Existence of A New Class of Impulsive Riemann-Liouville Fractional Partial Neutral Functional Differential Equations with Infinite Delay

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***Abstract*—In this paper, a new class of abstract impulsive Riemann-Liouville fractional partial neutral functional differential equations with infinite delay is introduced. We apply the suitable fixed point theorem together with the Hausdorff measure of noncompactness to investigate the existence of mild solutions for these equations in Banach spaces. Finally, two examples to illustrate the applications of main results are also given.**

***Index Terms*—impulsive fractional partial neutral functional differential equations, Hausdorff measure of noncompactness, infinite delay, Riemann-Liouville fractional integral operator, fixed point theorem.**

I. Introduction

FRACTIONAL differential equations occur more frequently in different research areas and engineering, such as physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, control of dynamical systems etc. (see [1], [2], [3]). Recently, many researchers paid attention to existence results of solution of fractional differential equations involves Riemann-Liouville as well as Caputo derivatives, such as [4], [5], [6], [7] and the references therein. The existence of mild solutions for fractional semilinear differential or integro-differential equations in Banach spaces is one of the theoretical fields that investigated by many authors [8], [9], [10], [11], [12], [13], [14], [15], [16].

The study of impulsive functional differential systems is linked to their utility in simulating processes and phenomena subject to short-time perturbations during their evolution. The perturbations are performed discretely and their duration is negligible in comparison with the total duration of the processes and phenomena. For the basic theory of impulsive differential equations the reader can refer to [17]. Now, the existence, uniqueness and other quantitative and qualitative properties of solutions to various fractional impulsive semilinear differential and integro-differential systems have been extensively studied in Banach spaces. For example, Shu et al. [18] investigated the existence and uniqueness of mild solutions for a class of impulsive fractional partial semilinear differential equations. Chauhan and Dabas [19] obtained the local and global existence of mild solution for an impulsive fractional functional integro-differential equations with nonlocal condition. The results concerned with the nonlinear function F is a Lipschitz continuous or completely continuous function, and A is a sectorial operator such that the operators families $\{S_\alpha(t) : t \geq 0\}$ and $\{T_\alpha(t) : t \geq 0\}$ are compact. Balachandran et al. [20] established some sufficient conditions for the existence of solutions of fractional differential equation of Sobolev type with impulse effect in Banach spaces, where the nonlinear function F is a Lipschitz continuous or continuous and the operator B^{-1} is compact. For the nonlinear function F only satisfies the Lipschitz condition, Debbouche and Baleanu [21] proved the controllability result of a class of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems. The existence and uniqueness of solutions of impulsive neutral fractional integro-differential equation with infinite delay is also presented in [22]. When the linear operator A generates a compact semigroup $\{T(t) : t \geq 0\}$, Balasubramaniam et al. [23] derived the approximate controllability of impulsive fractional integro-differential systems with nonlocal conditions in Hilbert spaces. Li and Liu [24] studied the solvability and optimal controls of impulsive fractional semilinear differential equations. Suganya et al. [25] investigated the existence of mild solutions for an impulsive fractional integro-differential equation with state-dependent delay. Moreover, we know that these abstract fractional impulsive differential or integro-differential systems considers basically problems for which the impulses are abrupt and instantaneous.

However, many impulsive systems arising from realistic models can be described as partial differential equations with not instantaneous impulses. Hernández and O'Regan [26] introduced a new class of first order abstract impulsive differential equations for which the impulses are not instantaneous. In the model, the impulses start abruptly at the points t_i and their action continue on a finite time interval $[t_i, s_i]$. This situation as an impulsive action which starts abruptly and stays active on a finite time interval. Pierri et al. [27] studied the existence of solutions for a class of first order semilinear abstract impulsive differential equations with not instantaneous impulses by using the theory of compact analytic semigroup $\{T(t) : t \geq 0\}$ and fractional power of closed operators. Pandey et al. [28] investigated existence of mild solution of second order partial neutral differential equation with state dependent delay and non-instantaneous impulses. The compactness condition of the operator families generated by A and other restrictive conditions have been omitted. For the nonlinear function F is a Lipschitz function, Gautam and Dabas [29], Kumar et al. [30] established the existence and

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uniqueness of mild solution for abstract fractional functional differential and integro-differential equations with not instantaneous impulse. Yu and Wang [31] discussed the periodic boundary value problems for nonlinear impulsive evolution equations with not instantaneous impulse on Banach spaces. In this paper we consider the existence of mild solutions for the following impulsive Riemann-Liouville fractional partial neutral functional differential equations with infinite delay and not instantaneous impulse

$$\begin{aligned} \frac{d}{dt}J_t^{1-\alpha}[x(t)-f(t,x_t)-\varphi(0)+f(0,\varphi)] \\ = Ax(t)+F(t,x_t), \\ t\in(s_i,t_{i+1}], i=0,1,\ldots,N, \end{aligned} \tag{1}$$

$$x(0)=\varphi(t),\ t\in(-\infty,0], \tag{2}$$

$$x(t)=g_i(t,x_t),\quad t\in(t_i,s_i], i=1,\ldots,N, \tag{3}$$

where the unknown $x(\cdot)$ takes values in a Banach space X with norm $\|\cdot\|$, and $0<\alpha<1$, $J_t^{1-\alpha}$ is the $(1-\alpha)$-order Riemann-Liouville fractional integral operator, A is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$ on X. The time history $x_t:(-\infty,0]\to X$ given by $x_t(\theta)=x(t+\theta)$ belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; and let $0=t_0=s_0<t_1\leq s_1\leq t_2<...<t_{N-1}\leq s_N\leq t_N\leq t_{N+1}=b$, are prefixed numbers, $g_i\in C((t_i,s_i]\times\mathcal{B};X)$ for all $i=1,...,N$, and $f,F:[0,b]\times\mathcal{B}\to X$ are suitable functions.

Among the previous works, the most authors supposed that the impulsive systems with the compactness assumption on associated operators and the nonlinear function F is a Lipschitz function or completely continuous, However, the above conditions are stronger restrictions, which are not satisfied usually in many practical problems [32], [33]. We can use measure of noncompactness to remove the assumptions for compactness of the operator and Lipschitz continuity of the nonlinear item F. This is one of our motivations. Although the papers [26], [27], [28], [29], [30], [31] obtained sufficient conditions for the existence of mild solution of the systems with not instantaneous impulses, besides the fact that these papers applies to the results by using the completely continuous assumptions and Lipschitz conditions, the class of systems is also different from the one studied here. To the best of our knowledge, there are no relevant reports on the impulsive fractional differential equations with not instantaneous impulses via the techniques of the measure of noncompactness. This is another of our motivations.

In this article, we shall study the existence of mild solutions of (1)-(3) by using the Hausdorff measure of noncompactness and fixed point theorems with the theory of Riemann-Liouville fractional integral operator and fractional power of closed operators. We do not assume $\{T(t),t\geq 0\}$ is a compact semigroup, and instead we assume that F satisfies a compactness condition involving the Hausdorff measure of noncompactness. Thus the compactness of $T(t)$ or F and the Lipschitz condition of F are the special cases of our conditions. The known results are generalized to the fractional functional neutral impulsive settings and the case of infinite delay without the assumptions of compactness. Therefore, the obtained results can be seen as a contribution to this emerging field.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and necessary preliminaries. In Section 3, we give our main results. In Section 4, two examples are given to illustrate our results.

II. Preliminaries

Let $(X,\|\cdot\|)$ be a Banach space. $C(J,X), J=[0,b]$, is the Banach space of all continuous functions from J into X with the norm $\|x\|_\infty=\sup\{\|x(t)\|: t\in J\}$ and $L(X)$ denotes the Banach space of bounded linear operators from X to X. A measurable function $x:J\to X$ is Bochner integrable if and only if $\|x\|$ is Lebesgue integrable. For properties of the Bochner integral see Yosida [34]. $L^1(J,X)$ denotes the Banach space of measurable functions $x:J\to X$ which are Bochner integrable normed by $\|x\|_{L^1}=\int_0^b\|x(t)\|\,dt$ for all $x\in L^1(J,X)$. The notation, $B_r(x,X)$ stands for the closed ball with center at x and radius $r>0$ in X.

Definition 1. Let A be the infinitesimal generator of an analytic semigroup $T(t)$. For every $\beta>0$, we define $(-A)^\beta=((-A)^{-\beta})^{-1}$. For $\beta=0$, $(-A)^\beta=I$.

We note that $D((-A)^\beta)$ is a Banach space equipped with the norm $\|x\|_\beta=\|(-A)^\beta x\|, x\in D((-A)^\beta)$. By X_β, we denote this Banach space. We collect some basic properties of fractional powers $(-A)^\beta$ appearing in Pazy [35].

Lemma 1 ([35]). Let A be the infinitesimal generator of an analytic semigroup $T(t)$. If $0\in\rho(A)$, then

(i) If $\beta\geq\alpha>0$ then $X_\beta\subset X_\alpha$.

(ii) If $\beta,\alpha>0$ then
$$(-A)^{\beta+\alpha}x=(-A)^\beta(-A)^\alpha x$$
for every $x\in D((-A)^\eta)$ where $\eta=\max(\beta,\alpha,\beta+\alpha)$.

(iii) For every $x\in D((-A)^\beta)$, then $T(t)(-A)^\beta x=(-A)^\beta T(t)x$.

(iv) For every $\beta>0$, there exists a positive constant M_β such that
$$\|(-A)^\beta T(t)\|\leq\frac{M_\beta}{t^\beta}.$$

We say that a function $x:[\mu,\tau]\to X$ is a normalized piecewise continuous function on $[\mu,\tau]$ if x is piecewise continuous and continuous on $[\mu,\tau]$. We denote by $\mathcal{PC}([\mu,\tau],X)$ the space formed by the normalized piecewise continuous from $[\mu,\tau]$ into X. In particular, we introduce the space $\mathcal{PC}$ formed by all the functions $x:[0,b]\to X$ such that $x(\cdot)$ is continuous at $t\neq t_i$, $x(t_i)=x(t_i^-)$ and $x(t_i^+)$ exists for all $i=1,...,N$. In this paper, we always assume that $\mathcal{PC}$ is endowed with the norm $\|x\|_{\mathcal{PC}}=\sup_{t\in[0,b]}\|x(t)\|$. Then $(\mathcal{PC},\|\cdot\|_{\mathcal{PC}})$ is a Banach space.

We assume that the phase space $(\mathcal{B},\|\cdot\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty,0]$ into X, and satisfying the following fundamental axioms due to Hale and Kato (see e.g., in [36]).

(A) If $x:(-\infty,\sigma+b]\to X$, $b>0$, is such that $x|_{[\sigma,\sigma+b]}\in C([\sigma,\sigma+b],X)$ and $x_\sigma\in\mathcal{B}$, then for every $t\in[\sigma,\sigma+b]$ the following conditions hold:
  (i) x_t is in $\mathcal{B}$;
  (ii) $\|x(t)\|\leq\tilde{H}\|x_t\|_{\mathcal{B}}$;

(iii) $\| x_t \|_{\mathcal{B}} \leq K(t-\sigma)\sup\{\| x(s) \|: \sigma \leq s \leq t\} + M(t-\sigma) \| x_\sigma \|_{\mathcal{B}}$, where $\tilde{H} \geq 0$ is a constant; $K, M : [0,\infty) \to [1,\infty)$, K is continuous and M is locally bounded, and $\tilde{H}, K, M$ are independent of $x(\cdot)$.

(B) For the function $x(\cdot)$ in (A), the function $t \to x_t$ is continuous from $[\sigma, \sigma+b]$ into $\mathcal{B}$.

(C) The space $\mathcal{B}$ is complete.

X be a Banach space, we recall some basic definitions in fractional calculus (see [3]).

Definition 2. The Riemann-Liouville fractional integral of the order $\alpha > 0$ of $g : J \to X$ is defined by

$$J_t^\alpha g(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)ds.$$

Definition 3. The Riemann-Liouville fractional derivative of the order $\alpha \in (0,1)$ of $g : J \to X$ is defined by

$$D_t^\alpha g(t) = \frac{d}{dt} J_t^{1-\alpha} g(t).$$

Definition 4. The Caputo fractional derivative of the order $\alpha \in (0,1)$ of $g : J \to X$ is defined by

$${}^C D_t^\alpha g(t) = D_t^\alpha (g(t) - g(0)).$$

Definition 5 ([37]). The Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}),\ \alpha,\beta > 0,\ z \in \tilde{C}, \qquad (4)$$

where $\tilde{C}$ denotes the complex plane. When $\beta = 1$, set $E_\alpha(z) = E_{\alpha,1}(z)$.

Definition 6 ([3]). The Mainardi's function is defined by

$$M_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n - \alpha + 1)},\ 0 < \alpha < 1,\ z \in \tilde{C}. \quad (5)$$

The Laplace transform of the Mainardi's function $M_\alpha(\xi)$ is (see [38]):

$$\int_0^\infty e^{-\xi\lambda} M_\alpha(\xi) d\xi = E_\alpha(-\lambda). \qquad (6)$$

By (4) and (6), it is clear that

$$\int_0^\infty M_\alpha(\xi) d\xi = 1,\ 0 < \alpha < 1. \qquad (7)$$

On the other hand, $M_\alpha(z)$ satisfies the following equality (see[38])

$$\int_0^\infty \frac{\alpha}{\xi^{\alpha+1}} M_\alpha(1/\xi^\alpha) e^{-\lambda\xi} d\xi = e^{-\lambda^\alpha}. \qquad (8)$$

and the equality (see [38])

$$\int_0^\infty \xi^\theta M_\alpha(\xi) d\xi = \frac{\Gamma(\theta+1)}{\Gamma(\alpha\theta+1)},\ \theta > -1,\ 0 < \alpha < 1. \quad (9)$$

According to Definitions 2, 3 and 4, the problem (1)-(3) can be written as

$$\begin{aligned} {}^C D_t^\alpha [x(t) - f(t,x_t)] &= Ax(t) + F(t,x_t), \qquad (10)\\ & t \in (s_i, t_{i+1}], i = 0, 1, \ldots, N, \end{aligned}$$

$$x(0) = \varphi(t),\ t \in (-\infty, 0], \qquad (11)$$

$$x(t) = g_i(t, x_t), \quad t \in (t_i, s_i], i = 1, \ldots, N, \qquad (12)$$

and it is suitable to rewrite the problem (10)-(12) in the equivalent integral equation

$$\begin{aligned} x(t) &= \varphi(0) - f(0,\varphi) + f(t,x_t) \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[Ax(s) + F(s,x_s)]ds \quad (13)\end{aligned}$$

for all $t \in [0, t_1]$ and

$$\begin{aligned} x(t) &= g_i(s_i, x(s_i)) - f(s_i, x(s_i)) + f(t,x_t) \\ &+ \frac{1}{\Gamma(\alpha)}\int_{s_i}^t (t-s)^{\alpha-1}[Ax(s) + F(s,x_s)]ds \quad (14)\end{aligned}$$

for all $t \in (s_i, t_{i+1}]$, $i = 1, ..., N$, and

$$x(t) = \varphi(t), \quad t \in (-\infty, 0], \qquad (15)$$

provided that the integral in (13)-(15) exists.

Lemma 2. If (13)-(15) holds, then we have

$$\begin{aligned} x(t) &= \int_0^\infty M_\alpha(\xi) T(t^\alpha \xi)[\varphi(0) - f(0,\varphi)]d\xi + f(t,x_t) \\ &+ \alpha\int_0^t\int_0^\infty \xi(t-s)^{\alpha-1} M_\alpha(\xi) AT((t-s)^\alpha\xi) \\ &\times f(s,x_s)d\xi ds \\ &+ \alpha\int_0^t\int_0^\infty \xi(t-s)^{\alpha-1} M_\alpha(\xi) T((t-s)^\alpha\xi) \\ &\times F(s,x_s)d\xi ds \end{aligned}$$

for all $t \in [0, t_1]$ and

$$\begin{aligned} x(t) &= \int_0^\infty M_\alpha(\xi) T((t-s_i)^\alpha \xi)[g_i(s_i, x(s_i)) \\ &- f(s_i, x(s_i))]d\xi + f(t,x_t) \\ &+ \alpha\int_{s_i}^t\int_0^\infty \xi(t-s)^{\alpha-1} M_\alpha(\xi) AT((t-s)^\alpha\xi) \\ &\times f(s,x_s)d\xi ds \\ &+ \alpha\int_{s_i}^t\int_0^\infty \xi(t-s)^{\alpha-1} M_\alpha(\xi) T((t-s)^\alpha\xi) \\ &\times F(s,x_s)d\xi ds \end{aligned}$$

for all $t \in (s_i, t_{i+1}]$, $i = 1, ..., N$, and

$$x(t) = \varphi(t), \quad t \in (-\infty, 0],$$

where M_α is the Mainardi's function defined on $(0, \infty)$.

The proof of Lemma 2 is similar to the proof of [9] for Lemma 3.1, and we omit the details here. We notice that ψ_α is the one-sided stable probability density [9], whose Laplace transform is given by $\int_0^\infty e^{-\lambda\xi}\psi_\alpha(\xi)d\xi = e^{-\lambda^\alpha}$, $\alpha \in (0,1)$. Combined with (8), we deduce that $M_\alpha(\xi) = \frac{1}{\alpha}\xi^{-1-1/\alpha}\psi_\alpha(\xi^{-1/\alpha})$.

Based on Lemma 2, we give the following definition of the mild solution of (1)-(3).

Definition 7. A function $x : (-\infty, b] \to X$ is called a mild solution of the problem (1)-(3) if $x_0 = \varphi \in \mathcal{B}$, $x(\cdot)|_{[0,b]} \in \mathcal{PC}$ and $x(t) = g_j(t, x(t))$ for all $t \in (t_j, s_j], j = 1, ..., N$, and for each $s \in [0, t)$ the function $(t-s)^{\alpha-1} AS_\alpha(t-s) f(s, x_s)$ is integrable and

$$\begin{aligned} x(t) &= T_\alpha(t)[\varphi(0) - f(0,\varphi)] + f(t,x_t) \\ &+ \int_0^t (t-s)^{\alpha-1} AS_\alpha(t-s) f(s,x_s)ds \\ &+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) F(s,x_s)ds \end{aligned}$$

for all $t \in [0, t_1]$ and

$$x(t) = T_\alpha(t - s_i)[g_i(s_i, x(s_i)) - f(s_i, x(s_i))] + f(t, x_t) + \int_{s_i}^{t} (t - s)^{\alpha-1} AS_\alpha(t - s)f(s, x_s)ds + \int_{s_i}^{t} (t - s)^{\alpha-1} S_\alpha(t - s)F(s, x_s)ds$$

for all $t \in (s_i, t_{i+1}]$, $i = 1, ..., N$, where

$$T_\alpha(t)x = \int_0^\infty M_\alpha(\xi)T(t^\alpha \xi)xd\xi,\ t \geq 0,\ x \in X, \quad (16)$$

and

$$S_\alpha(t)x = \int_0^\infty \alpha\xi M_\alpha(\xi)T(t^\alpha \xi)xd\xi,\ t \geq 0,\ x \in X. \quad (17)$$

It is easy to see that $T_\alpha(t)$ and $S_\alpha(t)$ are strongly continuous on R^+.

Now, we introduce the Hausdorff measure of noncompactness χ_Y defined by

$$\chi_Y(B) = \inf\{\varepsilon > 0; B \text{ has a finite } \varepsilon - \text{net in} Y\},$$

for bounded set B in any Banach space Y. Some basic properties of $\chi_Y(\cdot)$ are given in the following lemma.

Lemma 3 ([39]). Let Y be a real Banach space and $B, C \subseteq Y$ be bounded, the following properties are satisfied:

(1) B is pre-compact if and only if $\chi_Y(B) = 0$;
(2) $\chi_Y(B) = \chi_Y(\overline{B}) = \chi_Y(\text{conv } B)$, where $\overline{B}$ and conv$B$ are the closure and the convex hull of B respectively;
(3) $\chi_Y(B) \leq \chi_Y(C)$ when $B \subseteq C$;
(4) $\chi_Y(B+C) \leq \chi_Y(B)+\chi_Y(C)$ where $B+C = \{x+y : x \in B, y \in C\}$;
(5) $\chi_Y(B \cup C) = \max\{\chi_Y(B), \chi_Y(C)\}$;
(6) $\chi_Y(\lambda B) \leq |\lambda|\chi_Y(B)$ for any $\lambda \in R$;
(7) If the map $\Phi : D(\Phi) \subseteq Y \to Z$ is Lipschitz continuous with constant κ then $\chi_Z(\Phi B) \leq \kappa\chi_Y(B)$ for any bounded subset $B \subseteq D(\Phi)$, where Z is a Banach space;
(8) If $\{W_n\}_{n=1}^\infty$ is a decreasing sequence of bounded closed nonempty subset of Y and $\lim_{n\to\infty} \chi_Y\ (W_n) = 0$, then $\cap_{n=1}^\infty W_n$ is nonempty and compact in Y.

Definition 8 ([40]). The map $\Phi : W \subseteq Y \to Y$ is said to be a χ_Y-contraction if there exists a positive constant $\kappa < 1$ such that $\chi_Y(\Phi(C)) \leq \kappa\chi_Y(C)$ for any bounded close subset $C \subseteq W$ where Y is a Banach space.

In this paper we denote by χ_C the Hausdorff's measure of noncompactness of $C([0, b], X)$ and by $\chi_{\mathcal{PC}}$ the Hausdorff's measure of noncompactness of $\mathcal{PC}$. To discuss the existence we need the following results.

Lemma 4 ([40]). If $W \subset C([a, b], X)$ is bounded, then $\chi(W(t)) \leq \chi_C(W)$, for any $t \in [a, b]$, where $W(t) = \{u(t) : u \in W \subseteq X\}$. Furthermore, if W is equicontinuous on $[a, b]$, then $W(t)$ is continuous for $t \in [a, b]$, and $\chi_C(W) = \sup\{W(t) : t \in [a, b]\}$.

Lemma 5 ([39]). If $W \subset C([a, b], X)$ is bounded and equicontinuous, then $\chi(W(t))$ is continuous for $t \in [a, b]$, and $\chi(\int_a^t W(s)ds) \leq \int_a^t \chi(W(s))ds$ for all $t \in [a, b]$, where $\int_a^t W(s)ds = \{\int_a^t x(s)ds : x \in W\}$.

Obviously, the following lemma holds.

Lemma 6. If $W \subset \mathcal{PC}([0, b], X)$ is bounded, then $\chi(W(t)) \leq \chi_{\mathcal{PC}}(W)$, for any $t \in [0, b]$, where $W(t) = \{u(t) : u \in W \subseteq X\}$. Furthermore, if W is piecewise equicontinuous on $[0, b]$, then $W(t)$ is continuous for $t \in [0, b]$, and $\chi_{\mathcal{PC}}(W) = \sup\{W(t) : t \in [0, b]\}$.

Proof. Let $W \subset \mathcal{PC}$, and $W = (\cup_{i=0}^N W_i) \cup (\cup_{i=1}^N U_i)$, where $W_i \subset C(\overline{J_i}, X)$, $\overline{J_i} = [s_i, t_{i+1}]$, $i = 0, 1, \ldots, N$, and $U_i \subset C(\overline{I_i}, X)$, $\overline{I_i} = [t_i, s_i], i = 1, \ldots, N$. Using Lemmas 3 and 4, we have

$$\begin{aligned}
\chi(W(t)) &= \chi((\cup_{i=0}^N W_i(t)) \cup (\cup_{i=1}^N U_i(t))) \\
&= \max\{\max\{\chi(W_i(t)) : i = 0, 1, \ldots, N\}, \\
&\quad \max\{\chi(U_i(t)) : i = 1, \ldots, N\}\} \\
&\leq \max\{\max\{\chi_C(W_i) : i = 0, 1, \ldots, N\}, \\
&\quad \max\{\chi_C(U_i) : i = 1, \ldots, N\}\} \\
&= \max\{\max\{\chi_{\mathcal{PC}}(W_i) : i = 0, 1, \ldots, N\}, \\
&\quad \max\{\chi_{\mathcal{PC}}(U_i) : i = 1, \ldots, N\}\} \\
&= \max\{\chi_{\mathcal{PC}}(\cup_{i=0}^N W_i), \chi_{\mathcal{PC}}(\cup_{i=1}^N U_i)\} \\
&= \chi_{\mathcal{PC}}((\cup_{i=0}^N W_i) \cup (\cup_{i=1}^N U_i)) = \chi_{\mathcal{PC}}(W).
\end{aligned}$$

From Lemma 4, if W is piecewise equicontinuous on $[0, b]$, then $W(t)$ is continuous for $t \in [0, b]$. In addition, we have

$$\begin{aligned}
\chi_{\mathcal{PC}}(W) &= \max\{\max\{\chi_{\mathcal{PC}}(W_i) : i = 0, 1, \ldots, N\}, \\
&\quad \max\{\chi_{\mathcal{PC}}(U_i) : i = 1, \ldots, N\}\} \\
&= \max\{\max\{\sup\{\chi(W_i(t)), \\
&\quad t \in \overline{J_i} : i = 0, 1, \ldots, N\}\}, \\
&\quad \max\{\sup\{\chi(U_i(t)), t \in \overline{I_i} : i = 1, \ldots, N\}\}\} \\
&= \max\{\sup\{\chi(W(t)), t \in [0, b]\}\} \\
&= \sup\{\chi(W(t)), t \in [0, b]\}.
\end{aligned}$$

Lemma 7 ([41] Darbo). If $W \subseteq Y$ is closed and convex and $0 \in W$, the continuous map $\Phi : W \to W$ is a χ_Y-contraction, if the set $\{x \in W : x = \lambda\Phi x\}$ is bounded for $0 < \lambda < 1$, then the map Φ has at least one fixed point in W.

Lemma 8 ([39] Darbo-Sadovskii). If $W \subseteq Y$ is bounded closed and convex, the continuous map $\Phi : W \to W$ is a χ_Y-contraction, then the map Φ has at least one fixed point in W.

III. Main results

In this section we shall present and prove our main result. Let us list the following hypotheses:

(H1) The analytic semigroup $T(t)$ generated by A and there exists a constant $M \geq 1$ such that $\parallel T(t) \parallel \leq M$.

(H2) For $\beta \in (0, 1)$, the function f is X_β -valued, and there exist positive constants $L_1, L_2 > 0$ such that

$$\parallel (-A)^\beta f(t, \psi_1) - (-A)^\beta f(t, \psi_2) \parallel \leq L_1 \parallel \psi_1 - \psi_2 \parallel_{\mathcal{B}},$$

$$t \in [0, b], \psi_1, \psi_2 \in \mathcal{B},$$

$$\parallel (-A)^\beta f(t, \psi) \parallel \leq L_1 \parallel \psi \parallel_{\mathcal{B}} + L_2,\ t \in [0, b], \psi \in \mathcal{B}.$$

(H3) The function $F : [0, b] \times \mathcal{B} \to X$ satisfies the following conditions:

(i) For each $t \in [0, b]$, the function $F(t, \cdot) : \mathcal{B} \to X$ is continuous and for each $x \in \mathcal{B}$, the function $F(\cdot, \psi) : [0, b] \to X$ is strongly measurable.

(ii) There exist a continuous function $m : [0,b] \to [0,\infty)$ and a continuous nondecreasing function $\Theta : [0,\infty) \to (0,\infty)$ such that

$$\| F(t,\psi) \| \leq m(t)\Theta(\| \psi \|_{\mathcal{B}}), \quad t \in [0,b], \psi \in \mathcal{B}.$$

(iii) There exists a continuous function $\zeta : [0,b] \to [0,\infty)$ such that for each bounded $D \subset \mathcal{B}$,

$$\chi(F(t,D)) \leq \zeta(t) \sup_{-\infty<\theta\leq 0} \chi(D(\theta))$$

for a.e. $t \in [0,b]$, where $D(\theta) = \{u(\theta) : u \in D\}$.

(H4) The functions $g_i : (t_i, s_i] \times \mathcal{B} \to X, i = 1,...,N$, are continuous, and

(i) There exist constants $c_i, d_i > 0, i = 1,...,N$, such that

$$\| g_i(t,\psi) \| \leq c_i \| \psi \|_{\mathcal{B}} + d_i, \quad t \in (t_i, s_i], \psi \in \mathcal{B}.$$

(ii) There exist constants $\gamma_i > 0$ such that for each bounded $D \subset \mathcal{B}$,

$$\chi(g_i(t,D)) \leq \gamma_i \sup_{-\infty<\theta\leq 0} \chi(D(\theta))$$

for a.e. $t \in (t_i, s_i], i = 1,...,N$, where $D(\theta) = \{u(\theta) : u \in D\}$.

(H5) There exist constants $\nu_i > 0, i = 1,...,N$, such that

$$\| g_i(t_1,\psi_1) - g_i(t_2,\psi_2) \| \leq \nu_i[|t_1-t_2| + \| \psi_1 - \psi_2 \|_{\mathcal{B}}],$$
$$t_1, t_2 \in (t_i, s_i], \psi_1, \psi_2 \in \mathcal{B}.$$

(H6) $\max_{1\leq i\leq N}\{(M+1)\gamma_i + \frac{Mb^\alpha}{\alpha\Gamma(\alpha)}\int_0^b \zeta(s)ds\} < 1$.

Theorem 1. *If the assumptions* (H1)-(H6) *are satisfied, then the problem* (1)-(3) *has at least one mild solution on* $[0,b]$, *provided that* $\int_1^\infty \frac{1}{\Theta(s)}ds = \infty$, *and*

$$\max_{1\leq i\leq N}\left\{K_b\left[(1+M)c_i + \left((M+1) \| (-A)^{-\beta} \| + \frac{\alpha M_{1-\beta}\Gamma(\beta+1)b^{\alpha\beta}}{\alpha\beta\Gamma(\alpha\beta+1)}\right)L_1\right]\right\} < 1. \quad (18)$$

Proof. We introduce the space $\mathcal{B}_b$ of all functions $x : (-\infty,b] \to X$ such that $x_0 \in \mathcal{B}$ and the the restriction $x|_{[0,b]} \in \mathcal{PC}$. Let$\| \cdot \|_b$ be a seminorm in $\mathcal{B}_b$ defined by

$$\| x \|_b = \| x_0 \|_{\mathcal{B}} + \sup\{\| x(s) \| : 0 \leq s \leq b\}, \quad x \in \mathcal{B}_b.$$

We consider the operator $\Phi : \mathcal{B}_b \to \mathcal{B}_b$ defined by

$$(\Phi x)(t) = \begin{cases} \varphi(t), t \in (-\infty,0], \\ T_\alpha(t)[\varphi(0) - f(0,\varphi)] + f(t,x_t) \\ \quad + \int_0^t (t-s)^{\alpha-1} A S_\alpha(t-s) f(s,x_s)ds \\ \quad + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \\ \quad \times F(s,x_s)ds, \quad t \in [0,t_1], i = 0, \\ g_i(t,x_t), \quad t \in (t_i,s_i], i \geq 1, \\ T_\alpha(t-s_i)[g_i(s_i,x_{s_i}) - f(s_i,x_{s_i})] + f(t,x_t) \\ \quad + \int_{s_i}^t (t-s)^{\alpha-1} A S_\alpha(t-s) f(s,x_s)ds \\ \quad + \int_{s_i}^t (t-s)^{\alpha-1} S_\alpha(t-s) \\ \quad \times F(s,x_s)ds, \quad t \in (s_i,t_{i+1}], i \geq 1. \end{cases}$$

For $\varphi \in \mathcal{B}$, we define $\tilde{\varphi}$ by

$$\tilde{\varphi}(t) = \begin{cases} \varphi(t), & -\infty < t \leq 0, \\ T_\alpha(t)\varphi(0), & 0 \leq t \leq b, \end{cases}$$

then $\tilde{\varphi} \in \mathcal{B}_b$. Set $x(t) = y(t) + \tilde{\varphi}(t), -\infty < t \leq b$. It is clear to see that x satisfies Definition 7 if and only if y satisfies $y_0 = 0$ and

$$y(t) = \begin{cases} -T_\alpha(t)f(0,\varphi) + f(t,y_t+\tilde{\varphi}_t) \\ \quad + \int_0^t (t-s)^{\alpha-1} A S_\alpha(t-s) f(s,y_s+\tilde{\varphi}_s)ds \\ \quad + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \\ \quad \times F(s,y_s+\tilde{\varphi}_s)ds, \quad t \in [0,t_1], i = 0, \\ g_i(t,y_t+\tilde{\varphi}_t), \quad t \in (t_i,s_i], i \geq 1, \\ T_\alpha(t-s_i)[g_i(s_i,y_{s_i}+\tilde{\varphi}_{s_i}) - f(s_i,y_{s_i}+\tilde{\varphi}_{s_i})] \\ \quad + f(t,y_t+\tilde{\varphi}_t) \\ \quad + \int_{s_i}^t (t-s)^{\alpha-1} A S_\alpha(t-s) f(s,y_s+\tilde{\varphi}_s)ds \\ \quad + \int_{s_i}^t (t-s)^{\alpha-1} S_\alpha(t-s) \\ \quad \times F(s,y_s+\tilde{\varphi}_s)ds, \quad t \in (s_i,t_{i+1}], i \geq 1. \end{cases}$$

Let $\mathcal{B}_b^0 = \{y \in \mathcal{B}_b : y_0 = 0 \in \mathcal{B}\}$. For any $y \in \mathcal{B}_b^0$,

$$\begin{aligned} \| y \|_b &= \| y_0 \|_{\mathcal{B}} + \sup\{\| y(s) \| : 0 \leq s \leq b\} \\ &= \sup\{\| y(s) \| : 0 \leq s \leq b\}, \end{aligned}$$

thus $(\mathcal{B}_b^0, \| \cdot \|_b)$ is a Banach space. By (7), (9), (16), (17) and (H1), we get that

$$\| T_\alpha(t) \| \leq M, \quad \| S_\alpha(t) \| \leq \frac{M}{\Gamma(\alpha)}.$$

For $y \in B_r(0,\mathcal{B}_b^0)$, we have

$$\begin{aligned} \| y_t + \tilde{\varphi}_t \|_{\mathcal{B}} &\leq \| y_t \|_{\mathcal{B}} + \| \tilde{\varphi}_t \|_{\mathcal{B}} \\ &\leq K_b r + (K_b M\tilde{H} + M_b) \| \varphi \|_{\mathcal{B}} \\ &= r^*, \ t \in [0,b]. \end{aligned}$$

Define the map $\bar{\Phi} : \mathcal{B}_b^0 \to \mathcal{B}_b^0$ defined by $(\bar{\Phi}y)(t) = 0, t \in [-\infty,0]$ and

$$(\bar{\Phi}y)(t) = \begin{cases} -T_\alpha(t)f(0,\varphi) + f(t,y_t+\tilde{\varphi}_t) \\ \quad + \int_0^t (t-s)^{\alpha-1} A S_\alpha(t-s) f(s,y_s+\tilde{\varphi}_s)ds \\ \quad + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \\ \quad \times F(s,y_s+\tilde{\varphi}_s)ds, \quad t \in [0,t_1], i = 0, \\ g_i(t,y_t+\tilde{\varphi}_t), \quad t \in (t_i,s_i], i \geq 1, \\ T_\alpha(t-s_i)[g_i(s_i,y_{s_i}+\tilde{\varphi}_{s_i}) \\ \quad - f(s_i,y_{s_i}+\tilde{\varphi}_{s_i})] + f(t,y_t+\tilde{\varphi}_t) \\ \quad + \int_{s_i}^t (t-s)^{\alpha-1} A S_\alpha(t-s) f(s,y_s+\tilde{\varphi}_s)ds \\ \quad + \int_{s_i}^t (t-s)^{\alpha-1} S_\alpha(t-s) \\ \quad \times F(s,y_s+\tilde{\varphi}_s)ds, \quad t \in (s_i,t_{i+1}], i \geq 1. \end{cases}$$

Obviously, the operator Φ has a fixed point if and only if operator $\bar{\Phi}$ has a fixed point, to prove which we shall employ Lemma 7. For better readability, we break the proof into a sequence of steps.

Step 1. For $0 < \lambda < 1$, set $\{y \in \mathcal{PC} : y = \bar{\Phi}y\}$ is bounded.

Let y_λ be a solution of $y = \bar{\Phi}y$ for $0 < \lambda < 1$. We have

$$\| y_{\lambda t} + \tilde{\varphi}_t \|_{\mathcal{B}} \leq K_b \| y_\lambda \|_t + (K_b M\tilde{H} + M_b) \| \varphi \|_{\mathcal{B}}, \quad (19)$$

where $\| y_\lambda \|_t = \sup_{0\leq s\leq t} \| y_\lambda(s) \|$.

Let $v_\lambda(t) = K_b \| y_\lambda \|_t + (K_b M\tilde{H} + M_b) \| \varphi \|_{\mathcal{B}}$. From (9), (17), and Lemma 1, it is easy to show that for $\beta \in (0,1)$,

$$\| (-A)^{1-\beta} S_\alpha(t) \| \leq \frac{\alpha M_{1-\beta}\Gamma(\beta+1)t^{\alpha(\beta-1)}}{\Gamma(\alpha\beta+1)}.$$

Then, by (H1)-(H4), and Young inequality ([42], Page 6), from the above equation, we have for $t \in [0, t_1]$

$$\begin{aligned}
&\| y_\lambda(t) \| \\
&\quad \leq \| T_\alpha(t)(-A)^{-\beta}(-A)^{\beta} f(0,\varphi) \| \\
&\quad\quad + \| (-A)^{-\beta}(-A)^{\beta} f(t, y_t + \tilde{\varphi}_t) \| \\
&\quad\quad + \left\| \int_0^t (t-s)^{\alpha-1}(-A)^{1-\beta} S_\alpha(t-s) \right. \\
&\quad\quad \left. \times (-A)^{\beta} f(s, y_s + \tilde{\varphi}_s) ds \right\| \\
&\quad\quad + \left\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) F(s, y_s + \tilde{\varphi}_s) ds \right\| \\
&\quad \leq M \| (-A)^{-\beta} \| (L_1 \| \varphi \|_{\mathcal{B}} + L_2) \\
&\quad\quad + \| (-A)^{-\beta} \| (L_1 \| y_{\lambda t} + \tilde{\varphi}_t \|_{\mathcal{B}} + L_2) \\
&\quad\quad + \frac{\alpha M_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \int_0^t (t-s)^{\alpha\beta-1} \\
&\quad\quad \times (L_1 \| y_{\lambda s} + \tilde{\varphi}_s \|_{\mathcal{B}} + L_2) ds \\
&\quad\quad + \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m(s) \Theta(\| y_{\lambda s} + \tilde{\varphi}_s \|_{\mathcal{B}}) ds \\
&\quad \leq M \| (-A)^{-\beta} \| (L_1 \| \varphi \|_{\mathcal{B}} + L_2) \\
&\quad\quad + \| (-A)^{-\beta} \| (L_1 \| y_{\lambda t} + \tilde{\varphi}_t \|_{\mathcal{B}} + L_2) \\
&\quad\quad + \frac{\alpha M_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \frac{t_1^{\alpha\beta}}{\alpha\beta} (L_1 \| y_{\lambda t} + \tilde{\varphi}_t \|_{\mathcal{B}} + L_2) \\
&\quad\quad + \frac{M}{\Gamma(\alpha)} \frac{t_1^{\alpha}}{\alpha} \int_0^t m(s) \Theta(\| y_{\lambda s} + \tilde{\varphi}_s \|_{\mathcal{B}}) ds \\
&\quad \leq M \| (-A)^{-\beta} \| (L_1 \| \varphi \|_{\mathcal{B}} + L_2) \\
&\quad\quad + \| (-A)^{-\beta} \| (L_1 v_\lambda(t) + L_2) \\
&\quad\quad + \frac{\alpha M_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \frac{t_1^{\alpha\beta}}{\alpha\beta} (L_1 v_\lambda(t) + L_2) \\
&\quad\quad + \frac{M}{\Gamma(\alpha)} \frac{t_1^{\alpha}}{\alpha} \int_0^t m(s) \Theta(v_\lambda(s)) ds.
\end{aligned}$$

For any $t \in (t_i, s_i], i = 1, \dots, N$, we have

$$\begin{aligned}
&\| y_\lambda(t) \| \\
&\quad \leq c_i \| y_{\lambda t} + \tilde{\varphi}_t \|_{\mathcal{B}} + d_i \leq c_i v_\lambda(t) + d_i.
\end{aligned}$$

Similarly, for any $t \in (s_i, t_{i+1}], i = 1, \dots, N$, we have

$$\begin{aligned}
&\| y_\lambda(t) \| \\
&\quad \leq \| T_\alpha(t-s_i)[g_i(s_i, y_{s_i} + \tilde{\varphi}_{s_i}) \\
&\quad\quad - (-A)^{-\beta}(-A)^{\beta} f(s_i, y_{s_i} + \tilde{\varphi}_{s_i})] \| \\
&\quad\quad + \| (-A)^{-\beta}(-A)^{\beta} f(t, y_t + \tilde{\varphi}_t) \| \\
&\quad\quad + \left\| \int_{s_i}^t (t-s)^{\alpha-1}(-A)^{1-\beta} S_\alpha(t-s) \right. \\
&\quad\quad \left. \times (-A)^{\beta} f(s, y_s + \tilde{\varphi}_s) ds \right\| \\
&\quad\quad + \left\| \int_{s_i}^t S_\alpha(t-s) F(s, y_s + \tilde{\varphi}_s) ds \right\| \\
&\quad \leq M[c_i \| y_{\lambda s_i} + \tilde{\varphi}_{s_i} \|_{\mathcal{B}} + d_i \\
&\quad\quad + \| (-A)^{-\beta} \| (L_1 \| y_{\lambda s_i} + \tilde{\varphi}_{s_i} \|_{\mathcal{B}} + L_2)] \\
&\quad\quad + \| (-A)^{-\beta} \| (L_1 \| y_{\lambda t} + \tilde{\varphi}_t \|_{\mathcal{B}} + L_2) \\
&\quad\quad + \frac{\alpha M_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \int_{s_i}^t (t-s)^{\alpha\beta-1} \\
&\quad\quad \times (L_1 \| y_{\lambda s} + \tilde{\varphi}_s \|_{\mathcal{B}} + L_2) ds \\
&\quad\quad + \frac{M}{\Gamma(\alpha)} \int_{s_i}^t (t-s)^{\alpha-1} m(s) \Theta(\| y_{\lambda s} + \tilde{\varphi}_s \|_{\mathcal{B}}) ds \\
&\quad \leq M[c_i \| y_{\lambda s_i} + \tilde{\varphi}_{s_i} \|_{\mathcal{B}} + d_i \\
&\quad\quad + \| (-A)^{-\beta} \| (L_1 \| y_{\lambda s_i} + \tilde{\varphi}_{s_i} \|_{\mathcal{B}} + L_2)] \\
&\quad\quad + \| (-A)^{-\beta} \| (L_1 \| y_{\lambda t} + \tilde{\varphi}_t \|_{\mathcal{B}} + L_2) \\
&\quad\quad + \frac{\alpha M_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \frac{(t_{i+1}-s_i)^{\alpha\beta}}{\alpha\beta} \\
&\quad\quad \times (L_1 \| y_{\lambda t} + \tilde{\varphi}_t \|_{\mathcal{B}} + L_2) \\
&\quad\quad + \frac{M}{\Gamma(\alpha)} \frac{(t_{i+1}-s_i)^{\alpha}}{\alpha} \int_{s_i}^t m(s) \Theta(\| y_{\lambda s} + \tilde{\varphi}_s \|_{\mathcal{B}}) ds \\
&\quad \leq M[c_i v_\lambda(s_i) + d_i + \| (-A)^{-\beta} \| (L_1 v_\lambda(s_i) + L_2)] \\
&\quad\quad + \| (-A)^{-\beta} \| (L_1 v_\lambda(t) + L_2) \\
&\quad\quad + \frac{\alpha M_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \frac{(t_{i+1}-s_i)^{\alpha\beta}}{\alpha\beta} (L_1 v_\lambda(t) + L_2) \\
&\quad\quad + \frac{M}{\Gamma(\alpha)} \frac{(t_{i+1}-s_i)^{\alpha}}{\alpha} \int_{s_i}^t m(s) \Theta(v_\lambda(s)) ds.
\end{aligned}$$

Then, for all $t \in [0, b]$, we have

$$\begin{aligned}
&\| y_\lambda(t) \| \\
&\quad \leq M^* + c_i v_\lambda(t) + M[c_i v_\lambda(t) \\
&\quad\quad + \| (-A)^{-\beta} \| L_1 v_\lambda(t)] + \| (-A)^{-\beta} \| L_1 v_\lambda(t) \\
&\quad\quad + \frac{\alpha M_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \frac{b^{\alpha\beta}}{\alpha\beta} L_1 v_\lambda(t) \\
&\quad\quad + \frac{M}{\Gamma(\alpha)} \frac{b^{\alpha}}{\alpha} \int_0^t m(s) \Theta(v_\lambda(s)) ds,
\end{aligned}$$

where

$$\begin{aligned}
M^* = \max_{1 \leq i < N} \Big\{ & M \| (-A)^{-\beta} \| (L_1 \| \varphi \|_{\mathcal{B}} + L_2) + d_i \\
& + M(d_i + \| (-A)^{-\beta} \| L_2) \\
& + \| (-A)^{-\beta} \| L_2 + \frac{\alpha M_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \frac{b^{\alpha\beta}}{\alpha\beta} L_2 \Big\}.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
v_\lambda(t) \leq\ & K_b M^* + (K_b M \tilde{H} + M_b) \| \varphi \|_{\mathcal{B}} \\
& + K_b \Bigg[(1+M)c_i + \bigg((M+1) \| (-A)^{-\beta} \| \\
& + \frac{\alpha M_{1-\beta}\Gamma(\beta+1) b^{\alpha\beta}}{\alpha\beta\Gamma(\alpha\beta+1)} \bigg) L_1 \Bigg] v_\lambda(t) \\
& + \frac{K_b M b^{\alpha}}{\alpha\Gamma(\alpha)} \int_0^t m(s) \Theta(v_\lambda(s)) ds.
\end{aligned}$$

Since $L^* = \max_{1 \leq i \leq N} \{ K_b[(1+M)c_i + ((M+1) \| (-A)^{-\beta} \| + \frac{\alpha M_{1-\beta}\Gamma(\beta+1)b^{\alpha\beta}}{\alpha\beta\Gamma(\alpha\beta+1)}) L_1] \} < 1$, we have

$$\begin{aligned}
v_\lambda(t) \leq\ & \frac{1}{1-L^*} [K_b M^* + (K_b M \tilde{H} + M_b) \| \varphi \|_{\mathcal{B}}] \\
& + \frac{1}{1-L^*} \frac{K_b M b^{\alpha}}{\alpha\Gamma(\alpha)} \int_0^t m(s) \Theta(v_\lambda(s)) ds.
\end{aligned}$$

Denoting by $\eta_\lambda(t)$ the right-hand side of the above inequality, we have $v_\lambda(t) \leq \eta_\lambda(t)$ for all $t \in [0, b]$, and

$$\eta_\lambda(0) = \frac{1}{1-L^*} [K_b M^* + (K_b M \tilde{H} + M_b) \| \varphi \|_{\mathcal{B}}],$$

$$\eta'_\lambda(t) \leq \frac{1}{1-L^*} \frac{K_b M b^{\alpha}}{\alpha\Gamma(\alpha)} m(t) \Theta(\eta_\lambda(t)),$$

which implies that

$$\int_{\eta_\lambda(0)}^{\eta_\lambda(t)} \frac{ds}{\Theta(s)} \leq \frac{1}{1-L^*}\frac{K_b M b^\alpha}{\alpha\Gamma(\alpha)}\int_0^b m(s)ds < \infty.$$

This inequality shows that the functions $\eta_\lambda(t)$ are bounded on $[0,b]$. Thus, the functions $v_\lambda(t)$ are bounded on $[0,b]$, and $y_\lambda(\cdot)$ are bounded on $[0,b]$.

Step 2. $\bar{\Phi}:\mathcal{B}_b^0\to\mathcal{B}_b^0$ is continuous.

Let $\{y^{(n)}\}_{n=0}^\infty\subseteq\mathcal{B}_b^0$ with $y^{(n)}\to y(n\to\infty)$ in $\mathcal{B}_b^0$. Then there is a number $r>0$ such that $\| y^{(n)}(t)\|\leq r$ for all n and a.e. $t\in[0,b]$, so $y^{(n)}\in B_r(0,\mathcal{B}_b^0)$ and $y\in B_r(0,\mathcal{B}_b^0)$. Again, similarly to (19), we have $\| y_t^{(n)}+\tilde{\varphi}_t\|_{\mathcal{B}}\leq r^*, t\in[0,b]$. Furthermore, from axiom (A), we know that

$$\begin{aligned}
&\| y_t^{(n)}-y_t\|_{\mathcal{B}}\\
&\quad\leq K(t)\sup\{\| y^n(s)-y(s)\|: 0\leq s\leq t\}\\
&\qquad + M(t)\| y_0^{(n)}-y_0\|_{\mathcal{B}}\\
&\quad = K(t)\sup\{\| y^n(s)-y(s)\|: 0\leq s\leq t\}\\
&\quad\leq K_b\| y^n-y\|_b\to 0(n\to\infty), t\in[0,b].
\end{aligned}$$

Thus, by (H3), we have for $t\in[0,b]$

$$F(t,y_t^{(n)}+\tilde{\varphi}_t)\to F(t,y_t+\tilde{\varphi}_t) \quad\text{as}\quad n\to\infty,$$

$$\| F(t,y_t^{(n)}+\tilde{\varphi}_t)-F(t,y_t+\tilde{\varphi}_t)\|\leq 2\Theta(r^*)m(t).$$

By (H2), (H4), Young inequality and the dominated convergence theorem we have that for $t\in[0,t_1]$

$$\begin{aligned}
&\| (\bar{\Phi}y^{(n)})(t)-(\bar{\Phi}y)(t)\|\\
&\quad\leq\| (-A)^{-\beta}\|\| (-A)^\beta f(t,y_t^{(n)}+\tilde{\varphi}_t)\\
&\qquad -(-A)^\beta f(t,y_t+\tilde{\varphi}_t)\|\\
&\qquad +\Big\| \int_0^t (t-s)^{\alpha-1}(-A)^{1-\beta}S_\alpha(t-s)\\
&\qquad \times[(-A)^\beta f(s,y_s^{(n)}+\tilde{\varphi}_s)-(-A)^\beta f(s,y_s+\tilde{\varphi}_s)]ds\Big\|\\
&\qquad +\Big\| \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)\\
&\qquad \times[F(s,y_s^{(n)}+\tilde{\varphi}_s)-F(s,y_s+\tilde{\varphi}_s)]ds\Big\|\\
&\quad\leq\| (-A)^{-\beta}\|\| (-A)^\beta f(t,y_t^{(n)}+\tilde{\varphi}_t)\\
&\qquad -(-A)^\beta f(t,y_t+\tilde{\varphi}_t)\|\\
&\qquad +\frac{\alpha M_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)}\frac{t_1^{\alpha\beta}}{\alpha\beta}\int_0^t\| (-A)^\beta f(s,y_s^{(n)}+\tilde{\varphi}_s)\\
&\qquad -(-A)^\beta f(s,y_s+\tilde{\varphi}_s)\| ds\\
&\qquad +\frac{M}{\Gamma(\alpha)}\frac{t_1^\alpha}{\alpha}\int_0^t\| F(s,y_s^{(n)}+\tilde{\varphi}_s)\\
&\qquad -F(s,y_s+\tilde{\varphi}_s)\| ds\to 0 \quad\text{as}\quad n\to\infty.
\end{aligned}$$

For any $t\in(t_i,s_i], i=1,...,N$, we have

$$\begin{aligned}
&\| (\bar{\Phi}y^{(n)})(t)-(\bar{\Phi}y)(t)\|\\
&\quad=\| g_i(t,y_t^{(n)}+\tilde{\varphi}_t)-g_i(t,y_t+\tilde{\varphi}_t)\|\to 0 \quad\text{as}\quad n\to\infty.
\end{aligned}$$

Similarly, for any $t\in(s_i,t_{i+1}], i=1,\ldots,N$, we have

$$\begin{aligned}
&\| (\bar{\Phi}y^{(n)})(t)-(\bar{\Phi}y)(t)\|\\
&\quad\leq\| T_\alpha(t-s_i)\{[g_i(s_i,y_{s_i}^{(n)}+\tilde{\varphi}_{s_i})-g_i(s_i,y_{s_i}+\tilde{\varphi}_{s_i})]\\
&\qquad +(-A)^{-\beta}[(-A)^\beta f(s_i,y_{s_i}^{(n)}+\tilde{\varphi}_{s_i})\\
&\qquad -(-A)^\beta f(s_i,y_{s_i}+\tilde{\varphi}_{s_i})]\}\|\\
&\qquad +\| (-A)^{-\beta}\|\| (-A)^\beta f(t,y_t^{(n)}+\tilde{\varphi}_t)\\
&\qquad -(-A)^\beta f(t,y_t+\tilde{\varphi}_t)\|\\
&\qquad +\Big\| \int_{s_i}^t (t-s)^{\alpha-1}(-A)^{1-\beta}S_\alpha(t-s)\\
&\qquad \times[(-A)^\beta f(s,y_s^{(n)}+\tilde{\varphi}_s)-(-A)^\beta f(s,y_s+\tilde{\varphi}_s)]ds\Big\|\\
&\qquad +\Big\| \int_{s_i}^t (t-s)^{\alpha-1}S_\alpha(t-s)\\
&\qquad \times[F(s,y_s^{(n)}+\tilde{\varphi}_s)-F(s,y_s+\tilde{\varphi}_s)]ds\Big\|\\
&\quad\leq M[\| g_i(s_i,y_{s_i}^{(n)}+\tilde{\varphi}_{s_i})-g_i(s_i,y_{s_i}+\tilde{\varphi}_{s_i})\|\\
&\qquad +\| (-A)^{-\beta}\|\| (-A)^\beta f(s_i,y_{s_i}^{(n)}+\tilde{\varphi}_{s_i})\\
&\qquad -(-A)^\beta f(s_i,y_{s_i}+\tilde{\varphi}_{s_i})\|]\\
&\qquad +\| (-A)^{-\beta}\|\| (-A)^\beta f(t,y_t^{(n)}+\tilde{\varphi}_t)\\
&\qquad -(-A)^\beta f(t,y_t+\tilde{\varphi}_t)\|\\
&\qquad +\frac{\alpha M_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)}\frac{(t_{i+1}-s_i)^{\alpha\beta}}{\alpha\beta}\\
&\qquad \int_{s_i}^t\| (-A)^\beta f(s,y_s^{(n)}+\tilde{\varphi}_s)\\
&\qquad -(-A)^\beta f(s,y_s+\tilde{\varphi}_s)\| ds\\
&\qquad +\frac{M}{\Gamma(\alpha)}\frac{(t_{i+1}-s_i)^\alpha}{\alpha}\int_{s_i}^t\| F(s,y_s^{(n)}+\tilde{\varphi}_s)\\
&\qquad -F(s,y_s+\tilde{\varphi}_s)\| ds\to 0 \quad\text{as}\quad n\to\infty.
\end{aligned}$$

Then

$$\| \bar{\Phi}y^{(n)}-\bar{\Phi}y\|_b\to 0 \quad\text{as}\quad n\to\infty.$$

Thus $\bar{\Phi}$ is continuous.

Step 3. $\bar{\Phi}$ is χ-contraction.

To clarify this, we decompose $\bar{\Phi}$ in the form $\bar{\Phi}=\bar{\Phi}_1+\bar{\Phi}_2$ for $t\in[0,b]$, where

$$(\bar{\Phi}_1 y)(t)=\begin{cases}
-T_\alpha(t)f(0,\varphi)+f(t,y_t+\tilde{\varphi}_t)\\
\quad+\int_0^t(t-s)^{\alpha-1}AS_\alpha(t-s)\\
\quad\times f(s,y_s+\tilde{\varphi}_s)ds, \quad t\in[0,t_1], i=0,\\
0, \quad t\in(t_i,s_i], i\geq 1,\\
-T_\alpha(t-s_i)f(s_i,y_{s_i}+\tilde{\varphi}_{s_i})+f(t,y_t+\tilde{\varphi}_t)\\
\quad+\int_0^t(t-s)^{\alpha-1}AS_\alpha(t-s)\\
\quad\times f(s,y_s+\tilde{\varphi}_s)ds, \quad t\in[s_i,t_{i+1}], i\geq 1,
\end{cases}$$

$$(\bar{\Phi}_2 y)(t)=\begin{cases}
\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)\\
\quad F(s,y_s+\tilde{\varphi}_s)ds, \quad t\in[0,t_1], i=0,\\
g_i(t,y_t+\tilde{\varphi}_t), \quad t\in(t_i,s_i], i\geq 1,\\
T_\alpha(t-s_i)g_i(s_i,y_{s_i}+\tilde{\varphi}_{s_i})\\
\quad+\int_{s_i}^t(t-s)^{\alpha-1}S_\alpha(t-s)\\
\quad\times F(s,y_s+\tilde{\varphi}_s)ds, \quad t\in[s_i,t_{i+1}], i\geq 1.
\end{cases}$$

Claim 1. $\bar{\Phi}_1$ is Lipschitz continuous.

Let $t \in [0, t_1]$ and $u, v \in \mathcal{B}_b^0$. From (H2), we have

$$\begin{aligned}
&\| (\bar{\Phi}_1 u)(t) - (\bar{\Phi}_1 v)(t) \| \\
&\quad \leq \| (-A)^{-\beta} \| \| (-A)^{\beta} f(t, u_t + \tilde{\varphi}_t) \\
&\quad\quad -(-A)^{\beta} f(t, v_t + \tilde{\varphi}_t) \| \\
&\quad\quad + \left\| \int_0^t (t-s)^{\alpha-1} (-A)^{1-\beta} S_\alpha(t-s) \right. \\
&\quad\quad \left. \times [(-A)^{\beta} f(s, u_s + \tilde{\varphi}_s) - (-A)^{\beta} f(s, v_s + \tilde{\varphi}_s)] ds \right\| \\
&\quad \leq L_1 \| (-A)^{-\beta} \| \| u_t - v_t \|_{\mathcal{B}} \\
&\quad\quad + \frac{\alpha M_{1-\beta} \Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \frac{t_1^{\alpha\beta}}{\alpha\beta} L_1 \| u_t - v_t \|_{\mathcal{B}} \\
&\quad \leq K_b \left(\| (-A)^{-\beta} \| + \frac{\alpha M_{1-\beta} \Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \frac{t_1^{\alpha\beta}}{\alpha\beta} \right) L_1 \\
&\quad\quad \times \| u - v \|_b .
\end{aligned}$$

Similarly, for any $t \in (t_i, t_{i+1}], i = 1, \ldots, N$, we have

$$\begin{aligned}
&\| (\bar{\Phi}_1 u)(t) - (\bar{\Phi}_1 v)(t) \| \\
&\quad \leq \| T_\alpha(t - s_i)(-A)^{-\beta} [(-A)^{\beta} f(s_i, u_{s_i} + \tilde{\varphi}_{s_i}) \\
&\quad\quad -(-A)^{\beta} f(s_i, u_{s_i} + \tilde{\varphi}_{s_i})] \| \\
&\quad\quad + \| (-A)^{-\beta} \| \| (-A)^{\beta} f(t, u_t + \tilde{\varphi}_t) \\
&\quad\quad -(-A)^{\beta} f(t, v_t + \tilde{\varphi}_t) \| \\
&\quad\quad + \left\| \int_{s_i}^t (t-s)^{\alpha-1} (-A)^{1-\beta} S_\alpha(t-s) \right. \\
&\quad\quad \left. \times [(-A)^{\beta} f(s, u_s + \tilde{\varphi}_s) - (-A)^{\beta} f(s, v_s + \tilde{\varphi}_s)] ds \right\| \\
&\quad \leq M L_1 \| (-A)^{-\beta} \| \| u_t - v_t \|_{\mathcal{B}} \\
&\quad\quad + L_1 \| (-A)^{-\beta} \| \| u_t - v_t \|_{\mathcal{B}} \\
&\quad\quad + \frac{\alpha M_{1-\beta} \Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \frac{(t_{i+1} - s_i)^{\alpha\beta}}{\alpha\beta} L_1 \| u_t - v_t \|_{\mathcal{B}} \\
&\quad \leq K_b \left((1+M) \| (-A)^{-\beta} \| \right. \\
&\quad\quad \left. + \frac{\alpha M_{1-\beta} \Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \frac{(t_{i+1} - s_i)^{\alpha\beta}}{\alpha\beta} \right) L_1 \| u - v \|_b .
\end{aligned}$$

Thus, for all $t \in [0, b]$, we have

$$\| (\bar{\Phi}_1 u)(t) - (\bar{\Phi}_1 v)(t) \| \leq L_0 \| u - v \|_b .$$

Taking supremum over t,

$$\| \bar{\Phi}_1 u - \bar{\Phi}_1 v \|_b \leq L_0 \| u - v \|_b,$$

where

$$L_0 = K_b \left((1+M) \| (-A)^{-\beta} \| + \frac{\alpha M_{1-\beta} \Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \frac{b^{\alpha\beta}}{\alpha\beta} \right) L_1.$$

By (18), we see that $L_0 < 1$. Hence, $\bar{\Phi}_1$ is Lipschitz continuous.

Claim 2. $\bar{\Phi}_2$ maps bounded sets into equicontinuous sets of $\mathcal{B}_b^0$.

Let $0 < \tau_1 < \tau_2 \leq t_1$. For each $x \in B_r(0, \mathcal{B}_b^0)$, we have

$$\begin{aligned}
&\| (\bar{\Phi}_2 y)(\tau_2) - (\bar{\Phi}_2 y)(\tau_1) \| \\
&\quad = \left\| \int_0^{\tau_2} (\tau_2 - s)^{\alpha-1} S_\alpha(\tau_2 - s) F(s, y_s + \tilde{\varphi}_s) ds \right. \\
&\quad\quad \left. - \int_0^{\tau_1} (\tau_1 - s)^{\alpha-1} S_\alpha(\tau_1 - s) F(s, y_s + \tilde{\varphi}_s) ds \right\| \\
&\quad \leq \left\| \int_0^{\tau_1} [(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}] \right. \\
&\quad\quad \left. \times S_\alpha(\tau_2 - s) F(s, y_s + \tilde{\varphi}_s) ds \right\| \\
&\quad\quad + \left\| \int_0^{\tau_1} (\tau_1 - s)^{\alpha-1} [S_\alpha(\tau_2 - s) \right. \\
&\quad\quad \left. - S_\alpha(\tau_1 - s)] F(s, y_s + \tilde{\varphi}_s) ds \right\| \\
&\quad\quad + \left\| \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} S_\alpha(\tau_2 - s) F(s, y_s + \tilde{\varphi}_s) ds \right\| \\
&\quad \leq \frac{M}{\Gamma(\alpha)} \Theta(r^*) \int_0^{\tau_1} |(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}| m(s) ds \\
&\quad\quad + \Theta(r^*) \int_0^{\tau_1} (\tau_1 - s)^{\alpha-1} \\
&\quad\quad \times \| S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s) \| m(s) ds \\
&\quad\quad + \frac{M}{\Gamma(\alpha)} \Theta(r^*) \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} m(s) ds.
\end{aligned}$$

For any $\tau_1, \tau_2 \in (t_i, s_i], \tau_1 < \tau_2, i = 1, ..., N$, we have

$$\begin{aligned}
&\| (\bar{\Phi}_2 y)(\tau_2) - (\bar{\Phi}_2 y)(\tau_1) \| \\
&\quad = \| g_i(\tau_2, y_{\tau_2} + \tilde{\varphi}_{\tau_2}) - g_i(\tau_1, y_{\tau_1} + \tilde{\varphi}_{\tau_1}) \| \\
&\quad \leq \nu_i [|\tau_2 - \tau_1| + \| y_{\tau_2} - y_{\tau_1} \|_{\mathcal{B}} + \| \tilde{\varphi}_{\tau_2} - \tilde{\varphi}_{\tau_1} \|_{\mathcal{B}}].
\end{aligned}$$

Similarly, for any $\tau_1, \tau_2 \in (t_i, t_{i+1}], \tau_1 < \tau_2, i = 1, \ldots, N$, we have

$$\begin{aligned}
&\| (\bar{\Phi}_2 y)(\tau_2) - (\bar{\Phi}_2 y)(\tau_1) \| \\
&\quad \leq \| [S_\alpha(\tau_2 - s_i) - S_\alpha(\tau_1 - s_i)] g_i(s_i, y_{s_i} + \tilde{\varphi}_{s_i}) \| \\
&\quad\quad + \left\| \int_{s_i}^{\tau_2} (\tau_2 - s)^{\alpha-1} S_\alpha(\tau_2 - s) F(s, y_s + \tilde{\varphi}_s) ds \right. \\
&\quad\quad \left. - \int_{s_i}^{\tau_1} (\tau_1 - s)^{\alpha-1} S_\alpha(\tau_1 - s) F(s, y_s + \tilde{\varphi}_s) ds \right\| \\
&\quad \leq \| S_\alpha(\tau_2 - s_i) - S_\alpha(\tau_1 - s_i) \| [c_i r^* + d_i] \\
&\quad\quad + \left\| \int_{s_i}^{\tau_1} [(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}] \right. \\
&\quad\quad \left. \times S_\alpha(\tau_2 - s) F(s, y_s + \tilde{\varphi}_s) ds \right\| \\
&\quad\quad + \left\| \int_{s_i}^{\tau_1} (\tau_1 - s)^{\alpha-1} [S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)] \right. \\
&\quad\quad \left. \times F(s, y_s + \tilde{\varphi}_s) ds \right\| \\
&\quad\quad + \left\| \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} S_\alpha(\tau_2 - s) F(s, y_s + \tilde{\varphi}_s) ds \right\| \\
&\quad \leq \| S_\alpha(\tau_2 - s_i) - S_\alpha(\tau_1 - s_i) \| [c_i r^* + d_i] \\
&\quad\quad + \frac{M}{\Gamma(\alpha)} \Theta(r^*) \int_{s_i}^{\tau_1} |(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}| m(s) ds \\
&\quad\quad + \Theta(r^*) \int_{s_i}^{\tau_1} (\tau_1 - s)^{\alpha-1} \\
&\quad\quad \times \| S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s) \| m(s) ds
\end{aligned}$$

$$+\frac{M}{\Gamma(\alpha)}\Theta(r^*)\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}m(s)ds.$$

The right-hand side tends to zero as $\tau_2 \to \tau_1$, since $T(t)$ is equicontinuous for $t > 0$, it is easy to show that $S_\alpha(t), T_\alpha(t)$ are equicontinuous for $t > 0$ implies the continuity in the uniform operator topology, and in view of hypotheses (H3)-(H5). The equicontinuities for the cases $\tau_1 < \tau_2 \le 0$ or $\tau_1 \le 0 \le \tau_2$ are very simple. Hence $\bar{\Phi}_2$ maps $B_r(0,\mathcal{B}_b^0)$ into an equicontinuous family of functions.

Claim 3. $\bar{\Phi}_2$ is χ-contraction.

Let arbitrary bounded subset $W \subset \mathcal{B}_b^0$. Since (H1) and the definition of $S_\alpha(\cdot)$, we get that the operator $S_\alpha(\cdot)$ is equicontinuous, so $S_\alpha(t-s)F(s, W_s + \tilde{\varphi}_s)$ is piecewise equicontinuous on each $\overline{J_i} = [s_i, t_{i+1}], i = 0, 1, \dots, N$. Then for each bounded set $W \in \mathcal{PC}$. From (H3), (H4), Young inequality and Lemmas 5, 6, we have for $t \in [0, t_1]$

$$\begin{aligned}
&\chi(\bar{\Phi}_2 W)(t))\\
&\quad= \chi\Big(\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)F(s,W_s+\tilde{\varphi}_s)ds\Big)\\
&\quad\le \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\chi(F(s,W_s+\tilde{\varphi}_s))ds\\
&\quad\le \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\zeta(s)\\
&\qquad\times \sup_{-\infty<\theta\le 0}\chi(W(s+\theta)+\tilde{\varphi}(s+\theta))ds\\
&\quad\le \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\zeta(s)\sup_{0\le\tau\le s}\chi(W(\tau))ds\\
&\quad\le \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\zeta(s)ds\sup_{0\le s\le b}\chi(W(s))\\
&\quad\le \frac{M}{\Gamma(\alpha)}\frac{t_1^\alpha}{\alpha}\int_0^t \zeta(s)ds\chi_{\mathcal{PC}}(W).
\end{aligned}$$

For any $t \in (t_i, s_i], i = 1, ..., N$, we have

$$\begin{aligned}
&\chi(\bar{\Phi}_2 W)(t))\\
&\quad= \chi(g_i(t, W_t + \tilde{\varphi}_t))\\
&\quad\le \gamma_i \sup_{-\infty<\theta\le 0}\chi(W(t+\theta)+\tilde{\varphi}(t+\theta))\\
&\quad\le \gamma_i \sup_{t_i<\tau\le t}\chi(W(\tau))\\
&\quad\le \gamma_i \sup_{0\le\tau\le b}\chi(W(\tau)) \le \gamma_i\chi_{\mathcal{PC}}(W).
\end{aligned}$$

Similarly, for any $t \in (s_i, t_{i+1}], i = 1, ..., N$, we have

$$\begin{aligned}
&\chi(\bar{\Phi}_2 W)(t))\\
&\quad\le \chi(T_\alpha(t-s_i)g_i(s_i, W_{s_i}+\tilde{\varphi}_{s_i}))\\
&\qquad+\chi\Big(\int_{s_i}^t (t-s)^{\alpha-1}S_\alpha(t-s)F(s,W_s+\tilde{\varphi}_s)ds\Big)\\
&\quad\le M\chi(g_i(s_i, W_{s_i}+\tilde{\varphi}_{s_i}))\\
&\qquad+\frac{M}{\Gamma(\alpha)}\int_{s_i}^t (t-s)^{\alpha-1}\chi(F(s,W_s+\tilde{\varphi}_s))ds\\
&\quad\le M\gamma_i \sup_{-\infty<\theta\le 0}\chi(W(s_i+\theta)+\tilde{\varphi}(s_i+\theta))\\
&\qquad+\frac{M}{\Gamma(\alpha)}\int_{s_i}^t (t-s)^{\alpha-1}\zeta(s)\\
&\qquad\times \sup_{-\infty<\theta\le 0}\chi(W(s+\theta)+\tilde{\varphi}(s+\theta))ds\\
&\quad\le M\gamma_i \sup_{s_i<\tau\le t}\chi(W(\tau))\\
&\qquad+\frac{M}{\Gamma(\alpha)}\int_{s_i}^t (t-s)^{\alpha-1}\zeta(s)\sup_{s_i<\tau\le s}\chi(W(\tau))ds\\
&\quad\le M\gamma_i \sup_{0\le\tau\le b}\chi(W(\tau))\\
&\qquad+\frac{M}{\Gamma(\alpha)}\int_{s_i}^t (t-s)^{\alpha-1}\zeta(s)ds\sup_{0\le s\le b}\chi(W(s))\\
&\quad\le M\Big(\gamma_i + \frac{1}{\Gamma(\alpha)}\frac{(t_{i+1}-s_i)^\alpha}{\alpha}\int_{s_i}^t \zeta(s)ds\Big)\chi_{\mathcal{PC}}(W).
\end{aligned}$$

Thus, for all $t \in [0, b]$, we have

$$\begin{aligned}
&\chi(\bar{\Phi} W)(t))\\
&\quad\le \Big((M+1)\gamma_i + \frac{Mb^\alpha}{\alpha\Gamma(\alpha)}\int_0^b \zeta(s)ds\Big)\chi_{\mathcal{PC}}(W),
\end{aligned}$$

and

$$\chi(\bar{\Phi} W) \le \tilde{L}\chi_{\mathcal{PC}}(W) \le \chi_{\mathcal{PC}}(W),$$

where $\tilde{L} = \max_{1\le i\le N}\{(M+1)\gamma_i + \frac{Mb^\alpha}{\alpha\Gamma(\alpha)}\int_0^b \zeta(s)ds\} < 1$. Therefore, $\bar{\Phi}$ is $\chi-$contraction. By Lemma 7, we conclude that $\bar{\Phi}$ has at least one fixed point $y^* \in W \subset \mathcal{B}_b^0$. Let $x(t) = y^*(t) + \tilde{\varphi}(t), t \in (-\infty, b]$. Then, x is a fixed point of the operator Φ, which implies that x is a mild solution of the problem (1)-(3) and the proof is complete.

Theorem 2. *If the assumptions* (H1)-(H6) *are satisfied, then the problem* (1)-(3) *has at least one mild solution on* $[0, b]$, *provided that*

$$\begin{aligned}
\max_{1\le i\le N} K_b\Big\{\Big[&(1+M)c_i + \Big((M+1)\parallel(-A)^{-\beta}\parallel\\
&+\frac{\alpha M_{1-\beta}\Gamma(\beta+1)b^{\alpha\beta}}{\alpha\beta\Gamma(\alpha\beta+1)}\Big)L_1\Big]\\
&+\frac{Mb^\alpha}{\alpha\Gamma(\alpha)}\int_0^b m(s)ds\limsup_{\varsigma\to\infty}\frac{\Theta(\varsigma)}{\varsigma}\Big\} < 1. \quad (20)
\end{aligned}$$

Proof. Let $\Phi, \bar{\Phi}$ be defined as in the proof of Theorem 1. We claim that there exists an $k > 0$ such that $\bar{\Phi}(B_k(0,\mathcal{B}_b^0)) \subset (B_k(0,\mathcal{B}_b^0))$. If we assume that this assertion is false, then for each $k > 0$ we can choose $y^k \in B_k(0,\mathcal{B}_b^0)$ and $t^k \in [0, b]$ such that $k <\parallel (\Phi y^k)(t^k)\parallel$, Consequently, using (H1)-(H4) and Young inequality, we have for $t^k \in [0, t_1]$

$$\begin{aligned}
&k <\parallel (\Phi y^k)(t^k)\parallel\\
&\quad\le\parallel T_\alpha(t^k)(-A)^{-\beta}(-A)^\beta f(0,\varphi)\parallel\\
&\qquad+\parallel(-A)^{-\beta}(-A)^\beta f(t^k, y^k_{t^k}+\tilde{\varphi}_{t^k})\parallel\\
&\qquad+\Big\|\int_0^{t^k}(t^k-s)^{\alpha-1}(-A)^{1-\beta}S_\alpha(t^k-s)\\
&\qquad\times(-A)^\beta f(s, y^k_s+\tilde{\varphi}_s)ds\Big\|\\
&\qquad+\Big\|\int_0^{t^k}(t^k-s)^{\alpha-1}S_\alpha(t^k-s)F(s, y^k_s+\tilde{\varphi}_s)ds\Big\|\\
&\quad\le M\parallel(-A)^{-\beta}\parallel(L_1\parallel\varphi\parallel_{\mathcal{B}}+L_2)\\
&\qquad+\parallel(-A)^{-\beta}\parallel(L_1\parallel y^k_{t^k}+\tilde{\varphi}_{t^k}\parallel_{\mathcal{B}}+L_2)\\
&\qquad+\frac{\alpha M_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)}\int_0^{t^k}(t^k-s)^{\alpha\beta-1}\\
&\qquad\times(L_1\parallel y^k_{\lambda s}+\tilde{\varphi}_s\parallel_{\mathcal{B}}+L_2)ds\\
&\qquad+\frac{M}{\Gamma(\alpha)}\int_0^{t^k}(t^k-s)^{\alpha-1}m(s)\Theta(\parallel y^k_s+\tilde{\varphi}_s\parallel_{\mathcal{B}})ds\\
&\quad\le M\parallel(-A)^{-\beta}\parallel(L_1\parallel\varphi\parallel_{\mathcal{B}}+L_2)
\end{aligned}$$

$$
\begin{aligned}
&+ \| (-A)^{-\beta} \| (L_1 \| y^k_{t^k} + \tilde{\varphi}_{t^k} \|_{\mathcal{B}} + L_2) \\
&+ \frac{\alpha M_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \frac{t_1^{\alpha\beta}}{\alpha\beta}(L_1 \| y^k_{t^k} + \tilde{\varphi}_{t^k} \|_{\mathcal{B}} + L_2) \\
&+ \frac{M}{\Gamma(\alpha)} \frac{t_1^{\alpha}}{\alpha} \int_0^{t^k} m(s)\Theta(\| y^k_s + \tilde{\varphi}_s \|_{\mathcal{B}}) ds.
\end{aligned}
$$

For any $t^k \in (t_i, s_i], i = 1, \dots, N$, we have

$$
k < \| (\Phi y^k)(t^k) \| \leq c_i \| y^k_{t^k} + \tilde{\varphi}_{t^k} \|_{\mathcal{B}} + d_i.
$$

Similarly, for any $t^k \in (s_i, t_{i+1}], i = 1, \dots, N$, we have

$$
\begin{aligned}
k &< \| (\Phi y^k)(t^k) \| \\
&\leq \| T_\alpha(t^k - s_i)[g_i(s_i, y^k_{s_i} + \tilde{\varphi}_{s_i}) \\
&\quad -(-A)^{-\beta}(-A)^{\beta} f(s_i, y^k_{s_i} + \tilde{\varphi}_{s_i})] \| \\
&\quad + \| (-A)^{-\beta}(-A)^{\beta} f(t^k, y^k_{t^k} + \tilde{\varphi}_{t^k}) \| \\
&\quad + \left\| \int_{s_i}^{t^k} (t^k - s)^{\alpha-1}(-A)^{1-\beta} S_\alpha(t^k - s) \right. \\
&\quad \left. \times (-A)^{\beta} f(s, y^k_s + \tilde{\varphi}_s) ds \right\| \\
&\quad + \left\| \int_{s_i}^{t^k} S_\alpha(t^k - s) F(s, y^k_s + \tilde{\varphi}_s) ds \right\| \\
&\leq M[c_i \| y^k_{s_i} + \tilde{\varphi}_{s_i} \|_{\mathcal{B}} + d_i \\
&\quad + \| (-A) \|^{-\beta} (L_1 \| y^k_{s_i} + \tilde{\varphi}_{s_i} \|_{\mathcal{B}} + L_2)] \\
&\quad + \| (-A)^{-\beta} \| (L_1 \| y^k_{t^k} + \tilde{\varphi}_{t^k} \|_{\mathcal{B}} + L_2) \\
&\quad + \frac{\alpha M_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \int_{s_i}^{t^k} (t^k - s)^{\alpha\beta-1} \\
&\quad \times (L_1 \| y_{\lambda s} + \tilde{\varphi}_s \|_{\mathcal{B}} + L_2) ds \\
&\quad + \frac{M}{\Gamma(\alpha)} \int_{s_i}^{t^k} (t^k - s)^{\alpha-1} m(s)\Theta(\| y^k_s + \tilde{\varphi}_s \|_{\mathcal{B}}) ds \\
&\leq M[c_i \| y^k_{s_i} + \tilde{\varphi}_{s_i} \|_{\mathcal{B}} + d_i \\
&\quad + \| (-A)^{-\beta} \| (L_1 \| y^k_{s_i} + \tilde{\varphi}_{s_i} \|_{\mathcal{B}} + L_2)] \\
&\quad + \| (-A)^{-\beta} \| (L_1 \| y^k_{t^k} + \tilde{\varphi}_{t^k} \|_{\mathcal{B}} + L_2) \\
&\quad + \frac{\alpha M_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \frac{(t^k_{i+1} - s_i)^{\alpha\beta}}{\alpha\beta} \\
&\quad \times (L_1 \| y^k_{t^k} + \tilde{\varphi}_{t^k} \|_{\mathcal{B}} + L_2) \\
&\quad + \frac{M}{\Gamma(\alpha)} \frac{(t^k_{i+1} - s_i)^{\alpha}}{\alpha} \int_{s_i}^{t^k} m(s)\Theta(\| y^k_s + \tilde{\varphi}_s \|_{\mathcal{B}}) ds.
\end{aligned}
$$

Then, for all $t \in [0, b]$, we have

$$
\begin{aligned}
k &< \| (\Phi y^k)(t^k) \| \\
&\leq M^* + c_i[K_b k + (K_b M \tilde{H} + M_b) \| \varphi \|_{\mathcal{B}}] \\
&\quad + M[c_i(K_b k + (K_b M \tilde{H} + M_b) \| \varphi \|_{\mathcal{B}})] \\
&\quad + \| (-A)^{-\beta} \| L_1[K_b k + (K_b M \tilde{H} + M_b) \| \varphi \|_{\mathcal{B}}] \\
&\quad + \| (-A)^{-\beta} \| L_1[K_b k + (K_b M \tilde{H} + M_b) \| \varphi \|_{\mathcal{B}}] \\
&\quad + \frac{\alpha M_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \frac{b^{\alpha\beta}}{\alpha\beta} \\
&\quad \times L_1[K_b k + (K_b M \tilde{H} + M_b) \| \varphi \|_{\mathcal{B}}] \\
&\quad + \frac{M}{\Gamma(\alpha)} \frac{b^{\alpha}}{\alpha} \int_0^t m(s)\Theta(K_b k \\
&\quad + (K_b M \tilde{H} + M_b) \| \varphi \|_{\mathcal{B}}) ds,
\end{aligned}
$$

and hence,

$$
\begin{aligned}
1 < \max_{1 \leq i \leq N} K_b \Bigg\{ \Bigg[&(1+M)c_i + \Bigg((M+1) \| (-A)^{-\beta} \| \\
&+ \frac{\alpha M_{1-\beta}\Gamma(\beta+1) b^{\alpha\beta}}{\alpha\beta\Gamma(\alpha\beta+1)} \Bigg) L_1 \Bigg] \\
&+ \frac{M b^{\alpha}}{\alpha\Gamma(\alpha)} \int_0^b m(s) ds \limsup_{\varsigma \to \infty} \frac{\Theta(\varsigma)}{\varsigma} \Bigg\},
\end{aligned}
$$

which is contrary to our assumption (20). The proofs of the other steps are similar to those in Theorem 1. Therefore, we omit the details. By means of Lemma 8, we conclude that (1)-(3) has at least a mild solution and the proof is complete.

IV. EXAMPLES

Example 1. Consider the following fractional impulsive partial neutral functional differential equations

$$
\begin{aligned}
\frac{d}{dt}[J_t^{1-\alpha} &z(t,x) - f(t, z(t-\tau, x)) \\
&-\varphi(0,x) + f(0, z(-\tau, x))] \\
&= \frac{\partial^2}{\partial x^2} z(t,x) + F(t, z(t-\tau, x)), \qquad (21) \\
&\tau > 0, (t,x) \in \cup_{i=1}^{N} [s_i, t_{i+1}] \times [0,1],
\end{aligned}
$$

$$
z(t,0) = z(t,1) = 0, \quad t \in [0,1], \qquad (22)
$$

$$
z(t,x) = \varphi(t,x), \quad (t,x) \in (-\infty, 0] \times \in [0,1], \qquad (23)
$$

$$
\begin{aligned}
z(t,x) = \int_{-\infty}^{t} &\varpi_i(t, s-t, z(s,x)) ds, \qquad (24) \\
&(t,x) \in (t_i, s_i] \times [0,1],
\end{aligned}
$$

where $0 < \alpha < 1$, $J_t^{1-\alpha}$ is the $(1-\alpha)$-order Riemann-Liouville fractional integral operator. Let $X = L^2([0,1])$ with the norm $\| \cdot \|$ and define the operators $A : D(A) \subseteq X \to X$ by $Au = u''$ with the domain $D(A) := \{u \in X : u, u'$ are absolutely continuous, $u'' \in X, u(0) = u(1) = 0\}$. Then

$$
Au = \sum_{n=1}^{\infty} -n^2 \langle u, u_n \rangle u_n, \quad u \in D(A),
$$

where $u_n(x) = \sqrt{2}\sin(n\pi x), n = 1, 2, \dots,$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $T(t), t \geq 0$ in X and $T(t)u = \sum_{n=1}^{\infty} e^{-n^2 t} \langle u, u_n \rangle u_n, \quad u \in X$. There exists a constant $M > 0$ such that $\| T(t) \| \leq M$, this implies that (H1) is satisfied. If we choose $\beta = \frac{3}{4}$, then the operator $(-A)^{\frac{3}{4}}$ is given by

$$
(-A)^{\frac{3}{4}} u = \sum_{n=1}^{\infty} -n^{\frac{8}{3}} \langle u, u_n \rangle u_n, \quad u \in D((-A)^{\frac{3}{4}}),
$$

where $D((-A)^{\frac{3}{4}}) = \{\omega(\cdot) \in X : \sum_{n=1}^{\infty} n^{\frac{8}{3}} \langle \omega, z_n \rangle z_n \in X\}$. Then

$$
\begin{aligned}
S_\alpha(t)u &= \int_0^{\infty} \alpha\xi M_\alpha(\xi) T(t^\alpha \xi) u d\xi \\
&= \sum_{n=1}^{\infty} E_{\alpha,\alpha}(-n^2 t^\alpha) \langle u, u_n \rangle u_n, \ u \in X.
\end{aligned}
$$

In the sequel, $\mathcal{B}$ will be the phase space $\mathcal{PC}_0 \times L^2(h, X)$ (see [43]). Then $\mathcal{B}$ satisfies axioms (A)-(C). Moreover, when $r = 0$ and $p = 2$, we can take $\tilde{H} = 1$, $M(t) = \gamma(-t)^{\frac{1}{2}}$

and $K(t) = 1 + (\int_{-t}^{0} h(\tau)d\tau)^{\frac{1}{2}}$, for $t \geq 0$; see Theorem 1.3.8 in [44] for details. Set $\psi(\theta)(x) = \psi(\theta, x)$, $(\theta, x) \in (-\infty, 0] \times \mathcal{B}$ and let $z(s)(x) = z(s, x)$. Defining the maps $f, F : [0, 1] \times \mathcal{B} \to X$, $g_i : (t_i, s_i] \times \mathcal{B} \to X, i = 1, ..., N$, by

$$(-A)^{\frac{3}{4}} f(t, \psi)(x) = \int_{-\infty}^{0} a_0(t)a_1(s)\psi(s, x)ds,$$

$$F(t, \psi)(x) = \int_{-\infty}^{0} b_0(t)b_1(s)\psi(s, x)ds,$$

$$g_i(t, \psi)(x) = \int_{-\infty}^{0} \varpi_i(t, s, \psi(s, x))ds,$$

where, we will assume that

(i) The functions $a_i : R \to R, i = 0, 1$, are continuous, and $l_1 = (\int_{-\infty}^{0} \frac{(a_1(s))^2}{h(s)} ds)^{\frac{1}{2}} < 1$.

(ii) The functions $b_i : R \to R, i = 0, 1$, are continuous, and $l_2 = (\int_{-\infty}^{0} \frac{(b_1(s))^2}{h(s)} ds)^{\frac{1}{2}} < 1$.

(iii) The functions $\varpi_i : R^3 \to R, i = 1, \ldots, N$, are continuous and there exist continuous function $\tilde{c}_i : R \to R$ such that

$$|\varpi_i(t, s, y)| \leq \tilde{c}_i(s)|y|, \quad (t, s, y) \in R^3,$$

$$\begin{aligned} &|\varpi_i(t_1, s, y_1) - \varpi_i(t_2, s, y_2)| \\ &\quad \leq \tilde{c}_i(s)[|t_1 - t_2| + |y_1 - y_2|], \\ &\quad (t_j, s, y_j) \in R^3, j = 1, 2 \end{aligned}$$

with $\tilde{l}_i = (\int_{-\infty}^{0} \frac{(\tilde{c}_i(s))^2}{h(s)} ds)^{\frac{1}{2}} < \infty$ for every $i = 1, \ldots, N$.

Then the problem (21)-(24) can be written as (1)-(3). For any $t \in [0, 1]$, we have

$$\begin{aligned} &\| (-A)^{\frac{3}{4}} f(t, \psi) \| \\ &= \Big(\int_0^1 \Big(\int_{-\infty}^{0} a_0(t)a_1(s)\psi(s, x)ds \Big)^2 dx \Big)^{\frac{1}{2}} \\ &\leq \| a_0 \|_\infty \left(\int_{-\infty}^{0} \frac{(a_1(s))^2}{h(s)} ds \right)^{\frac{1}{2}} \\ &\quad \times \Big(\int_{-\infty}^{0} h(s) \| \psi(s) \|^2 ds \Big)^{\frac{1}{2}} \\ &\leq L_f \| \psi \|_{\mathcal{B}}, \ \psi \in \mathcal{B}, \end{aligned}$$

and

$$\begin{aligned} &\| (-A)^{\frac{3}{4}} f(t, \psi_1) - (-A)^{\frac{3}{4}} f(t, \psi_2) \| \\ &= \Big(\int_0^1 \Big(\int_{-\infty}^{0} a_0(t)a_1(s) \\ &\quad \times [\psi_1(s, x) - \psi_2(s, x)]ds \Big)^2 dx \Big)^{\frac{1}{2}} \\ &\leq \| a_0 \|_\infty \left(\int_{-\infty}^{0} \frac{(a_1(s))^2}{h(s)} ds \right)^{\frac{1}{2}} \\ &\quad \times \Big(\int_{-\infty}^{0} h(s) \| \psi_1(s) - \psi_2(s) \|^2 ds \Big)^{\frac{1}{2}} \\ &\leq L_f \| \psi_1 - \psi_1 \|_{\mathcal{B}}, \ \psi_1, \psi_2 \in \mathcal{B}, \end{aligned}$$

where $L_f = \| a_0 \|_\infty l_1$. Similarly,

$$\| F(t, \psi) \| \leq L_F \| \psi \|_{\mathcal{B}}, \ \psi \in \mathcal{B},$$

and

$$\| F(t, \psi_1) - F(t, \psi_2) \| \leq L_F \| \psi_1 - \psi_1 \|_{\mathcal{B}}, \ \psi_1, \psi_2 \in \mathcal{B},$$

we can see that each bounded set $D \subset \mathcal{B}$,

$$\chi(F(t, D)) \leq L_F \sup_{-\infty < \theta \leq 0} \chi(D(\theta)),$$

where $L_F = \| b_0 \|_\infty l_2$.

Moreover, we have for any $t \in (t_i, s_i], i = 1, ..., N$,

$$\| g_i(t, \psi) \| \leq \tilde{l}_i \| \psi \|_{\mathcal{B}}, \ \psi \in \mathcal{B},$$

and for $t_1, t_2 \in (t_i, s_i], i = 1, ..., N$, we get

$$\begin{aligned} &\| g_i(t_1, \psi_1) - g_i(t_2, \psi_2) \| \\ &\quad \leq \vartheta_i[|t_1 - t_2| + \| \psi_1 - \psi_1 \|_{\mathcal{B}}], \ \psi_1, \psi_2 \in \mathcal{B}, \end{aligned}$$

we can see that for each bounded set $D \subset \mathcal{B}$,

$$\chi(g_i(t, D)) \leq \vartheta_i \sup_{-\infty < \theta \leq 0} \chi(D(\theta)),$$

where $\vartheta_i = \max\{\tilde{l}_i, \int_{-\infty}^{0} \tilde{c}_i(s)ds\}, i = 1, ..., N$.

Therefore (H2)-(H5) are all satisfied and $\Theta(s) = s, \int_1^\infty \frac{1}{\Theta(s)} ds = \infty$. If also the the associated conditions (H6) and (18) hold. Then, by Theorem 1, we can conclude that the problem (21)-(24) has at least one mild solution on $[0, 1]$.

Example 2. We consider the following impulsive fractional partial neutral functional differential equation of the form

$$\begin{aligned} &\frac{d}{dt} J_t^{1-\alpha} \Big[z(t, x) - \int_{-\infty}^{0} \int_0^\pi \tilde{b}_1(s, v, x) z(t + s, v) dv ds \\ &\quad - \varphi(0, x) + \int_{-\infty}^{0} \int_0^\pi \tilde{b}_1(s, v, x) z(s, v) dv ds \Big] \\ &= \frac{\partial^2}{\partial x^2} z(t, x) + \int_{-\infty}^{0} \tilde{b}_2(t, s) z(t + s, x) ds, \quad (25) \\ &\quad (t, x) \in \cup_{i=1}^N [s_i, t_{i+1}] \times [0, \pi], \end{aligned}$$

$$z(t, 0) = z(t, \pi) = 0, \quad t \in [0, \pi], \quad (26)$$

$$z(\tau, x) = \varphi(\tau, x), \quad (\tau, x) \in (-\infty, 0] \times [0, \pi], \quad (27)$$

$$z(t, x) = \int_{-\infty}^{0} \eta_i(s) z(t + s, x) ds, \quad (t, x) \in (t_i, s_i] \times [0, \pi], \quad (28)$$

where $\alpha, J_t^{1-\alpha}$ are the same with those in Example 4.1. Let $X = L^2([0, \pi])$ with the norm $\| \cdot \|$ and define the operators $A : D(A) \subseteq X \to X$ by $A\omega = \omega''$ with the domain $D(A) := \{\omega \in X : \omega, \omega''$ are absolutely continuous,$\omega'' \in X, \omega(0) = \omega(\pi) = 0\}$. Then

$$A\omega = \sum_{n=1}^{\infty} n^2 \langle \omega, \omega_n \rangle \omega_n, \quad \omega \in D(A),$$

where $\omega_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx), n = 1, 2, \ldots$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $T(t), t \geq 0$ in X and is given by $T(t)\omega = \sum_{n=1}^{\infty} e^{-n^2 t} (\omega, \omega_n) \omega_n, \omega \in X$, and $\| T(t) \| \leq e^{-t}$ for all $t \geq 0$. For each $\omega \in X$, $(-A)^{-\frac{1}{2}} \omega = \sum_{n=1}^{\infty} \frac{1}{n} \langle \omega, \omega_n \rangle \omega_n$ and $\| (-A)^{-\frac{1}{2}} \| = 1$. The operator $(-A)^{\frac{1}{2}}$ is given by

$$(-A)^{\frac{1}{2}} \omega = \sum_{n=1}^{\infty} n \langle \omega, \omega_n \rangle \omega_n$$

on the space $D((-A)^{\frac{1}{2}}) = \{\omega(\cdot) \in X, \sum_{n=1}^{\infty} n\langle\omega, \omega_n\rangle\omega_n \in X\}$. Moreover, $\| (-A)^{\frac{1}{2}}T(t) \| \leq \frac{1}{\sqrt{2e}}t^{-\frac{1}{2}}$ for all $t > 0$. In the next applications, the phase space $\mathcal{B} = \mathcal{PC}_0 \times L^2(h, X)$ is the space introduced in Example 1.

Additionally, we will assume that

(i) The functions $\frac{\partial^i b(s,v,x)}{\partial x^i}, i = 0, 1$, are measurable, $\tilde{b}_1(s, v, \pi) = \tilde{b}_1(s, v, 0) = 0$ for every (s, v) and

$$\tilde{L}_f = \max\Bigg\{\Bigg(\int_0^{\pi}\int_{-\infty}^0\int_0^{\pi}\frac{1}{h(s)} \times\bigg(\frac{\partial^i\tilde{b}_1(s,v,x)}{\partial x^i}\bigg)^2 dvdsdx\Bigg)^{\frac{1}{2}} : i = 0, 1\Bigg\} < \infty.$$

(ii) The function $\tilde{b}_2 : R^2 \to R$ is continuous with $\tilde{L}_F(t) = (\int_{-\infty}^0 \frac{(\tilde{b}_2(t,s))^2}{h(s)}ds)^{\frac{1}{2}} < \infty$ for all $t \in [0, b]$.

(iii) The functions $\eta_i : R \to R, i = 1, \ldots, N$, are continuous and $\rho_i = (\int_{-\infty}^0 \frac{(\eta_i(s))^2}{h(s)}ds)^{\frac{1}{2}} < \infty$ for every $i = 1, 2, \ldots, N$.

Defining the maps $f, F : [0, 1]\times\mathcal{B} \to X$, $g_i : (t_i, s_i]\times\mathcal{B} \to X, i = 1, ..., N$, by

$$f(t, \psi)(x) = \int_{-\infty}^0\int_0^{\pi}\tilde{b}_1(s, v, x)\psi(s, v)dvds,$$

$$F(t, \psi)(x) = \int_{-\infty}^0\tilde{b}_2(t, s)\psi(s, x)ds,$$

$$g_i(t, \psi)(x) = \int_{-\infty}^0\eta_i(s)\psi(s, x)ds.$$

Then the problem (25)-(28) can be written as (1)-(3). Noting that

$$\begin{aligned}&\langle f(t,\psi), \omega_n\rangle\\ &= \int_0^{\pi}\omega_n(x)\Big(\int_{-\infty}^0\int_0^{\pi}\tilde{b}_1(s,v,x)\psi(s,v)dvds\Big)dx\\ &= \frac{1}{n}\Big\langle\int_{-\infty}^0\int_0^{\pi}\frac{\partial}{\partial x}\tilde{b}_1(s,v,x)\\ &\quad\times\psi(s,v)dvds, \sqrt{\frac{2}{\pi}}\cos(nx)\Big\rangle,\end{aligned}$$

and for any $t \in [0, b]$,

$$\begin{aligned}&\| (-A)^{\frac{1}{2}}f(t,\psi) \|\\ &= \Big\|\sum_{n=1}^{\infty}n\langle f(t,\psi),\omega_n\rangle\omega_n\Big\|\\ &\leq \Big[\int_0^{\pi}\Big(\int_{-\infty}^0\Big(\int_0^{\pi}\Big(\frac{\partial b(s,v,x)}{\partial x}\Big)^2dv\Big)^{\frac{1}{2}}\\ &\quad\times \| \psi(s) \| ds\Big)^2dx\Big]^{\frac{1}{2}}\\ &\leq \Big(\int_0^{\pi}\int_{-\infty}^0\int_0^{\pi}\frac{1}{h(s)}\Big(\frac{\partial b(s,v,x)}{\partial x}\Big)^2dvdsdx\Big)^{\frac{1}{2}}\\ &\quad\times\Big(\int_{-\infty}^0 h(s) \| \psi(s) \|^2 ds\Big)^{\frac{1}{2}}\\ &\leq \tilde{L}_f \| \psi \|_{\mathcal{B}},\ \psi \in \mathcal{B},\end{aligned}$$

and

$$\begin{aligned}&\| (-A)^{\frac{1}{2}}f(t,\psi_1) - (-A)^{\frac{1}{2}}f(t,\psi_2) \|\\ &= \Big\|\sum_{n=1}^{\infty}n\langle f(t,\psi_1) - f(t,\psi_2),\omega_n\rangle\omega_n\Big\|\\ &\leq \Big[\int_0^{\pi}\Big(\int_{-\infty}^0\Big(\int_0^{\pi}\Big(\frac{\partial b(s,v,x)}{\partial x}\Big)^2dv\Big)^{\frac{1}{2}}\\ &\quad\times \| \psi_1(s) - \psi_2(s) \| ds\Big)^2dx\Big]^{\frac{1}{2}}\\ &\leq \Big(\int_0^{\pi}\int_{-\infty}^0\int_0^{\pi}\frac{1}{h(s)}\Big(\frac{\partial b(s,v,x)}{\partial x}\Big)^2dvdsdx\Big)^{\frac{1}{2}}\\ &\quad\times\Big(\int_{-\infty}^0 h(s) \| \psi_1(s) - \psi_2(s) \|^2 ds\Big)^{\frac{1}{2}}\\ &\leq \tilde{L}_f \| \psi_1 - \psi_2 \|_{\mathcal{B}},\ \psi_1,\psi_2 \in \mathcal{B}.\end{aligned}$$

Similarly to the Example 1, for any $t \in [0, b]$, we have

$$\| F(t, \psi) \| \leq \tilde{L}_F(t) \| \psi \|_{\mathcal{B}},\ \psi \in \mathcal{B},$$

and

$$\| F(t, \psi_1) - F(t, \psi_2) \| \leq \tilde{L}_F(t) \| \psi_1 - \psi_1 \|_{\mathcal{B}},\ \psi_1, \psi_2 \in \mathcal{B},$$

we can see that each bounded set $D \subset \mathcal{B}$,

$$\chi(F(t, D)) \leq \tilde{L}_F(t) \sup_{-\infty<\theta\leq 0}\chi(D(\theta)).$$

Moreover, we have for $t \in (t_i, s_i], i = 1, \ldots, N$,

$$\| g_i(t, \psi) \| \leq \rho_i \| \psi \|_{\mathcal{B}},\ \psi \in \mathcal{B},$$

and for $t_1, t_2 \in (t_i, s_i], i = 1, \ldots, N$, we get

$$\| g_i(t_1, \psi_1) - g_i(t_2, \psi_2) \| \leq \rho_i \| \psi_1 - \psi_1 \|_{\mathcal{B}},\ \psi_1, \psi_2 \in \mathcal{B},$$

we can see that each bounded set $D \subset \mathcal{B}$,

$$\chi(g_i(t, D)) \leq \rho_i \sup_{-\infty<\theta\leq 0}\chi(D(\theta)).$$

Therefore (H1)-(H5) are all satisfied. If also the the associated conditions (H6) and (20) hold. Then, by Theorem 2, the problem (25)-(28) admits a mild solution on $[0, b]$.

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