Degree Reduction of Disk Rational Bézier Curves Using Multi-objective Optimization Techniques

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Abstract—**In this paper, we start by introducing a novel disk rational Bézier based on parallel projection, whose properties are also discussed. Then applying weighted least squares, multi-objective optimization techniques and constrained quadratic programming, we achieve multi-degree reduction of this kind of disk rational Bézier curve. The paper also gives error estimation and shows some numerical examples to illustrate the validity of theoretical reasoning.**

Index Terms—**Disk rational Bézier curve, Multi-degree reduction, Weighted least squares, Constrained quadratic programming, Multi-objective optimization methods.**

I. Introduction

BECAUSE operations of geometric objects in current Computer-Aided Design systems are based on floating point arithmetic, representations of geometric objects are inaccurate and geometrical computations are approximate. In order to deal with this problem, interval arithmetic is used in the fields. In 1992, Sederberg and Farouki [1] formally introduced the concept of interval Bézier curve that can transfer a complete description of approximation errors along with the curves to applications in other systems. Inspired by Sederberg's work, Hu et al. [2] [3] [4] researched the algorithms for curve and surface intersections and solid modeling. Chen and Lou [5] discussed the problem of bounding interval Bézier curve with lower degree interval Bézier curve. However, as Chen pointed out [5], interval curve possesses two shortcomings: interval generally enlarge rapidly in a computational process and rectangular intervals are not rotationally symmetric. To overcome these shortcomings, Lin and Rokne [6] applied a disk to replace a rectangle. The corresponding interval curve are called disk Bézier curve. Since interval Bézier curve can't represent conic precisely, Hu et al. [3] [4] introduced interval non-uniform rational B-splines (INURBS) curve based on perspective projection. In 2011, using parallel projection, the first author of [7] defined a novel disk rational Bézier curve, whose error radius functions are Bézier polynomial functions.

One of the important theme for rational Bézier curve is degree reduction. This problem arises because of the limit of maximum degree for polynomial and the need of data compression [8]. In 1983, Farin [9] described a degree reduction method for rational Bézier curve for interactive interpolation and approximation. Later, Sederberg and Chang [10] achieved one degree reduction based on perturbing the numerator and denominator polynomials of a rational curve such that the best linear common divisor is canceled. Chen [11] applied the shifted Chebyshev polynomials to achieve the degree reduction of rational Bézier with $C^{(0,0)}$-continuity at end points. A deficiency of above methods is that reduced-weights may be negative. in 2010, Cai and Wang [12] researched $C^{(r,s)}$-continuity at end points using the Steepest Descent algorithm. In fact, the degree reduction of rational Bézier curve is a vector-valued optimization problem, so the multi-objective optimization method is used in this paper. For the details about multi-objective optimization method, the reader can see [13] or [16]. On the other hand, for the degree reduction of the error radius curve, we can transform it into solving a constrained quadratic programming problem.

This paper has the following structure: To ensure the structural integrity of this paper, in section 2, we review the definition of disk rational Bézier curve and its properties. In section 3, we propose an efficient algorithm to the problem of degree reduction of rational disk Bézier curve. In section 4, some examples are provided.

II. Disk rational Bézier curves

A. Disk rational arithmetic

A disk in the plane is defined to be the set

$$\begin{aligned}(\boldsymbol{q}) &= (x_0, y_0)_r \\ &= \{\boldsymbol{x} \in \mathbb{R}^2 \mid \|\boldsymbol{x} - \boldsymbol{q}\| \le r, r \in \mathbb{R}^+ \},\end{aligned}$$

whose centric point is $\boldsymbol{q}$ and radius is r.

For any two disks $(\boldsymbol{q}_i) = (x_i, y_i)_{r_i}, i = 1, 2$, the two operations are defined as follow

$$\begin{aligned}k(\boldsymbol{q}_i) &= (k\boldsymbol{q}_i) \\ &= (kx_i, ky_i)_{|k|r_i}, \forall\ k \in \mathbb{R}, i = 1, 2, \quad (1)\\ (\boldsymbol{q}_1) + (\boldsymbol{q}_2) &= (x_1 + x_2, y_1 + y_2)_{r_1 + r_2}. \quad (2)\end{aligned}$$

Equations (1) and (2) can be generalized as

$$\sum_{i=0}^{n} k_i(\boldsymbol{q}_i) = \left(\sum_{i=0}^{n} k_i x_i, \sum_{i=0}^{n} k_i y_i\right)_{\sum\limits_{i=0}^{n} |k_i| r_i}. \quad (3)$$

In homogeneous coordinates, a disk can be defined as

$$\begin{aligned}(\boldsymbol{P}^\omega) &= (\omega x_0, \omega y_0, \omega)_{\omega r} \\ &= \{\boldsymbol{x}^\omega = (\omega x, \omega y, \omega) \in \mathbb{R}^3 \mid \|\boldsymbol{x}^\omega - \boldsymbol{P}^\omega\| \le \omega r\}.\end{aligned}$$

Applying the perspective projection $\mathrm{H}(\cdot)$ to the disk $(\boldsymbol{P}^\omega)$ can yields a corresponding rational disk in plane $\omega = 1$. That is

$$(\boldsymbol{q}) = \mathrm{H}((\boldsymbol{P}^\omega)) = \left(\frac{X_0}{\omega}, \frac{Y_0}{\omega}\right)_{\frac{R}{\omega}} = (x_0, y_0)_r. \quad (4)$$

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In addition, the disk can also be represented by the homogeneous coordinates

$$\begin{aligned}(\boldsymbol{P}^{\omega}) &= (\omega x_0, \omega y_0, \omega)_r \\ &= \{\boldsymbol{x}^{\omega} = (\omega x, \omega y, \omega) \in \mathbb{R}^3 \mid \|\boldsymbol{x}^{\omega} - \boldsymbol{P}^{\omega}\| \le r\}.\end{aligned}$$

Applying the oblique projection $\mathrm{I}(\cdot)$ to the disk $(\boldsymbol{P}^{\omega})$ can yields another corresponding rational disk in plane $\omega = 1$. That is

$$(\boldsymbol{q}) = \mathrm{I}((\boldsymbol{P}^{\omega})) = \left(\frac{X_0}{\omega}, \frac{Y_0}{\omega}\right)_r = (x_0, y_0)_r. \tag{5}$$

Both equations (4) and (5) obviously agree with operation (3). Based on equations (3) and (4), a kind of disk rational Bézier curve can be defined and has been researched in [3] [4]. However, using oblique projection $\mathrm{I}(\cdot)$, we can define a novel kind of disk rational curve and its properties, such as end interpolation, affine invariant etc., are similar to the classic disk rational Bézier curve.

B. Disk rational Bézier curves

A disk rational Bézier curve of degree n with control disk points $(\boldsymbol{p}_i) = (x_i, y_i)_{r_i}$ and corresponding weights $\omega_i \in \mathbb{R}^+, i = 0, ..., n,$ is defined by

$$\begin{aligned}(\boldsymbol{p})(t) &= [\boldsymbol{p}(t); r(t)] \\ &= \left[\frac{\sum\limits_{i=0}^{n} \boldsymbol{p}_i \omega_i B_i^n(t)}{\sum\limits_{i=0}^{n} \omega_i B_i^n(t)}; \sum_{i=0}^{n} r_i B_i^n(t)\right],\end{aligned}$$

or be written in the basis form

$$\begin{aligned}(\boldsymbol{p})(t) &= [\boldsymbol{p}(t); r(t)] \\ &= \left[\sum_{i=0}^{n} \boldsymbol{p}_i R_i^n(t); \sum_{i=0}^{n} r_i B_i^n(t)\right],\end{aligned}$$

where

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}, (0 \le t \le 1),\ i = 0, ..., n,$$

are Bernstein polynomials,

$$R_i^n(t) = \frac{\omega_i B_i^n(t)}{\sum_{j=0}^{n} \omega_j B_j^n(t)},\ i = 0, 1, ..., n,$$

are the rational basis functions. $\boldsymbol{p}(t)$ and $r(t)$ are respectively called the center curve and the radius of the disk rational Bézier curve $(\boldsymbol{p})(t)$.

C. Properties of disk rational Bézier curves

A disk rational Bézier curve satisfies the following properties.

• **End interpolation:**

$$(\boldsymbol{p})(0) = (\boldsymbol{p}_0) \text{ and } (\boldsymbol{p})(1) = (\boldsymbol{p}_n).$$

• **Affine invariant:** Let $\mathcal{A}$ be an affine transformation (for example, a rotation, reflection, translation, or scaling), then

$$\mathcal{A}\frac{\sum\limits_{i=0}^{n} \omega_i(\boldsymbol{p}_i) B_i^n(t)}{\sum\limits_{i=0}^{n} \omega_i B_i^n(t)} = \frac{\sum\limits_{i=0}^{n} \omega_i \mathcal{A}(\boldsymbol{p}_i) B_i^n(t)}{\sum\limits_{i=0}^{n} \omega_i B_i^n(t)}$$

• **Convex hull:** The disk rational Bézier curve lies in the convex hull of the control disks.

Since the convex hull of control disks $(\boldsymbol{p}_i), i = 0, 1, ..., n,$ are the set of all convex combinations $\sum_{i=0}^{n} \alpha_i (\boldsymbol{p}_i)$, where $\alpha_i = R_i^n(t)$ and $\sum_{i=0}^{n} R_i^n(t) = 1$, and the property is desired.

• **Non-uniform convergence:** Since

$$\lim_{\omega_i \to +\infty} R_i^n(t) = \begin{cases} 0 & t = 0, i \ne 0, \\ 1 & t = 0, i = 0, \\ 1 & 0 < t < 1, i = 0, ..., n, \\ 1 & t = 1, i = n, \\ 0 & t = 1, i \ne n, \end{cases}$$

the disk rational Bézier curve converges non-uniformly on colsed interval $[0, 1]$,

• **De Casteljau algorithm:** For any $t \in [0, 1]$, $(\boldsymbol{p})(t)$ can be computed as follows:

$$\begin{cases} \boldsymbol{p}_i^j = (1-t)\frac{\omega_i^{j-1}}{\omega_i^j}\boldsymbol{p}_i^{j-1} + t\frac{\omega_{i+1}^{j-1}}{\omega_i^j}\boldsymbol{p}_{i+1}^{j-1}, \\ \omega_i^j = (1-t)\omega_i^{j-1} + t\omega_{i+1}^{j-1}, \\ r_i^j = (1-t) r_i^{j-1} + t r_{i+1}^{j-1}, \end{cases}$$

where $j = 1, ..., n$ and $i = 0, ..., n-j$.

• **Subdivision:** Let $c \in (0, 1)$ be a real number. Then $(\boldsymbol{p})(t)$ can be subdivided into two segments:

$$(\boldsymbol{p})(t) = \begin{cases} \left[\frac{\sum\limits_{i=0}^{n} \boldsymbol{p}_i^i(c)\omega_i^i(c) B_i^n(\frac{t}{c})}{\sum\limits_{i=0}^{n} \omega_i^i(c) B_i^n(\frac{t}{c})};\ \sum\limits_{i=0}^{n} r_i^i(c) B_i^n(\frac{t}{c})\right], \\ 0 \le t \le c, \\ \left[\frac{\sum\limits_{i=0}^{n} \boldsymbol{p}_n^{n-i}(c)\omega_n^{n-i}(c) B_i^n(\frac{t-c}{1-c})}{\sum\limits_{i=0}^{n} \omega_n^{n-i}(c) B_i^n(\frac{t-c}{1-c})};\ \sum\limits_{i=0}^{n} r_n^{n-i}(c) B_i^n(\frac{t-c}{1-c})\right], \\ c \le t \le 1, \end{cases}$$

• **Degree elevation:** A disk rational Bézier curve $(\boldsymbol{p})(t)$ of degree m can be represented as a disk rational Bézier curve of degree $m+s$ as follows

$$\begin{aligned}(\boldsymbol{p})(t) &= \frac{\sum\limits_{i=0}^{m} (\boldsymbol{p}_i)\omega_i B_i^m(t)}{\sum\limits_{i=0}^{m} \omega_i B_i^m(t)} \\ &= \frac{\sum\limits_{i=0}^{m+s} (\hat{\boldsymbol{p}}_i)\hat{\omega}_i B_i^{m+s}(t)}{\sum\limits_{i=0}^{m+s} \hat{\omega}_i B_i^{m+s}(t)}, 0 \le t \le 1,\end{aligned}$$

where

$$\hat{\omega}_i = \sum_{j=max(0,i-s)}^{min(m,i)} \omega_j \binom{m}{j} \frac{\binom{s}{i-j}}{\binom{m+s}{i}},$$

$$\hat{\boldsymbol{p}}_i = \frac{1}{\hat{\omega}_i} \sum_{j=max(0,i-s)}^{min(m,i)} \boldsymbol{p}_j \omega_j \binom{m}{j} \frac{\binom{s}{i-j}}{\binom{m+s}{i}},$$

and

$$\hat{r}_i = \sum_{j=max(0,i-s)}^{min(m,i)} r_j \binom{m}{j} \frac{\binom{s}{i-j}}{\binom{m+s}{i}}, \tag{6}$$

$i = 0, 1, ..., m+s.$

• **Exact degree reduction:**

A degree n disk rational Bézier curve $(\boldsymbol{p})(t)$ represents exactly a degree m $(m < n)$ disk ratioanl Bézier curve $(\check{\boldsymbol{p}})(t)$ with control disks $(\check{\boldsymbol{p}}_i)$ and weights $\check{\omega}_i \in \mathbb{R}^+$, $i = 0, 1, ..., m$, if only if the following equations are satisfied

$$\begin{aligned}&\sum_{j=max(0,i-n)}^{min(m,i)} \frac{\binom{m}{j}\binom{n}{i-j}}{\binom{m+n}{i}}\check{\omega}_j\omega_{i-j}\boldsymbol{p}_{i-j}\\ =&\sum_{j=max(0,i-n)}^{min(m,i)} \frac{\binom{m}{j}\binom{n}{i-j}}{\binom{m+n}{i}}\check{\omega}_j\omega_{i-j}\check{\boldsymbol{p}}_j,\\ &i = 0, 1, ..., n+m,\end{aligned} \tag{7}$$

and

$$r_k = \sum_{j=max(0,k-n+m)}^{min(m,k)} \check{r}_j \binom{m}{j} \frac{\binom{n-m}{k-j}}{\binom{n}{k}}, \tag{8}$$

$k = 0, 1, \ldots, n$.

For the center curve, by

$$\boldsymbol{p}(t) = \check{\boldsymbol{p}}(t),$$

we have

$$\sum_{i=0}^{n}\boldsymbol{p}_i\omega_i B_i^n(t)\sum_{i=0}^{m}\check{\omega}_i B_i^m(t) = \sum_{i=0}^{n}\check{\boldsymbol{p}}_i\check{\omega}_i B_i^n(t)\sum_{i=0}^{m}\omega_i B_i^m(t),$$

which, after some rearrangement of the equation, gives

$$\begin{aligned}&\sum_{i=0}^{m+n}\sum_{j=max(0,i-m)}^{min(m,i)} \frac{\binom{m}{j}\binom{n}{i-j}}{\binom{m+n}{i}}\check{\omega}_j\omega_{i-j}\boldsymbol{p}_{i-j}B_i^{m+n}(t)\\ =&\sum_{i=0}^{m+n}\sum_{j=max(0,i-m)}^{min(m,i)} \frac{\binom{m}{j}\binom{n}{i-j}}{\binom{m+n}{i}}\omega_{i-j}\check{\omega}_j\check{\boldsymbol{p}}_j B_i^{m+n}(t).\end{aligned}$$

Comparing coefficients of like terms on both sides of the equation, this establishes the equation (7).

The proof for the radius (8) is straightforward by equation (6). By equation (7), it is clear that if

$$\begin{aligned}C_j &= \sum_{j=max(0,i-m)}^{min(m,i)} \frac{\binom{m}{j}\binom{n}{i-j}}{\binom{m+n}{i}}(\check{\omega}_j\omega_{i-j}\boldsymbol{p}_{i-j} - \omega_{i-j}\check{\omega}_j\check{\boldsymbol{p}}_j)\\ &= 0,\end{aligned} \tag{9}$$

where $j = 0, ..., \mu - 1$, and $j = n + n - \nu + 1, ..., n + m$, then the curves $\boldsymbol{p}(t)$ and $\check{\boldsymbol{p}}(t)$ satisfy $C^{(\mu,\nu)}$ - continuity.

III. Degree reduction of disk rational Bézier curves

The problem of degree reduction of disk Rational Béziers curve can be stated as follows:

Given a degree n disk rational Bézier curve $(\boldsymbol{p})(t)$, find a degree $m < n$ disk rational Bézier curve $(\check{\boldsymbol{p}})(t)$ such that $(\check{\boldsymbol{p}})(t)$ is a closure of $(\boldsymbol{p})(t)$.

The above problem can be decomposed into two parts.

A) *Degree reduction approximation of center curve.*

Using weighted least squares, the degree reduction of rational Bézier curve with $C^{(\mu,\nu)}$ - continuity can be expressed as the following mathematical formula

$$\begin{cases}\min & \int_0^1 \rho(t)(\boldsymbol{p}(t) - \check{\boldsymbol{p}}(t))^2 dt\\ s.t. & C_j = 0, \quad j = 0, ..., \mu - 1,\\ & C_j = 0, \quad j = n + n - \nu + 1, ..., n + m,\\ & \check{\omega}_i > 0, \quad i = 0, ..., m,\end{cases} \tag{10}$$

where C_j are given by equation (9).

Specifically, let

$$\boldsymbol{p}(t) = \frac{\boldsymbol{q}(t)}{\omega(t)} = \frac{\sum_{i=0}^{n}\omega_i\boldsymbol{p}_i B_i^n(t)}{\sum_{i=0}^{n}\omega_i B_i^n(t)},$$

$$\check{\boldsymbol{p}}(t) = \frac{\check{\boldsymbol{q}}(t)}{\check{\omega}(t)} = \frac{\sum_{i=0}^{m}\check{\omega}_i\check{\boldsymbol{p}}_i B_i^m(t)}{\sum_{i=0}^{m}\check{\omega}_i B_i^m(t)}$$

and $\rho(t) = (\omega(t)\check{\omega}(t))^2$. The objective functions of equation (10) can be written as

$$\begin{aligned}(f_1, f_2) &= \int_0^1 \rho(t)(\boldsymbol{p}(t) - \check{\boldsymbol{p}}(t))^2 dt\\ &= \int_0^1 (\check{\omega}(t)\boldsymbol{q}(t) - \omega(t)\check{\boldsymbol{q}}(t))^2 dt\\ &= \frac{1}{2m+2n+1}\sum_{i=0}^{2n+2m}\sum_{j=max(0,i-2n)}^{min(2m,i)} H_{ij}\end{aligned}$$

where

$$\begin{aligned}H_{ij} &= \frac{\binom{2m}{j}\binom{2n}{i-j}}{\binom{2m+2n}{i}}(A_j B_{i-j} - 2C_j D_{i-j} + E_j F_{i-j}),\\ A_k &= \sum_{j=max(0,k-m)}^{min(m,k)} \frac{\binom{m}{j}\binom{m}{k-j}}{\binom{2m}{k}}\check{\omega}_j\check{\omega}_{k-j},\\ B_k &= \sum_{j=max(0,k-n)}^{min(n,k)} \frac{\binom{n}{j}\binom{n}{k-j}}{\binom{2n}{k}}\omega_j\boldsymbol{p}_j\omega_{k-j}\boldsymbol{p}_{k-j},\\ C_k &= \sum_{j=max(0,k-m)}^{min(m,k)} \frac{\binom{m}{j}\binom{m}{k-j}}{\binom{2m}{k}}\check{\omega}_j\check{\boldsymbol{p}}_j\check{\omega}_{k-j},\\ D_k &= \sum_{j=max(0,k-n)}^{min(n,k)} \frac{\binom{n}{j}\binom{n}{k-j}}{\binom{2n}{k}}\omega_j\boldsymbol{p}_j\omega_{k-j},\\ E_k &= \sum_{j=max(0,k-m)}^{min(m,k)} \frac{\binom{m}{j}\binom{m}{k-j}}{\binom{2m}{k}}\check{\omega}_j\check{\boldsymbol{p}}_j\check{\omega}_{k-j}\check{\boldsymbol{p}}_{k-j}\end{aligned}$$

and

$$F_k = \sum_{j=max(0,k-n)}^{min(n,k)} \frac{\binom{n}{j}\binom{n}{k-j}}{\binom{2n}{k}}\omega_j\omega_{k-j}.$$

Many methods can be used to solve the above equation . The weighted-sum-of-objective-functions method [13] is used in this paper. That is, a new objective function is

$$f = \frac{1}{2}(f_1 + f_2).$$

Accordingly, we use the *fmincon* procedure of MATLAB to solve the nonlinear programming and the algorithm option is the *interior-point* method [14] [15].

B) *Degree reduction approximation of error radius curve*

The problem of degree reduction approximation of error radius curve can be expressed as the following formula

$$\begin{cases} min & \|\check{r}(t) - r(t)\|_2^2\\ s.t. & \check{r}(t) \geq r(t) + dist(\boldsymbol{p}(t), \check{\boldsymbol{p}}(t))\\ & \check{r}_j > 0, \ j = 0, 1, ..., m.\end{cases} \tag{11}$$

TABLE I
ERROR AND WEIGHT COMPARISONS OF THE SEDERBERG'S, CHEN'S AND OUR METHODS

Method	Constraint	Weights	Error
Sederberg's method	N/A	-0.4774, -2.4585, -0.6294, -0.9129	0.0276
Chen's method	$C^{(0,0)}$	-0.4768, -2.4570, -0.6293, -0.9112	0.0376
Cai and Wang's method	$C^{(0,0)}$	0.3719,1.8149, 0.5698,0.7186	0.0138
Our method	$C^{(0,0)}$	1.6962, 8.3632,2.5549,3.1139	0.0168

where $dist(\boldsymbol{p}(t),\check{\boldsymbol{p}}(t)), t \in [0,1]$, is the Hausdorff distance between the curve $\check{\boldsymbol{p}}(t)$ and the curve $\boldsymbol{p}(t)$.

In practice, the equation (11) can be further simplified as quadratic programming.

For the constraint function, elevating the degree of the error radius curve $\check{r}(t)$ from m to n by equation (8), we have

$$\check{r}(t) = \sum_{i=0}^{m} \check{r}_i B_i^m(t) = \sum_{i=0}^{n} \hat{r}_i B_i^n(t),$$

where

$$\hat{r}_i = \sum_{j=max(0,i-n+m)}^{min(m,i)} \check{r}_j \binom{m}{j} \frac{\binom{n-m}{i-j}}{\binom{n}{i}}.$$

Then one of a sufficient condition to satisfy the equation (11) can be stated as

$$\hat{r}_i > r_i + d, \quad i = 0, 1, ..., n,$$

where

$$d = max\{ \max_{\boldsymbol{p}(t_i)\in\boldsymbol{p}(t),\, \check{\boldsymbol{p}}(t_j)\in\check{\boldsymbol{p}}(t)} \min \|\boldsymbol{p}(t_i) - \check{\boldsymbol{p}}(t_j)\|, \max_{\check{\boldsymbol{p}}(t_i)\in\check{\boldsymbol{p}}(t),\, \boldsymbol{p}(t_j)\in\boldsymbol{p}(t)} \min \|\boldsymbol{p}(t_i) - \boldsymbol{p}(t_j)\|\} \quad (12)$$

and $\|\boldsymbol{p}(t_i) - \check{\boldsymbol{p}}(t_j)\|$ is discrete Euclidean distance, $\boldsymbol{p}(t_i)$ and $\check{\boldsymbol{p}}(t_i)$, $i = 0, 1, \cdots, M$, are discrete sample points on curves $\boldsymbol{p}(t)$ and $\check{\boldsymbol{p}}(t)$.

For the objective function, we have

$$\begin{aligned} & \|r(t) - \check{r}(t)\|_2^2 \\ &= \int_0^1 (r(t) - \check{r}(t))^2 \, dt \\ &= \int_0^1 \check{r}^2(t)dt - 2\int_0^1 r(t)\check{r}(t)dt + \int_0^1 r^2(t)dt \\ &= \sum_{i=0}^{m}\sum_{j=0}^{m} \check{r}_i\check{r}_j H_{ij} - 2\sum_{i=0}^{m}\sum_{j=0}^{n} \check{r}_i r_j S_{ij} + \sum_{i=0}^{n}\sum_{j=0}^{n} r_i r_j G_{ij}, \end{aligned}$$

where $H_{ij} = \frac{\binom{m}{i}\binom{m}{j}}{(2m+1)\binom{2m}{i+j}}$, $S_{ij} = \frac{\binom{m}{i}\binom{n}{j}}{(m+n+1)\binom{m+n}{i+j}}$ and $G_{ij} = \frac{\binom{n}{i}\binom{n}{j}}{(2n+1)\binom{2n}{i+j}}$. The third term $\sum_{i=0}^{n}\sum_{j=0}^{n} r_i r_j G_{ij}$ is constant and can be omitted. So the problem of degree reduction of error radius function can be transformed to find the optimal solution of the following problem:

$$\begin{cases} min & \sum_{i=0}^{m}\sum_{j=0}^{m} \check{r}_i\check{r}_j H_{ij} - 2\sum_{i=0}^{m}\sum_{j=0}^{n} \check{r}_i r_j S_{ij} \\ s.t. & \hat{r}_i \geq r_i + d, \quad i = 0, 1, \ldots, n, \\ & \check{r}_j > 0, \qquad j = 0, 1, ..., m. \end{cases}$$

Similarity to the equation (12), the error between curves $r(t)$ and $\check{r}(t)$ is defined as

$$\begin{aligned} e = max\{ & \max_{\boldsymbol{r}(t_i)\in\boldsymbol{r}(t),\, \check{\boldsymbol{r}}(t_j)\in\check{\boldsymbol{r}}(t)} \min \|\boldsymbol{r}(t_i) - \check{\boldsymbol{r}}(t_j)\|, \\ & \max_{\check{\boldsymbol{r}}(t_i)\in\check{\boldsymbol{r}}(t),\, \boldsymbol{r}(t_j)\in\boldsymbol{r}(t)} \min \|\boldsymbol{r}(t_i) - \boldsymbol{r}(t_j)\|\} \end{aligned}$$

where $r(t_j)$ and $\check{r}(t_j)$, $j = 0, 1, \cdots, M$, are discrete sample points on curves $r(t)$ and $\check{r}(t)$.

IV. EXAMPLES

Example 1 (Also example in [10]). Given a 4 degree rational Bézier curve with control points $(0,0)$, $(2,2)$, $(3,0)$, $(4,-2)$, $(4,0)$ and associated weights $1, 4, 2, 1, 1$, to find a 1-degree reduced rational Bézier curve to a approximate the original curve. See Table 1 for comparisons of approximation error, and Fig. 1 for illustration. Although our method's error is larger than Cai and Wang's method [12], we find that our resulting curve may be better than others by Fig. 2 and Fig. 3.

Example 2. Given a disk rational Bézier $(\boldsymbol{p})(t)$ of eight degree with control disks $(6, 14.9)_1$, $(8.6, 25)_{0.4}$, $(20.3, 30)_1$,$(35, 31)_{1.5}$, $(40.2, 25)_2$, $(37.5, 11.5)_{1.8}$, $(47.2, 8.1)_{0.8}$,$(65.1, 11.2)_1$, $(71.5, 25)_{0.5}$ and associated weights 1.88, 1.68, 1.63, 1.73, 1.79, 2.18, 1.24, 1.08, 1.9. The best 3-degree reduction curve satisfying $C^{(1,1)}$-continuity with the given curve has control disks $(6.0, 14.9)_{1.2577}$, $(10.3896, 31.9518)_{0.2977}$, $(44.3244, 39.6231)_{2.8827}$, $(38.4055, 12.9182)_{2.0760}$, $(48.2014, -25.2377)_{1.5577}$, $(71.5, 25.0)_{0.7577}$, and associated weights 1.8658, 1.5801, 1.4595, 2.7315, 0.4741, 1.8977. (See Fig. 4 and Fig. 5). The error distances of center curve and the error radius curve are 0.2577 and 0.3938, respectively (See Fig. 6 and Fig. 7).

V. CONCLUSION

In this paper, we discussed the problem of degree reduction of disk rational Bézier curves and proposed an efficient method to solve the problem. Theoretic results and experiments show that the proposed algorithm produces is very effective. The idea presented in this paper can be easily generalized to solve the degree reduction problem of disk rational Bézier surfaces and NURBS curves and surfaces.

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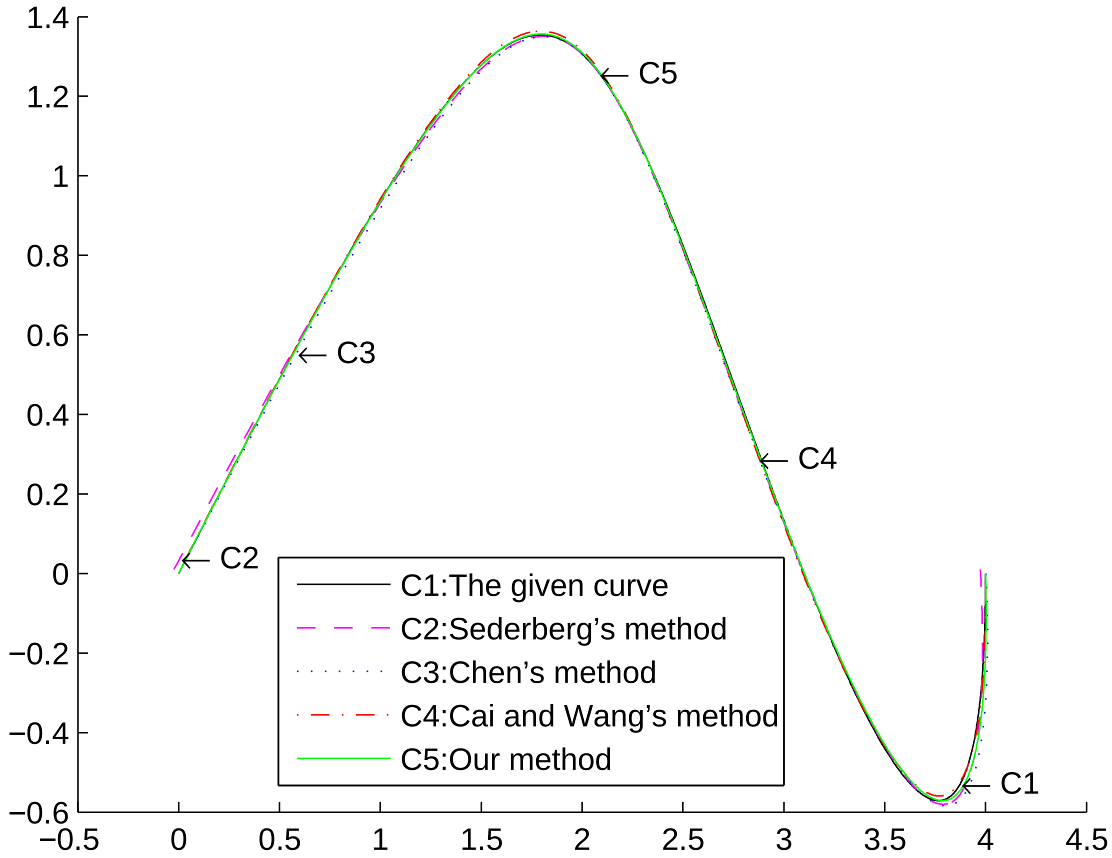

Fig. 1. Comparison of four degree reduction methods. (For interpretation of the reference to color in this figure legend, the reader is referred to the web version of this article.)

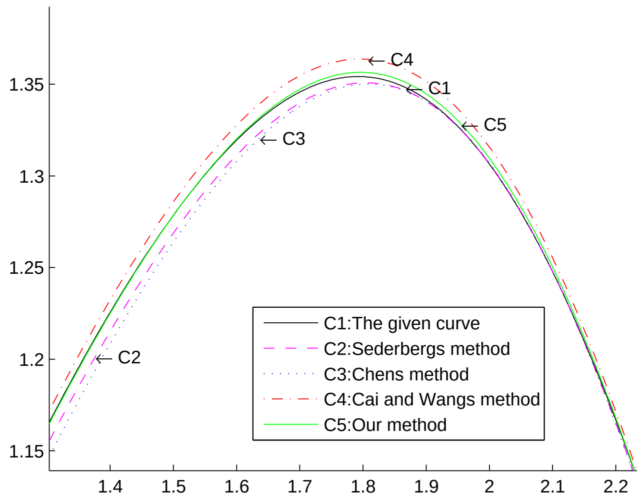

Fig. 2. The photomicrograph of Fig. 1 on the interval [1.3, 2.2]. (For interpretation of the reference to color in this figure legend, the reader is referred to the web version of this article.)

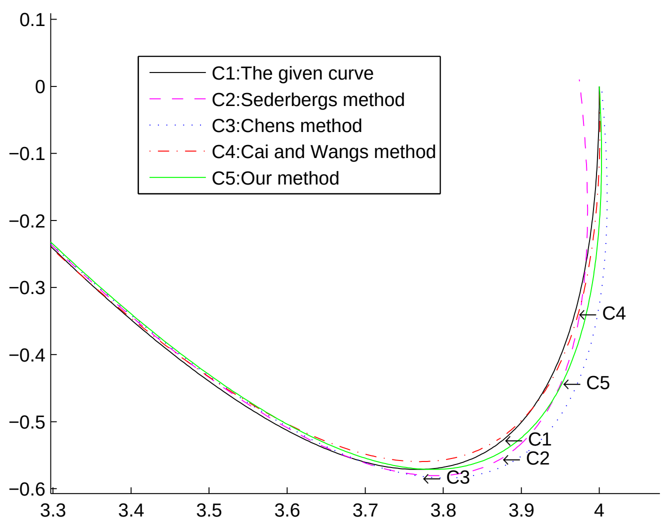

Fig. 3. The photomicrograph of Fig. 1 on the interval [3.3, 4.1]. (For interpretation of the reference to color in this figure legend, the reader is referred to the web version of this article.)

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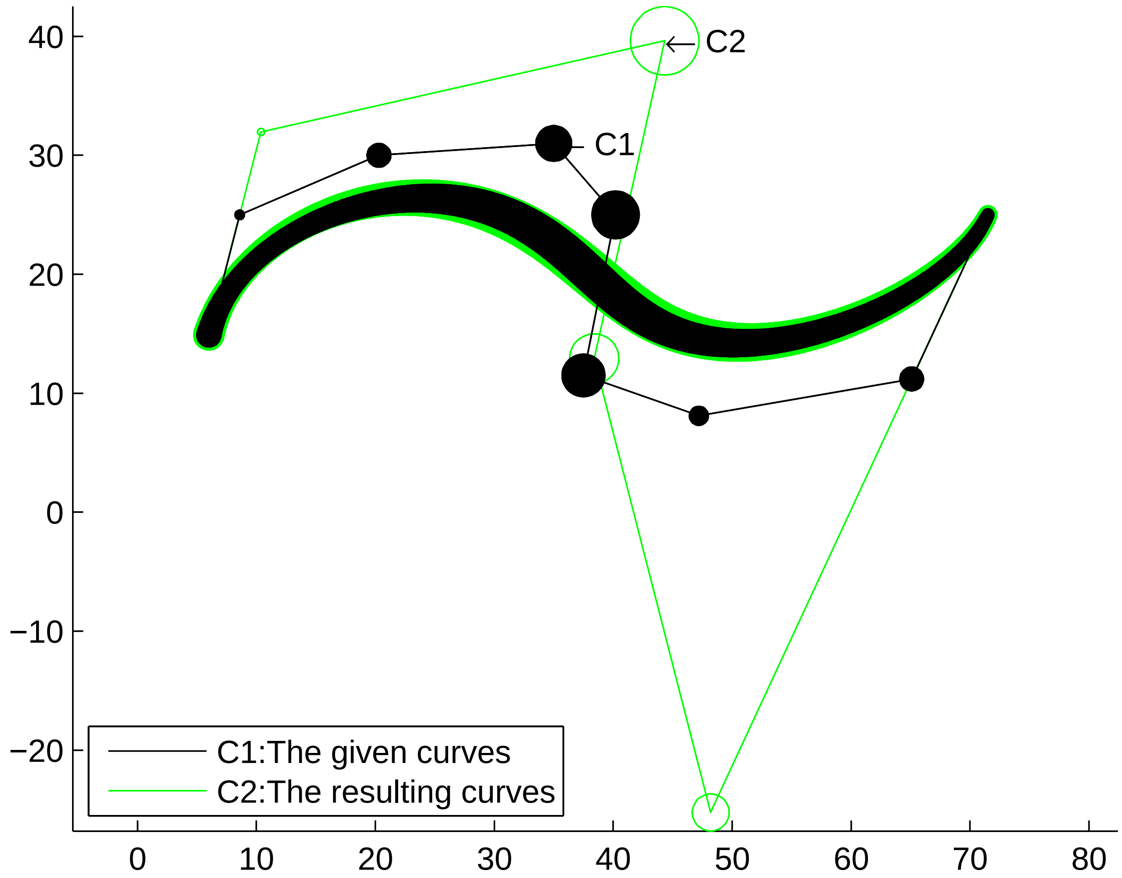

Fig. 4. A degree 8 disk rational Bézier curve (with black) and its three-degree reduced curve (with blue)with $C^{(1,1)}$continuity at two endpoints. (For interpretation of the reference to color in this figure legend, the reader is referred to the web version of this article.)

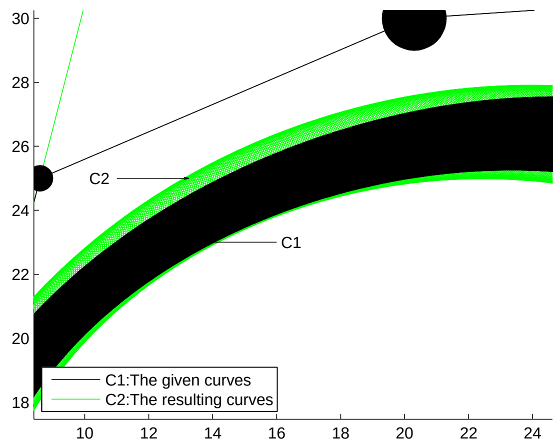

Fig. 5. The photomicrograph of Fig. 1 on the interval [9.0, 24.0]. (For interpretation of the reference to color in this figure legend, the reader is referred to the web version of this article.)

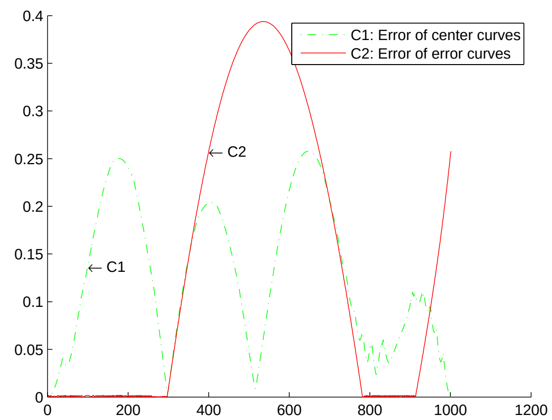

Fig. 6. The corresponding error distance curve of center rational Bézier curve and error curve. (For interpretation of the reference to color in this figure legend, the reader is referred to the web version of this article.)

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