Advance online publication: 14 November 2015)
and
\[ D_{0,t}^{\alpha} \phi(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T \frac{\phi(s)}{(s-t)^\alpha} ds, \]
where \( \Gamma \) denotes the gamma function.

**Definition 2:** Let \( 0 < \alpha < 1 \) and \( \phi' \in L^1(0,T) \). The left-sided and respectively right-sided Caputo derivatives of order \( \alpha \) for \( \phi' \) are defined as:
\[ D_{0,t}^{\alpha} \phi(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\phi'(s)}{(t-s)^\alpha} ds, \]
and
\[ D_{t,T}^{\alpha} \phi(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^T \frac{\phi'(s)}{(s-t)^\alpha} ds, \]
where \( \Gamma \) denotes, as usual, the gamma function. The relation between Caputo and Riemann-Liouville derivatives is written as
\[ D_{0,t}^{\alpha} \phi(t) = D_{0,t}^{\alpha}[\phi(t) - \phi(0)]. \]

Finally, taking into account the following integration by parts formula:
\[ \int_0^T f(t)(D_{0,t}^{\alpha} g)(t) dt = \int_0^T (D_{t,T}^{\alpha} f)(t)g(t) dt, \]
we adopt the following definition concerning the weak formulation for the problem (1).

**Definition 3:** Let \( Q_T = (0,T) \times R^N, 0 < T < +\infty \). We say that \((u,v) \in (L^1_{loc}(Q_T))^2\) is a local weak solution to problem (1) on \( Q_T \), if \( u^p_i v^q_i \in L^1_{loc}(Q_T), i = 1, 2 \), and it is such that
\[ \int_{Q_T} u_0 D_{t,T}^{\alpha_i} \varphi_1 dx = \int_{Q_T} u |D_{t_i}^{\alpha_i} \varphi_1| dx, \]
for all test functions \( \varphi_1 \in C_T^{1,2}(Q_T) \) such that \( \varphi_1(T,x) = 0, i = 1, 2 \).

III. MAIN RESULTS

We now state our main result as follows.

**Theorem 1:** Let \( p_1 \geq 0, q_2 \geq 0, p_2 > 1, q_1 > 1, p_2 q_2' = p_2 + p_2', q_1 q_1' = q_1 + q_1' \). Let \( u_0, v_0 \) in \( L^\infty(R^N) \) such that \( u_0, v_0 \geq 0 \) and \( u_0, v_0 \neq 0 \). If
\[ N \leq \max \left\{ \frac{q_2}{p_2} + \alpha_1 - (1 - \frac{1}{p_2 q_1}), \frac{q_1}{q_1'}, \frac{\alpha_1}{q_1'}, \frac{\alpha_2}{q_2}, \frac{\alpha_2}{q_2'} \right\}, \]
then the problem (1) does not admit global weak solutions.

**Proof:** The proof is by contradiction. Suppose that \((u,v)\) is a global weak solution to problem (1). Since \( u_0, v_0 \geq 0 \) and \( u_0, v_0 \neq 0 \), then \( u(t), v(t) > 0 \) for all \( t \in (0,T^*) \) for any arbitrary \( T^* > 0 \).

Let \( T \) and \( \theta \) be two real numbers such that
\[ 0 < T < T^* \] and \( \theta = \min \left\{ \frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2} \right\}. \]

Let \( \Phi \in C^2_0(R^+), 0 \leq \Phi(r) \leq 1 \) for all \( r \geq 0 \), a smooth nonnegative nonincreasing function such that
\[ \Phi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } r \geq 2, \end{cases} \]
and
\[ \Psi(s) = \begin{cases} (1-s)^\gamma & \text{if } 0 \leq s \leq 1, \\ 0 & \text{if } s \geq 1, \end{cases} \]
where \( \gamma \geq \max \left\{ q_1 q_1', \alpha_2 q_2' \right\}. \)

We choose
\[ \varphi_i(t,x) = \phi_i(t) \psi(t), \quad i = 1, 2 \]

with
\[ \phi(x) = \Phi(|x| T) \psi(t) = \Psi \left( \frac{t}{T} \right) \]

Applying Hölder inequality to the right hand sides of the weak formulations (3) and (4) on \( \Sigma \), where
\[ \Sigma = (0,T) \times \{ x \in R^N : |x| \leq 2T^\theta \}, \]
we obtain
\[ \int_{Q_T} u |D_{t,T}^{\alpha} \varphi_1| \leq \left( \int_{Q_T} |u|^p_i |v|^{q_2'} \varphi_2 \right)^{\frac{1}{p_2}} \cdot A, \]
and
\[ \int_{Q_T} u |(-\Delta)^{\frac{\alpha_1}{2}} \varphi_1| \leq \left( \int_{Q_T} |u|^p_i |v|^{q_2} \varphi_2 \right)^{\frac{1}{p_2}} \cdot B, \]
where
\[ A = \left( \int_{Q_T} |D_{t,T}^{\alpha_1} \varphi_1|^2 |v|^{q_2} \Phi^{\frac{q_2'}{q_2}} \right)^{\frac{1}{q_2}} \]
and
\[ B = \left( \int_{Q_T} |(-\Delta)^{\frac{\alpha_1}{2}} \varphi_1|^2 |v|^{q_2'} \Phi^{\frac{q_2}{q_2'}} \right)^{\frac{1}{q_2'}} \]

Consequently
\[ \int_{Q_T} |u|^p_i |v|^{q_1} \varphi_1 \leq \left( \int_{Q_T} |u|^p_i |v|^{q_2} \varphi_2 \right)^{\frac{1}{p_2}} \cdot A_1, \]
where
\[ A_1 = A + B. \]

The same way give us the next estimate
\[ \int_{Q_T} |u|^p_i |v|^{q_2} \varphi_2 \leq \left( \int_{Q_T} |u|^p_i |v|^{q_1} \varphi_1 \right)^{\frac{1}{p_1}} \cdot A_2, \]
where
\[ A_2 = \left( \int_{Q_T} |D_{t,T}^{\alpha_1} \varphi_2|^2 |v|^{q_1} \Phi^{\frac{q_1'}{q_1}} \right)^{\frac{1}{q_1}} + \left( \int_{Q_T} |(-\Delta)^{\frac{\alpha_1}{2}} \varphi_2|^2 |v|^{q_1'} \Phi^{\frac{q_1}{q_1'}} \right)^{\frac{1}{q_1'}}. \]
Using (5) and (6) one can write
\[
\left(\int_{\Sigma} |u|^{p_1} |v|^{q_1} \varphi_1 \right)^{\frac{1}{\gamma_1}} \leq A_1 A_2^{\gamma_1}, \tag{7}
\]
and
\[
\left(\int_{\Sigma} |u|^{p_2} |v|^{q_2} \varphi_2 \right)^{\frac{1}{\gamma_2}} \leq A_1^{\gamma_2} A_2. \tag{8}
\]
Therefore, as \( u > 0 \) and \( v > 0 \), then using Ju’s inequality
\((\Delta)^{1/2} \phi^p \leq \int \Delta^{1/2} \phi^p \) (see [7]), and introducing the change of variables \( t = T \tau, \ x = T^{\frac{\alpha}{2}} \xi \) in \( A_1 \), we obtain
\[
\left(\int_{\Sigma} |u|^{p_1} |v|^{q_1} \varphi_1 \right)^{\frac{1}{\gamma_1}} \leq CT \left( \frac{2\gamma_1}{\gamma_1^2 + \gamma_2^2} \right)^{\frac{1}{\gamma_1}}, \tag{9}
\]
and
\[
\left(\int_{\Sigma} |u|^{p_2} |v|^{q_2} \varphi_2 \right)^{\frac{1}{\gamma_2}} \leq CT \left( \frac{2\gamma_2}{\gamma_1^2 + \gamma_2^2} \right)^{\frac{1}{\gamma_2}}, \tag{10}
\]
where
\[
\gamma_1 = \alpha_1 p_2 - \frac{\alpha_1}{\beta_1} N - 1, \quad \gamma_2 = \alpha_2 q_1 - \frac{\alpha_2}{\beta_2} N - 1.
\]
Now, if we choose \( N < N^* \) and pass to the limit in (9) and (10), as \( T \) goes to infinity, we get
\[
\int_{R^+ \times RN^} |u|^{p_1} |v|^{q_1} \varphi_1 = 0,
\]
or
\[
\int_{R^+ \times RN^} |u|^{p_2} |v|^{q_2} \varphi_2 = 0.
\]
Using the dominated convergence theorem and the continuity in time and space of \( u \) and \( v \), we infer that
\[
\int_{R^+ \times RN^} |u|^{p_1} |v|^{q_1} = 0,
\]
or
\[
\int_{R^+ \times RN^} |u|^{p_2} |v|^{q_2} = 0.
\]
This implies that \( u \equiv 0 \) or \( v \equiv 0 \), which is a contradiction.

In the case \( N = N^* \), we modify the previous function \( \phi \) by introducing a new number \( R, 0 < R < T \), such that
\[
\phi(x) = \Phi \left( \frac{|x|}{(T/R)^\theta} \right),
\]
and we set
\[
\Sigma_R = (0, T) \times \left\{ x \in RN^ : |x| \leq 2(T/R)^\theta \right\}, \quad \Delta_R = (0, T) \times \left\{ x \in RN^ : (T/R)^\theta \leq |x| \leq 2(T/R)^\theta \right\}.
\]
Since, from (9) and (10), we find that
\[
\left(\int_{R^+ \times RN^} |u|^{p_1} |v|^{q_1} \varphi_1 \right)^{\frac{1}{\gamma_1}} \leq C,
\]
or
\[
\left(\int_{R^+ \times RN^} |u|^{p_2} |v|^{q_2} \varphi_2 \right)^{\frac{1}{\gamma_2}} \leq C,
\]
Then we have
\[
\lim_{T \to +\infty} \int_{\Delta_R} |u|^{p_1} |v|^{q_1} \varphi_1 dt dx = 0,\tag{11}
\]
or
\[
\lim_{T \to +\infty} \int_{\Delta_R} |u|^{p_2} |v|^{q_2} \varphi_2 dt dx = 0.\tag{12}
\]
Applying Hölder inequality to the right hand sides of the weak formulations (3) and (4) on \( \Sigma_R \), we obtain
\[
\int_{\Sigma_R} u|D_{1/2}^{i/2} \varphi_1| \leq \left( \int_{\Sigma_R} |u|^{p_2} |v|^{q_2} \varphi_2 \right)^{\frac{1}{p_2}} \cdot B_1, \tag{16}
\]
and
\[
\int_{\Sigma_R} u|(-\Delta)^{\frac{1}{2}} \varphi_1| \leq \left( \int_{\Sigma_R} |u|^{p_2} |v|^{q_2} \varphi_2 \right)^{\frac{1}{p_2}} \cdot C_1, \tag{17}
\]
where
\[
B_1 = \left( \int_{\Sigma_R} |D_{1/2}^{i/2} \varphi_1|^{p_2} |v|^{\frac{p_2 q_2}{p_2 r_2} - \frac{p_2 r_2}{r_2} \varphi_2} \right)^{\frac{1}{p_2}} \tag{18}
\]
and
\[
C_1 = \left( \int_{\Delta_R} |(-\Delta)^{\frac{1}{2}} \varphi_1|^{p_2} |v|^{\frac{p_2 q_2}{p_2 r_2} - \frac{p_2 r_2}{r_2} \varphi_2} \right)^{\frac{1}{p_2}}. \tag{19}
\]
Consequently
\[
\int_{\Sigma_R} |u|^{p_1} |v|^{q_1} \varphi_1 \leq \left( \int_{\Sigma_R} |u|^{p_2} |v|^{q_2} \varphi_2 \right)^{\frac{1}{p_2}} \cdot B_1 + \left( \int_{\Delta_R} |u|^{p_2} |v|^{q_2} \varphi_2 \right)^{\frac{1}{p_2}} \cdot C_1, \tag{20}
\]
The same way give us the next estimate
\[
\int_{\Sigma_R} |u|^{p_2} |v|^{q_2} \varphi_2 \leq \left( \int_{\Sigma_R} |u|^{p_1} |v|^{q_1} \varphi_1 \right)^{\frac{1}{q_1}} \cdot B_2 + \left( \int_{\Delta_R} |u|^{p_1} |v|^{q_1} \varphi_1 \right)^{\frac{1}{q_1}} \cdot C_2, \tag{21}
\]
where
\[
B_2 = \left( \int_{\Sigma} |D_{1/2}^{i/2} \varphi_2|^{q_2} |u|^{\frac{p_1 q_2}{p_2 q_1} - \frac{q_2 q_1}{q_1}} \right)^{\frac{1}{q_1}}, \tag{22}
\]
and
\[
C_2 = \left( \int_{\Delta_R} |(-\Delta)^{\frac{1}{2}} \varphi_2|^{q_2} |u|^{\frac{p_1 q_2}{p_2 q_1} - \frac{q_2 q_1}{q_1}} \right)^{\frac{1}{q_1}}. \tag{23}
\]
If we introduce the change of variables
\[
t = T \tau, \quad x = (T/R)^{\frac{\alpha}{2}} \xi
\]
in \( B_1 \) and \( C_1 \), and using (13) and (14), we obtain via (11) and (12), after passing the limit as \( T \) goes to infinity
\[
\left( \int_{R^+ \times RN^} |u|^{p_1} |v|^{q_1} \varphi_1 \right)^{\frac{1}{\gamma_1}} \leq CR \left( \frac{\gamma_1}{\gamma_2^2 + \gamma_1^2} \right)^{\frac{1}{\gamma_1}}, \tag{24}
\]
or
\[
\left( \int_{R^+ \times RN^} |u|^{p_2} |v|^{q_2} \varphi_2 \right)^{\frac{1}{\gamma_2}} \leq CR \left( \frac{\gamma_2}{\gamma_1^2 + \gamma_2^2} \right)^{\frac{1}{\gamma_2}}. \tag{25}
\]

(Advance online publication: 14 November 2015)
where

\[ \gamma_i' = \frac{\alpha_i}{\beta_i} N, \quad i = 1, 2. \]

Then, taking the limit when \( R \) goes to infinity, we obtain

\( u \equiv 0 \) or \( v \equiv 0 \), contradiction.

**Remark 1:** When \( p_1 = q_2 = 0 \), we recover the case studied by Kirane et al. [8], however we have to impose the constraints \( u_0 \geq 0, \neq 0 \) and \( v_0 \geq 0, \neq 0 \), while Kirane et al. require only the conditions \( u_0 \geq 0 \) and \( v_0 \geq 0 \).

**Remark 2:** We can extend our result to the more general system

\[
\begin{aligned}
D^\alpha_0 u + (-\Delta)^{\beta}/2 (|u|^{n-1}u) &= h|u|^{p_1}|v|^q_1 + g|u|^{r_1}|v|^{s_1}, \\
D^\beta_0 v + (-\Delta)^{\alpha}/2 (|v|^{m-1}v) &= k|u|^{p_2}|v|^q_2 + l|u|^{r_2}|v|^{s_2},
\end{aligned}
\]

under some suitable conditions on \( h(t, x) \), \( g(t, x) \), \( k(t, x) \) and \( l(t, x) \).

### IV. Conclusion

We have established some new results for a class of nonlinear fractional differential systems with nonlinear terms of the form \( |u|^{p_i}|v|^q_i \), \( i = 1, 2 \) where \( p_1, q_2 \geq 0 \) and \( p_2, q_1 > 1 \) are constants. We note that these results can also be applied to study the problem (1) for a high order \( \alpha_i > 1 \).

Finally, we wish that the present manuscript will open wide avenues for further research in the field of fractional calculus and other domains.

### References


