

Construction of the Transreal Numbers and Algebraic Transfields

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Abstract—The transreal numbers, introduced by James Anderson, are an extension of the real numbers. The four arithmetical operations of addition, subtraction, multiplication and division are closed on the set of transreal numbers. Transreal arithmetic has engendered controversy because it allows division by zero and is proposed as a replacement for real arithmetic. Anderson introduced the transreals intuitively and axiomatically. In the history of mathematics, constructive proofs have ended controversies. We construct the transreal numbers and transreal arithmetic from the very well accepted real numbers and real arithmetic. This construction proves consistency. We then extend the very well accepted algebraic structure of a field to a transfield. We show that, just as the rationals are the smallest, ordered field and reals are the unique, ordered, complete field, so, under suitable conditions, transrationals are the smallest, ordered transfield and transreals are the smallest, ordered, complete transfield. Thus we both prove consistency and demonstrate the wider applicability of the transreals. We hope this does enough to end controversy about the correctness of the transreals, leaving an assessment of their usefulness to future experience.

Index Terms—transreal numbers, division by zero, nonstandard arithmetic, algebraic transfields.

I. INTRODUCTION

CONTEMPORARY computers have a processing limitation arising from the need to have control mechanisms to handle division by zero. Providing detection and processing of such divisions causes an excessive expenditure of space in the processor, slows processing time, wastes electrical energy and wastes programmer time in anticipating and handling errors. James Anderson designed a supercomputer which must handle physical faults but which does not need any special mechanism to deal with division by zero or any logical exceptions [3][11].

According to Anderson, his computer has several advantages. Every operation can be applied to any arguments giving a valid result, every syntactically correct arithmetical sentence is semantically correct and, consequently, there are no exceptions. Because of this, pipelines never break and programs can crash only on a physical fault. Compared to the IEEE standard for floating-point arithmetic, transreal arithmetic doubles absolute, not relative, numerical precision by re-using a redundant binade. Furthermore, it also removes superfluous relational operations. Anderson states that the

unordered relation is logically redundant, having utility only in the IEEE 754 model of error handling. Thus transreal arithmetic simplifies the relational operators, simplifies programming and removes an entire class of errors. Therefore, transreal arithmetic is better suited to safety critical applications [8]. Moreover, Anderson proposes the use of transreal arithmetic to define pipeline machines that are infinitely scalable. With these machines exascale processing may well be technologically achievable. Using his total arithmetic, Anderson makes unbreakable pipelines where, apart from physical errors, a core is guaranteed to pass on its data every clock tick. Anderson arrived at this position in a succession of three architectures. First he exploited a single instruction implemented in a fixed-point arithmetic. Then the architecture was modified, by the introduction of a second operation, to operate on trans-floating-point numbers. Finally the architecture was completely pipelined [11].

Having outlined the technological background, let us now concentrate on mathematical details. In the 2000s Anderson proposed the set of transreal numbers. This set, denoted by \mathbb{R}^T , is the set of real numbers, together with three new elements: $-1/0$, $1/0$ and $0/0$ [3]. Like many axiomatic theories, the transreals have raised doubts and controversies, as Martinez relates in his book about an interview with Anderson on BBC television: “Many readers have replied that division by zero is clearly ‘impossible.’ Some have complained that the everyday examples Anderson gives are defective or ‘obviously ridiculous’ because airplanes and heart pacemakers have control mechanisms that are programmed to handle exceptions, to prevent internal calculators from dividing by zero” [17]. Herein lies the importance of our present paper. We emphasise that, although it does not deal directly with Anderson’s computer, our paper is in the interest of Computer Science because it provides a proof of consistency of the arithmetic used in Anderson’s computer. Our paper may be of interest to the History of Mathematics because it draws parallels between the development of the transreals and other number systems. Finally our paper may be of interest to contemporary Mathematics because it shows how the very well accepted algebraic structure of a field, generalises to a transfield, which we introduce here.

Before examining the issue of division by zero, let us apprise the reader of the development of transreal numbers. Division by zero has been considered for more than a thousand years, during which time many approaches have been explored.

Anderson provided a solution to division by zero inspired by projective geometry [1]. Since then several studies have developed the transreal numbers. In [2] Anderson considers the syntactic application of the rules for adding and multiplying fractions, notwithstanding the fact that fractions may have a zero denominator. In [4] Anderson proposes the set

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of transrational numbers, $\mathbb{Q}^T := \mathbb{Q} \cup \{-1/0, 1/0, 0/0\}$. In [5], a list of axioms that establish the set and arithmetic of the transreal numbers is given. In [6] Anderson extends the trigonometric, logarithmic and exponential functions to the transreal numbers and, in [7], Anderson proposes a topology for transreal space and establishes the transmetric.

Martinez, in his book on the history of mathematics [17], talks about the paradigm of division by zero, about how it was treated throughout history until we arrive at Anderson's proposal. In [20], Reis, Gomide and Kubrusly draw an analogy between the present moment in the development of the transreal numbers and the historical development of several other kinds of numbers. Further, in [15], Gomide and Reis make a study of the motivations for Anderson's conception of nullity and compare the transfinite numbers of Cantor to the transreals, stating that the latter allow the extension of the concept of a metric to infinite and indeterminate distances.

Anderson argues [8] that trans-floating-point arithmetic removes nine quadrillion redundancies from 64-bit, IEEE 754, floating-point arithmetic. Anderson and Gomide [10] propose an arithmetisation of paraconsistent logic using the transreals. Anderson and Reis [9] formalise the concepts of limit and continuity in transreal space and elucidate the transreal tangent. Arguing from this firm foundation, they maintain that there are three category errors in the IEEE 754 standard. Firstly the claim that IEEE infinities are limits of real arithmetic confuses limiting processes with arithmetic. Secondly a defence of IEEE negative zero confuses the limit of a function with the value of a function. Thirdly the definition of IEEE NaN confuses *undefined* with *unordered*. All of which speaks to the value of transreal numbers in developing computer hardware.

Reis and Anderson [22] rehearse the concepts of derivative and integral in transreal space. This paper won an award for best paper at the International Conference on Computer Science and Applications 2014. In [23] they develop the set of transcomplex numbers. Reis [21] proposes a contextual interpretation for arithmetical operations between transreals. In [25] Reis and Anderson continue the development of transreal calculus. They extend the real derivative and integral to a transreal derivative and integral. This demonstrates that transreal calculus contains real calculus and operates at singularities where real calculus fails. Hence software that implements calculus, such as is used in scientific, engineering and financial applications, is extended to operate at singularities where it currently fails. This promises to make software both more competent and more reliable.

Reis and Anderson [24] extend all elementary functions of real numbers to transreal numbers. Gomide, Reis and Anderson [16] propose a logical space based on total semantics. This is a semantics containing the classical truth values, fuzzy values, a contradiction value and a gap value. This logical space of propositions is inspired by Wittgenstein's conception of a logical space [31]. It is a trans-Cartesian space where axes are possible worlds, co-ordinates are transreal numbers and points are propositions. Besides the above mentioned texts, transmathematics was the subject of the first author's doctoral thesis. In addition to the above matters, dos Reis's thesis makes a digression about the challenge for transmathematics of entering the select group of theories that

are well accepted by academia [26]. This is the world's first doctorate to be awarded for transmathematics. Transmathematics was also the subject of postdoctoral research by the second author. Gomide focused on the philosophical study of transmathematics in the study of logic. This is the world's first post-doctoral research in transmathematics.

Anderson proposes the existence of transreal numbers in an axiomatic way. By contrast, in our current paper, we propose a construction of the transreals from the reals. Thus the transreal numbers and their arithmetic arise as a consequence of the reals and not via free-standing axioms. We define transreal numbers by a particular class of subsets of pairs of real numbers and then define addition and multiplication operations (by using addition and multiplication of real numbers) on these pairs and show that there is a "copy" of the reals within this class. This class of ordered pairs plays the role of the set of transreal numbers. Then we establish on \mathbb{R}^T some arithmetical and ordering properties of a field. We then generalise algebraic fields to transfields so that they support arithmetics where the operations of addition, subtraction, multiplication, and division are total functions. We give several different examples of transfields, derive some of their properties. We show that, just as the rationals are the smallest, ordered field and reals are the unique ordered, complete field, so, under suitable conditions, transrationals are the smallest, ordered transfield and transreals are the smallest, ordered, complete transfield.

II. INITIAL CONSIDERATIONS

The impossibility of division by zero, in the real numbers, is well known. One of the difficulties in defining such division is that both historical and currently popular interpretations of the division operation are not valid when the divisor is zero. For example the integral equality $n/d = m$ can be interpreted as follows: n objects can be divided into (set out as) d groups of m objects. This account of division makes no sense, for non-zero n , when d is zero because there is no number such that zero groups ($d = 0$), of any m objects, sum to $n \neq 0$. Even if we dispense with historical and pedagogical models, by operating formally, a problem remains. Division in the real numbers, \mathbb{R} , is multiplication by the multiplicative inverse. That is, if $a, b \in \mathbb{R}$ and $b \neq 0$ then a/b means $a \times b^{-1}$, where b^{-1} is a real number such that $b \times b^{-1} = 1$. Now if we wish to allow a denominator of zero, we must have a multiplicative inverse of zero. This is not possible in the usual definition of multiplication because, if there is $c \in \mathbb{R}$ such that $0 \times c = 1$, we would have $0 = 0 \times c = 1$, which is absurd! That said, it is clear that if we want to divide by zero, we need to extend the definition of division and, perhaps, the definition of number.

We believe that the transreal numbers are going through a common process in the history of mathematics. The real numbers themselves were initially conceived intuitively. Positive integers and positive rationals are present in the earliest records of mathematics but the recognition of irrational numbers is attributed to the Greeks of the fourth century BC [28]. Over time the real numbers were widely used and were informally understood to be in a bijective correspondence with the set of points on a straight line. Despite this understanding, for many people the irrational numbers were not accepted as numbers, but as convenient objects in certain

studies [28]. The advent of the differential and integral calculus, around seventeenth century, brought new ideas and, together with these new ideas, controversies about their methods. These controversies were partially responsible for causing a move toward the formalisation of the mathematical concepts of number, in other words, the establishment of numbers without the assumption of geometrical intuition.

In the eighteenth century, efforts were made to formalise the real numbers, but their consolidation occurred only in the nineteenth century with a construction from the rational numbers by Dedekind. Dedekind's motivation was to establish the set of real numbers, not just by the admission of its existence, but by constructing the real numbers from numbers that were already established.

Another example of this constructive process occurred with the complex numbers. When, in the sixteenth century, Bombelli found the square root of a negative number, while solving an equation of the third degree, he had the courage to operate on this object by assuming that it followed the arithmetical properties of the real numbers. He found, indeed, at the end of his calculation, a solution to the equation in question. At that moment, Bombelli was not preoccupied with rigour nor with the interpretation of the strange object, he was just brave and supposed the existence of new entities that could be called numbers.

Complex numbers were studied by other mathematicians but with the phenomenological quality of being *imaginary* numbers. Only in the nineteenth century, did Hamilton give a rigorous definition of the complex numbers and their arithmetic, deducing their properties from the properties of real numbers.

The odyssey of the hyperreal numbers should also be mentioned. One of the fundamental concepts of the differential and integral calculus is now understood to be the limit. But Leibniz, one of the founders of modern calculus, did not use the idea of a limit. For instance he did not take the limit of a number tending to zero. Leibniz took a fixed number that was infinitely close to zero [14]. Even without a rigorous definition of infinitesimal numbers, formalising the idea of infinitely small numbers, Leibniz was still able to deduced several results of modern calculus. The infinitesimals suffered severe criticism and only in the 1960s did Robinson [27] construct the infinitesimal numbers from the real numbers and deduce the properties foreseen by Leibniz.

Just as the aforementioned numbers were introduced intuitively, before being formalised with constructive proofs, so the transreals were initially proposed, by Anderson, using intuitions about the syntax of the rules for adding and multiplying fractions. Now the time has come for the transreals to receive a rigorous, constructive proof of consistency.

III. CONSTRUCTION OF THE TRANSREAL NUMBERS FROM THE REAL NUMBERS

In what follows, we propose a construction of the transreal numbers from the real numbers. We define, for a given class of subsets of pairs of the real numbers, arithmetical operations (using real arithmetical operations) and we show that there is a copy of the real numbers in this class. Thus the transreal numbers and the arithmetic proposed by Anderson become consequences of these definitions and of the properties of real numbers.

Definition 1: Let $T := \{(x, y); x, y \in \mathbb{R} \text{ and } y \geq 0\}$. Given $(x, y), (w, z) \in T$, we say that $(x, y) \sim (w, z)$, that is (x, y) is equivalent to (w, z) , with respect to \sim , if and only if there is a positive $\alpha \in \mathbb{R}$ such that $x = \alpha w$ and $y = \alpha z$.

Proposition 2: The relation \sim is an equivalence relation on T .

Proof: The reflexivity of \sim is immediate. Now let $(x, y), (w, z), (u, v) \in T$ such that $(x, y) \sim (w, z)$ and $(w, z) \sim (u, v)$. Then there are positive $\alpha, \beta \in \mathbb{R}$ such that $x = \alpha w, y = \alpha z, w = \beta u$ and $z = \beta v$. Symmetry follows from $w = \frac{1}{\alpha}x$ and $z = \frac{1}{\alpha}y$. Transitivity follows from $x = \alpha\beta u$ and $y = \alpha\beta v$. ■

For each $(x, y) \in T$, let us denote by $[x, y]$ the equivalence class of (x, y) , that is

$$[x, y] := \{(w, z) \in T; (w, z) \sim (x, y)\}.$$

Let us denote by T/\sim the quotient set of T with respect to \sim , that is $T/\sim := \{[x, y]; (x, y) \in T\}$.

Proposition 3: It follows that

$$T/\sim = \{[t, 1]; t \in \mathbb{R}\} \cup \{[0, 0], [1, 0], [-1, 0]\}.$$

Furthermore the elements $[t, 1], [0, 0], [1, 0], [-1, 0]$ are pairwise distinct and for each $t, s \in \mathbb{R}$, it is the case that $[t, 1] \neq [s, 1]$ whenever $t \neq s$.

Proof: If $[x, y] \in T/\sim$ then either $y > 0$ or $y = 0$. If $y > 0$ then $[x, y] = [x/y, 1] \in \{[t, 1]; t \in \mathbb{R}\}$ because $x = y \frac{x}{y}$ and $y = y \times 1$. On the other hand

$$y = 0 \Rightarrow \begin{cases} \text{if } x = 0 \text{ then } [x, y] = [0, 0] \\ \text{if } x > 0 \text{ then } [x, y] = [1, 0] \\ \text{if } x < 0 \text{ then } [x, y] = [-1, 0] \end{cases}.$$

The rest of the proof follows immediately. ■

Now let us define operations on T/\sim which extend the arithmetical operations between real numbers.

Definition 4: Given $[x, y], [w, z] \in T/\sim$ let us define the operations addition, multiplication, opposite, reciprocal, subtraction and division, respectively, by:

- a) $[x, y] \oplus [w, z] := \begin{cases} [2x, y] & , \text{ if } [x, y] = [w, z] \\ [xz + wy, yz] & , \text{ if } [x, y] \neq [w, z] \end{cases}$
- b) $[x, y] \otimes [w, z] := [xw, yz]$.
- c) $\ominus[x, y] := [-x, y]$.
- d) $[x, y]^{(-1)} := \begin{cases} [y, x] & , \text{ if } x \geq 0 \\ [-y, -x] & , \text{ if } x < 0 \end{cases}$
- e) $[x, y] \ominus [w, z] := [x, y] \oplus (\ominus[w, z])$.
- f) $[x, y] \oslash [w, z] := [x, y] \otimes [w, z]^{(-1)}$.

Proposition 5: The operations \oplus, \otimes, \ominus and $^{(-1)}$ are well defined. That is $[x, y] \oplus [w, z], [x, y] \otimes [w, z], \ominus[x, y]$ and $[x, y]^{(-1)}$ are independent of the choice of representatives of the classes $[x, y]$ and $[w, z]$.

Proof: Let $[x, y], [w, z] \in T/\sim$, $(x', y') \in [x, y]$ and $(w', z') \in [w, z]$. Then there are positive $\alpha, \beta \in \mathbb{R}$ such that $x = \alpha x'$, $y = \alpha y'$, $w = \beta w'$ and $z = \beta z'$.

- a) If $[x, y] = [w, z]$ then $[x', y'] = [w', z']$. Thus $[x, y] \oplus [w, z] = [2x, 2y] = [2x', 2y'] = [x', y'] \oplus [w', z']$. If $[x, y] \neq [w, z]$ then $[x', y'] \neq [w', z']$ and $xz + wy = \alpha x' \beta z' + \beta w' \alpha y' = \alpha \beta (x' z' + w' y')$ and $yz = \alpha y' \beta z' = \alpha \beta y' z'$. Thus $[x, y] \oplus [w, z] = [xz + wy, yz] = [x' z' + w' y', y' z'] = [x', y'] \oplus [w', z']$.
- b) Notice that $xw = \alpha x' \beta w' = \alpha \beta x' w'$ and $yz = \alpha y' \beta z' = \alpha \beta y' z'$, whence $[x, y] \otimes [w, z] = [xw, yz] = [x' w', y' z'] = [x', y'] \otimes [w', z']$.
- c) Note that $-x = -(\alpha x') = \alpha(-x')$ and $y = \alpha y'$. Thus $\ominus[x, y] = [-x, y] = [-x', y'] = \ominus[x', y']$.
- d) Notice that $y = \alpha y'$, $x = \alpha x'$, $-y = \alpha(-y')$ and $-x = \alpha(-x')$. Thus if $x \geq 0$ then $[x, y]^{(-1)} = [y, x] = [y', x'] = [x', y']^{(-1)}$ and if $x < 0$ then $[x, y]^{(-1)} = [-y, -x] = [-y', -x'] = [x', y']^{(-1)}$.

Now let us define an order relation on T/\sim .

Definition 6: Let arbitrary $[x, y], [w, z] \in T/\sim$. We say that $[x, y] \prec [w, z]$ if and only if either $[x, y] = [-1, 0]$ and $[w, z] = [1, 0]$ or else $xz < wy$. Furthermore we say that $[x, y] \preceq [w, z]$ if and only if $[x, y] \prec [w, z]$ or $[x, y] = [w, z]$.

Notice that the relation \preceq is well defined and is an order relation on T/\sim .

The following theorem assures us that, in an appropriate sense, \mathbb{R} is subset of T/\sim .

Theorem 7: The set $R := \{[t, 1]; t \in \mathbb{R}\}$ is an ordered, complete field.

Proof: The result follows from the fact that $\pi : \mathbb{R} \rightarrow R$, $\pi(t) = [t, 1]$ is bijective and, for any $t, s \in \mathbb{R}$,

- i) $\pi(t) \oplus \pi(s) = \pi(t + s)$,
- ii) $\pi(t) \otimes \pi(s) = \pi(ts)$ and
- iii) $\pi(t) \preceq \pi(s)$ if and only if $t \leq s$,

and from the fact that \mathbb{R} is an ordered, complete field. ■

Note that for each $t \in \mathbb{R}$, $\ominus[t, 1] = [-t, 1]$ and if $t \neq 0$ then $[t, 1]^{(-1)} = [t^{-1}, 1]$.

Observation 8: Since π is an isomorphism of ordered, complete fields between R and \mathbb{R} , we can say that R is a “copy” of \mathbb{R} in T/\sim . Therefore let us abuse language and notation: henceforth R will be denoted by \mathbb{R} and will be called the set of real numbers and each $[t, 1] \in R$ will be denoted, simply, by t and will be called a real number. In this sense we can say that $\mathbb{R} \subset T/\sim$ and we replace the symbols $\oplus, \otimes, \ominus, \oslash, ^{(-1)}, \prec$ and \preceq , respectively, by $+, \times, -, /, ^{-1}, <$ and \leq .

We define *negative infinity, infinity, nullity*, respectively, by $-\infty := [-1, 0]$, $\infty := [1, 0]$, $\Phi := [0, 0]$. The elements of T/\sim are *transreal numbers*, hence T/\sim is the *set of transreal numbers*. $\mathbb{R}^T := T/\sim$, whence $\mathbb{R}^T = \mathbb{R} \cup \{-\infty, \infty, \Phi\}$ and $-\infty, \infty$ and Φ are *strictly transreal numbers*.

The next theorem gives transreal arithmetic and ordering.

Theorem 9: For each $x \in \mathbb{R}^T$, it follows that:

- a) $-\Phi = \Phi$, $-\infty = -\infty$ and $-\infty = \infty$.
- b) $0^{-1} = \infty$, $\Phi^{-1} = \Phi$, $(-\infty)^{-1} = 0$ and $\infty^{-1} = 0$.
- c) $\Phi + x = \Phi$, $\infty + x = \begin{cases} \Phi & \text{, if } x \in \{-\infty, \Phi\} \\ \infty & \text{, otherwise} \end{cases}$ and $-\infty + x = -(\infty - x)$.
- d) $\Phi \times x = \Phi$, $\infty \times x = \begin{cases} \Phi & \text{, if } x \in \{0, \Phi\} \\ -\infty & \text{, if } x < 0 \\ \infty & \text{, if } x > 0 \end{cases}$ and $-\infty \times x = -(\infty \times x)$.
- e) If $x \in \mathbb{R}$ then $-\infty < x < \infty$.
- f) The following does not hold $x < \Phi$ or $\Phi < x$.

Proof: Denote $x = [x_1, x_2]$.

- a) $-\Phi = -[0, 0] = [0, 0] = \Phi$, $-\infty = -[1, 0] = [-1, 0] = -\infty$ and $-\infty = -[-1, 0] = [1, 0] = \infty$.
- b) $0^{-1} = [0, 1]^{-1} = [1, 0] = \infty$, $\Phi^{-1} = [0, 0]^{-1} = [0, 0] = \Phi$, $(-\infty)^{-1} = [-1, 0]^{-1} = [-0, -(-1)] = [0, 1] = 0$ and $\infty^{-1} = [1, 0]^{-1} = [0, 1] = 0$.
- c) $\Phi + x = [0, 0] + [x_1, x_2] = [0 \times x_2 + x_1 \times 0, 0 \times x_2] = [0, 0] = \Phi$, $\infty + (-\infty) = [1, 0] + [-1, 0] = [1 \times 0 + (-1) \times 0, 0 \times 0] = [0, 0] = \Phi$, $\infty + \Phi = [1, 0] + [0, 0] = [1 \times 0 + 0 \times 0, 0 \times 0] = [0, 0] = \Phi$ and $\infty + \infty = [1, 0] + [1, 0] = [2, 0] = [1, 0] = \infty$. If $x \in \mathbb{R}$, $\infty + x = [1, 0] + [x, 1] = [1 \times 1 + x \times 0, 0 \times 1] = [1, 0] = \infty$. The addition $-\infty + x$ holds analogously.
- d) $\Phi \times x = [0, 0] \times [x_1, x_2] = [0 \times x_1, 0 \times x_2] = [0, 0] = \Phi$, $\infty \times 0 = [1, 0] \times [0, 1] = [1 \times 0, 0 \times 1] = [0, 0] = \Phi$ and $\infty \times \Phi = [1, 0] \times [0, 0] = [1 \times 0, 0 \times 0] = [0, 0] = \Phi$. If $x < 0$ then $x_1 < 0$, whence $\infty \times x = [1, 0] \times [x_1, x_2] = [1 \times x_1, 0 \times x_2] = [x_1, 0] = [-1, 0] = -\infty$. If $x > 0$ then $x_1 > 0$, whence $\infty \times x = [1, 0] \times [x_1, x_2] = [1 \times x_1, 0 \times x_2] = [x_1, 0] = [1, 0] = \infty$. The multiplication $-\infty \times x$ holds analogously.
- e) If $x = [x_1, x_2] \in \mathbb{R}$ then $x_2 > 0$, whence $-1 \times x_2 = -x_2 < 0 = x_1 \times 0$ and $x_1 \times 0 = 0 < x_2 = 1 \times x_2$.
- f) $\Phi \neq [-1, 0]$, $\Phi \neq [1, 0]$ and the inequality $x_1 \times 0 = 0 < 0 = 0 \times x_2$ does not hold.

If $x, y \in \mathbb{R}^T$, we write $x \not\leq y$ if and only if $x \leq y$ does not hold.

Corollary 10: Let $x, y \in \mathbb{R}$ where $x > 0$ and $y < 0$. It follows that:

- a) $x/0 = \infty$.
- b) $y/0 = -\infty$.
- c) $0/0 = \Phi$.

Proof:

- a) $x/0 = x \times 0^{-1} = x \times \infty = \infty$.
- b) $y/0 = y \times 0^{-1} = y \times \infty = -\infty$.
- c) $0/0 = 0 \times 0^{-1} = 0 \times \infty = \Phi$.

In the next theorem we establish on \mathbb{R}^T some arithmetical and ordering properties that are true on \mathbb{R} . Regarding the properties that are not true for all transreal numbers, we indicate the necessary restrictions.

Theorem 11: Let $x, y, z \in \mathbb{R}^T$. It follows that:

- a) (Additive Commutativity) $x + y = y + x$.
- b) (Additive Associativity) $(x + y) + z = x + (y + z)$.
- c) (Additive Identity) $x + 0 = 0 + x = 0$.
- d) (Additive Inverse) if $x \notin \{-\infty, \infty, \Phi\}$ then $x - x = 0$.
- e) (Multiplicative Commutativity) $x \times y = y \times x$.
- f) (Multiplicative Associativity) $(x \times y) \times z = x \times (y \times z)$.
- g) (Multiplicative Identity) $x \times 1 = 1 \times x = x$.
- h) (Multiplicative Inverse) if $x \notin \{0, -\infty, \infty, \Phi\}$ then $\frac{x}{x} = 1$.
- i) (Distributivity) if $x \notin \{-\infty, \infty\}$ or $yz > 0$ or $y + z = 0$ or $x, y, z \in \{-\infty, \infty\}$ then $x \times (y + z) = (x \times y) + (x \times z)$ and $(y + z) \times x = (y \times x) + (z \times x)$.
- j) (Additive Monotonicity) if not simultaneously $z = -\infty, x = -\infty$ and $y = \infty$ and not simultaneously $z = -\infty, x \in \mathbb{R}$ and $y = \infty$ and not simultaneously $z = \infty, x = -\infty$ and $y = \infty$ and not simultaneously $z = \infty, x = -\infty$ and $y \in \mathbb{R}$ then

$$x \leq y \Rightarrow x + z \leq y + z.$$

- k) (Multiplicative Monotonicity) if not simultaneously $z = 0, x = -\infty$ and $y \in \mathbb{R}$ and not simultaneously $z = 0, x \in \mathbb{R}$ and $y = \infty$ then

$$x \leq y \text{ and } z \geq 0 \Rightarrow xz \leq yz.$$

- l) (Existence of Supremum) if $A \subset \mathbb{R}^T \setminus \{\Phi\}$ is non-empty then A has supremum in \mathbb{R}^T .

Notice that, as show in the following examples, the restrictions on items (d), (h), (i), (j) and (k) of Theorem 11 are indeed necessary.

Example 12: From Theorem 9, $\Phi - \Phi = -\infty - (-\infty) = \infty - \infty = \Phi$.

Example 13: From Theorem 9, $\frac{0}{0} = \frac{\Phi}{\Phi} = \frac{-\infty}{-\infty} = \frac{\infty}{\infty} = \Phi$.

Example 14: $\infty \times (-2 + 3) = \infty \times 1 = \infty \neq \Phi = -\infty + \infty = (\infty \times (-2)) + (\infty \times 3)$.

$\infty \times (0 + 3) = \infty \times 3 = \infty \neq \Phi = \Phi + \infty = (\infty \times 0) + (\infty \times 3)$.

$\infty \times (-\infty + 3) = \infty \times (-\infty) = -\infty \neq \Phi = -\infty + \infty = (\infty \times (-\infty)) + (\infty \times 3)$.

Example 15: $-\infty \leq \infty$ and $-\infty + (-\infty) = -\infty \not\leq \Phi = \infty + (-\infty)$.

If $x \in \mathbb{R}$ then $x \leq \infty$ and $x + (-\infty) = -\infty \not\leq \Phi = \infty + (-\infty)$.

$-\infty \leq \infty$ and $-\infty + \infty = \Phi \not\leq \infty = \infty + \infty$.

If $y \in \mathbb{R}$ then $-\infty \leq y$ and $-\infty + \infty = \Phi \not\leq \infty = y + \infty$.

Example 16: If $y \in \mathbb{R}$ then $-\infty \leq y$ and $-\infty \times 0 = \Phi \not\leq 0 = y \times 0$.

If $x \in \mathbb{R}$ then $x \leq \infty$ and $x \times 0 = 0 \not\leq \Phi = \infty \times 0$.

The transreal numbers have not been easily accepted. We believe that one reason for the resistance to Anderson's proposal is the fact that, in his presentation [5], the set of transreals is defined by $\mathbb{R}^T := \mathbb{R} \cup \{-1/0, 1/0, 0/0\}$. By defining \mathbb{R}^T in this way, Anderson presents a cyclical argument. He defines the transreals as being the reals joined to the elements $-1/0, 1/0$ and $0/0$ and defines these elements as transreal numbers not real. That is, the objects $-1/0, 1/0$ and $0/0$ are used to define themselves.

Another reason for transreals appearing strange is that, in the new objects, the symbol “/” is undefined in context. Usually this symbol means division and a fraction with denominator zero has no sense in the set of real numbers (which is the set for which the arithmetical properties are already established). It is used as a symbol in the “old” representation to denote a “new” operation. That is, Anderson uses the division symbol between real numbers to represent something not yet defined, the division between transreal numbers.

It is worth mentioning that in our text, after a certain moment, we use also the symbol “/” to represent division between transreal numbers, but this is justified by Observation 8. We emphasise that this procedure is quite common in mathematics. Dedekind defines operations for addition and multiplication in the set of cuts and uses the same symbols for addition and for multiplication in the “old” arithmetic of the rationals and in the new operations because there is an isomorphism of ordered fields between a certain subset of

cuts and the set of rational numbers. The same happens in many other cases, such as: the construction of the complexes from the reals, the construction of the hyperreals from the reals, the rationals from the integers and the integers from the naturals.

To solve the problem of cyclical arguments, we used the concept of an equivalence relation. Notice that we want the fractions $-1/0$, $1/0$ and $0/0$ to be elements of the new set. Each fraction is determined by two real numbers, each one in a specific position. So the starting point was to think of each transreal number as an ordered pair of real numbers. The next step was to establish the criteria to consider two “fractions” (ordered pairs) as equivalent fractions. This justifies the relation created in Definition 1. And so we came to consider the quotient set T/\sim and not just T . That is, a transreal number is not a pair of real numbers, but a certain class of ordered pairs of real numbers.

In Definition 4 we extended the arithmetical operations to the transreals. Note that the rules for obtaining the results of these operations are the same rules that customary practice dictates are used between fractions of real numbers, except for addition, whose definition was dismembered into two cases. Even so, addition can be obtained similarly to the well known practical rules for adding fractions of real numbers:

To sum two fractions, x and y , of real numbers. If x and y have the same denominator then copy the denominator into the result and add up the two numerators to give the numerator of the result. Otherwise create new fractions, with a common denominator, by multiplying the numerator and the denominator of x by the denominator of y and by multiplying the numerator and the denominator of y by the denominator of x then, as before, copy the new common denominator into the result and add up the numerators of the two new fractions to give the numerator of the result.

In the transreal case:

To sum transreal numbers x and y . If $x = y$ then copy the second element into the result and add up the first elements to give the first element of the result. Otherwise multiply the first element and the second element of x by the second element of y , multiply the first element and the second element of y by the second element of x then, as before, copy the new second element into the result and add up the first elements of the two new pairs to give the first element of the result.

We note that, of course, opposite does not mean additive inverse and reciprocal does not mean multiplicative inverse. However, we stress that changing the meaning of operations, when it extends the concept of number, is a common occurrence. For example, for the natural numbers 3 and 6, the result of $6/3$ is the number of instalments, all equal to 3, whose sum is 6. This interpretation is meaningless when we operate on $3/6$. There is no number of instalments, all equal to 6, whose sum is 3. Of course, in the set of rational numbers, $3/6 = 0.5$, but this division no longer has the previous meaning. It makes no sense to say that the sum of 0.5 parts, all equal to 6, is equal to 3. We observe that if $[x, y] \in T/\sim$ then $-[x, y]$ does not mean the additive

inverse of $[x, y]$, instead it means the image of $[x, y]$ in the function $[x, y] \mapsto [-x, y]$. Likewise $[x, y]^{-1}$ does not mean the multiplicative inverse of $[x, y]$, it means the image of $[x, y]$ in the function $[x, y] \mapsto \begin{cases} [y, x], & x \geq 0 \\ [-y, -x], & x < 0 \end{cases}$. Nevertheless we also observed that, when restricted to real numbers, the arithmetical operations defined on transreals coincide with the “old” operations of the reals.

It should be mentioned that the equivalence and arithmetic proposed here were motivated by the arithmetic of function limit theory. Note that if $k \in \mathbb{R}$ and $k > 0$ then $\lim_{x \rightarrow 0^+} \frac{k}{x} = \infty$ [29]. This motivates an equivalence relation so that if $k \in \mathbb{R}$ and $k > 0$ then $k/0 = \infty$. Among many other examples, we highlight that if $a \in \mathbb{R}$ and f and g are real functions such that $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = \infty$ then $\lim_{x \rightarrow a} (f(x) + g(x)) = \infty$. This motivates an arithmetic such that $\infty + \infty = \infty$.

We are proposing the enlargement of the number concept. As already mentioned, this is not a new process in the development of mathematics. We are aware that the new set of numbers, \mathbb{R}^T , has some properties that appear somewhat unnatural in numbers. To cite one example, the distributive property does not hold for all transreal numbers, as seen in Example 14. However, at various moments in the extension of concepts, some properties are lost. Among many other examples we can point out that the set of complex numbers is not an ordered field, as the reals are, the hyperreals do not have the Archimedean property that the reals have, the matrix product and product between Hamilton’s quaternions are not commutative, unlike the reals, and in Cantor’s transfinite arithmetic, addition is not commutative, unlike the reals.

IV. ALGEBRAIC TRANSFIELDS

The transreal numbers were created with the aim of extending the real numbers to a set where the arithmetic is total. The arithmetical operations were extended to fractions of denominator zero, inspired by the usual arithmetic of fractions. This inspiration led to the arithmetic described in Theorem 9. In Theorem 11 we looked at what properties of real numbers are valid in the transreals. In fact, we know that the real numbers form a more abstract structure, namely the algebraic *field*. We now investigate how transreal numbers fit into a more general algebraic structure. In this section we propose the *transfield* as a generalisation of the field – not in the sense that every transfield is a field, but in the sense that a field is a system of axioms that establishes properties of addition and multiplication operations which are extended by the special axioms of a transfield so that addition, multiplication, subtraction and division are total operations. Thus a transfield contains a field and supplies total operations.

We start by defining a transfield so that the smallest number of axioms are used to admit the maximum possible structure of real arithmetic, subject to the constraint that a transfield describes both the transreals and the transcomplexes [23]. In particular, associativity of addition is not required so that the transcomplexes are admitted and neither absorptivity nor extremals are required because these are not parts of real arithmetic and enforcing them would increase the number of axioms.

We then elucidate various properties of transfields. The reals are an important algebraic structure: the set of real numbers is, up to isomorphism, the unique, ordered, complete field. We examine whether the transreals occupy a similar place with respect to transfields. Unfortunately (or fortunately!) the transreals are not the unique, ordered, complete transfield, but they are the smallest of these transfields under suitable conditions, established in Theorem 39. Similarly we find that the transrationals are the smallest, ordered transfield.

Definition 17: A *transfield* is a set T provided with two binary operations $+$ and \times and two unary operations $-$ and $^{-1}$ (named, respectively, *addition*, *multiplication*, *opposite* and *reciprocal*) such that:

- a) T is closed for $+$, \times , $-$ and $^{-1}$.
- b) $+$ and \times are commutative, each has an identity element and \times is associative.
- c) There is $F \subset T$ such that F is a field with respect to $+$ and \times and F and T have common additive and multiplicative identities.
- d) For each $x \in F$, $-x$ coincides with the additive inverse of x in F and, for each $x \in F$, different from the additive identity, x^{-1} coincides with the multiplicative inverse of x in F .

When we refer to a transfield just by T , the operations $+$, \times , $-$, $^{-1}$ and the field F are implied.

Observation 18: Let us use the following notation: the additive and multiplicative identities are denoted, respectively, by $\mathbf{0}$ and $\mathbf{1}$. Further ∞ denotes $\mathbf{0}^{-1}$ and Φ denotes $\mathbf{0} \times \mathbf{0}^{-1}$. We emphasise that $\mathbf{0}$ and $\mathbf{1}$ are not necessarily the real number zero, 0 , and one, 1 , ∞ is not necessarily the transreal number infinity, ∞ , and Φ is not necessarily the transreal number nullity, Φ . The symbols $\mathbf{0}$, $\mathbf{1}$, ∞ and Φ just denote, respectively, the additive identity, the multiplicative identity, the reciprocal of the additive identity and the product of the additive identity with its reciprocal.

If T is a transfield and $x, y \in T$, we define *subtraction* as $x - y := x + (-y)$ and *division* as $x \div y := x \times y^{-1}$. As usual $\frac{1}{x}$ and $\mathbf{1}/x$ denote x^{-1} while $\frac{x}{y}$ and x/y denote $x \div y$.

Example 19: Let Θ be an arbitrary, atomic element and let $T_{19} := \mathbb{R} \cup \{\Theta\}$ with the operations $+$, \times , $-$ and $^{-1}$, defined as follows. For each $x, y \in T_{19}$, if $x, y \in \mathbb{R}$ then $x + y$, $x \times y$, $-x$ and, if $x \neq 0$, x^{-1} are defined in the usual way. Further $-\Theta := \Theta$, $0^{-1} := \Theta$, $\Theta^{-1} := \Theta$, $\Theta + x := x + \Theta := \Theta$ and $\Theta \times x := x \times \Theta := \Theta$. Notice that T_{19} is a transfield with, among other options, $F = \mathbb{R}$ or $F = \mathbb{Q}$. Notice also that $\infty = \Phi = \Theta$.

Definition 20: A transfield T is said to be an *ordered transfield* if and only if there is $P \subset T$ such that:

- a) Given $x \in T$, exactly one of these four alternatives occurs: either $x = \Phi$ or $x = \mathbf{0}$ or $x \in P$ or $-x \in P$.
- b) If $x, y \in P$ then $x + y, x \times y \in P$.

In this way, we say that T is ordered by P . Further, if $x, y \in T$ we denote $x < y$ if and only if $y - x \in P$. We also denote $x \leq y$ if and only if $x < y$ or $x = y$. We call P the *set of positive elements of T*. When we refer to an ordered transfield just by T , the set P is implied.

Example 21: Notice that T_{19} is ordered by $P = \mathbb{R}^+$.

Example 22: Let Θ be an arbitrary, atomic element and let $T_{22} := \mathbb{C} \cup \{\Theta\}$ with the operations $+$, \times , $-$ and $^{-1}$, defined as follows. For each $x, y \in T_{22}$, if $x, y \in \mathbb{C}$ then $x + y$, $x \times y$, $-x$ and, if $x \neq 0$, x^{-1} are defined in the usual way. Further $-\Theta := \Theta$, $0^{-1} := \Theta$, $\Theta^{-1} := \Theta$, $\Theta + x := x + \Theta := \Theta$ and $\Theta \times x := x \times \Theta := \Theta$. Notice that T_{22} is a transfield with, among other options, $F = \mathbb{C}$, but T_{22} is not ordered.

Example 23: Let a, b, c, d and e be arbitrary, atomic elements and $T_{23} := \{a, b, c, d, e\}$ with the operations $+$, \times , $-$ and $^{-1}$, defined by the tables below.

+	a	b	c	d	e
a	a	b	c	d	e
b	b	a	c	d	e
c	c	c	c	e	e
d	d	d	e	d	e
e	e	e	e	e	e

$-a = a$
$-b = b$
$-c = d$
$-d = c$
$-e = e$

\times	a	b	c	d	e
a	a	a	e	e	e
b	a	b	c	d	e
c	e	c	d	c	e
d	e	d	c	d	e
e	e	e	e	e	e

$a^{-1} = d$
$b^{-1} = b$
$c^{-1} = a$
$d^{-1} = a$
$e^{-1} = e$

Notice that T_{23} is a transfield, with $F = \{a, b\}$, but T_{23} is not ordered. Notice also that $\mathbf{0} = a$, $\mathbf{1} = b$, $\infty = d$ and $\Phi = e$.

Definition 24: An ordered transfield, T , is said to be an *ordered, complete transfield* if and only if every nonempty subset of T , which has an upper bound, has a supremum in T .

Example 25: Notice that T_{19} is complete.

Example 26: Let Θ be an arbitrary, atomic element and let $T_{26} := \mathbb{Q} \cup \{\Theta\}$ with the operations $+$, \times , $-$ and $^{-1}$, defined as follows. For each $x, y \in T_{19}$, if $x, y \in \mathbb{Q}$ then $x + y$, $x \times y$, $-x$ and, if $x \neq 0$, x^{-1} are defined in the usual way. Further $-\Theta := \Theta$, $0^{-1} := \Theta$, $\Theta^{-1} := \Theta$, $\Theta + x := x + \Theta := \Theta$ and $\Theta \times x := x \times \Theta := \Theta$. Notice that T_{26} is an ordered transfield but is not complete.

Definition 27: Let T be an ordered transfield. We say that T has the *extremal property* if and only if, for all $x \in T$:

- a) $\infty \in P$.
- b) $-(-\infty) = \infty$.
- c) $\infty^{-1} = \mathbf{0}$ and $(-\infty)^{-1} = \mathbf{0}$.

$$\begin{aligned}
 \text{d) } \mathfrak{O} + x &= \begin{cases} \Phi & , \text{ if } x \in \{-\mathfrak{O}, \Phi\} \\ \mathfrak{O} & , \text{ otherwise} \end{cases} & \text{ and } -\mathfrak{O} + x = \\
 & -(\mathfrak{O} - x). \\
 \text{e) } \mathfrak{O} \times x &= \begin{cases} \Phi & , \text{ if } x \in \{0, \Phi\} \\ -\mathfrak{O} & , \text{ if } x < 0 \\ \mathfrak{O} & , \text{ if } x > 0 \end{cases} & \text{ and } -\mathfrak{O} \times x = \\
 & -(\mathfrak{O} \times x).
 \end{aligned}$$

If T has the extremal property, we call \mathfrak{O} the extremal element.

Example 28: Notice that T_{19} and T_{26} do not have the extremal property since, for example, $\mathfrak{O} = \Theta = \Phi \notin P$. Further T_{22} and T_{23} also do not have the extremal property since they are not ordered.

Example 29: Let $\Phi, -\mathfrak{O}, \mathfrak{O}$ be arbitrary, atomic elements and let $T_{29} := \mathbb{R} \cup \{\Phi, -\mathfrak{O}, \mathfrak{O}\}$ with the operations $+, \times, -$ and $^{-1}$ defined as follows. For each $x, y \in T_{29}$, if $x, y \in \mathbb{R}$ then $x + y, x \times y, -x$ and, if $x \neq 0, x^{-1}$ are defined in the usual way. Further:

$$\begin{aligned}
 -\Phi &:= \Phi, & -(\mathfrak{O}) &:= -\mathfrak{O} & \text{ and } & -(-\mathfrak{O}) &:= \mathfrak{O}, \\
 0^{-1} &:= \mathfrak{O}, & \Phi^{-1} &:= 0, & (-\mathfrak{O})^{-1} &:= 0 & \text{ and } & \mathfrak{O}^{-1} &:= 0, \\
 \Phi + x &:= x + \Phi := \Phi, \\
 \mathfrak{O} + x &:= x + \mathfrak{O} := \begin{cases} \Phi & , \text{ if } x \in \{-\mathfrak{O}, \Phi\} \\ \mathfrak{O} & , \text{ otherwise} \end{cases} & \text{ and } \\
 -\mathfrak{O} + x &:= x + (-\mathfrak{O}) := -(\mathfrak{O} - x), \\
 \Phi \times x &:= x \times \Phi := \Phi, \\
 \mathfrak{O} \times x &:= x \times \mathfrak{O} := \begin{cases} \Phi & , \text{ if } x \in \{0, \Phi\} \\ -\mathfrak{O} & , \text{ if } x < 0 \\ \mathfrak{O} & , \text{ if } x > 0 \end{cases} & \text{ and } \\
 -\mathfrak{O} \times x &:= x \times (-\mathfrak{O}) := -(\mathfrak{O} \times x).
 \end{aligned}$$

Notice that T_{29} is an ordered transfield which has the extremal property.

Proposition 30: If T is an ordered transfield with the extremal property then:

- a) $x < \mathfrak{O}$ for all $x \in T \setminus \{\Phi, \mathfrak{O}\}$.
- b) $-\mathfrak{O} < x$ for all $x \in T \setminus \{\Phi, -\mathfrak{O}\}$.

Proof: Let T be an ordered transfield with the extremal property.

- a) Let $x \in T \setminus \{\Phi, \mathfrak{O}\}$. From Definition 27, items (d) and (a), it follows that $\mathfrak{O} - x = \mathfrak{O} + (-x) = \mathfrak{O} \in P$. Hence, from Definition 20, $x < \mathfrak{O}$.
- b) Let $x \in T \setminus \{\Phi, -\mathfrak{O}\}$. From Definition 27, items (b), (d) and (a), it follows that $x - (-\mathfrak{O}) = x + \mathfrak{O} = \mathfrak{O} + x = \mathfrak{O} \in P$. Hence, from Definition 20, $-\mathfrak{O} < x$. ■

If T is an ordered transfield with the extremal property then, because of Proposition 30, we can say that $-\mathfrak{O} < x < \mathfrak{O}$ for all $x \in T \setminus \{\Phi, -\mathfrak{O}, \mathfrak{O}\}$. Similarly $-\mathfrak{O} \leq x \leq \mathfrak{O}$ for all $x \in T \setminus \{\Phi\}$.

Definition 31: Let T be a transfield. We say that T has the absorptive property if and only if, for all $x \in T$,

$$-\Phi = \Phi, \quad \Phi^{-1} = \Phi, \quad \Phi + x = \Phi \quad \text{and} \quad \Phi \times x = \Phi.$$

If T has the absorptive property, we call Φ the absorptive element.

Example 32: Notice that T_{19}, T_{23} and T_{26} have the absorptive property. While T_{29} does not have the absorptive property, since $\Phi^{-1} = 0 \neq \Phi$.

Example 33: Notice that \mathbb{R}^T , the set of the transreal numbers, is an ordered, complete transfield with extremal and absorptive properties.

Example 34: Notice that \mathbb{C}^T , the set of the transcomplex numbers [23], is a transfield with the absorptive property. Notice also that \mathbb{C}^T is not ordered and, therefore, it does not have the extremal property.

Observation 35: As a curious fact, note that addition is associative in \mathbb{R}^T but is not associative in \mathbb{C}^T .

Proposition 36: If T is an ordered transfield with the absorptive property then for all $x \in T$, neither $x < \Phi$ nor $\Phi < x$.

Proof: Suppose T is an ordered transfield with the absorptive property and let $x \in T$ be arbitrary. By Definition 31 and Definition 20, $\Phi - x = x - \Phi = \Phi \notin P$, whence neither $x < \Phi$ nor $\Phi < x$. ■

Let T be an ordered transfield. If $x, y \in T$, we write $x \not< y$ if and only if $x < y$ does not hold and we write $x \not> y$ if and only if $x > y$ does not hold. Notice that $\not<$ is not equivalent to \leq . For example, if T has the absorptive property, $\Phi \not> 0$ but $\Phi \leq 0$ does not hold.

Proposition 37: If T is an ordered transfield with the absorptive property then Φ is the unique element $y \in T$ such that $y + x = y$ for all $x \in T$ or $y \times x = y$ for all $x \in T$. In other words, if $y \in T$ and $y + x = y$ for all $x \in T$ or $y \times x = y$ for all $x \in T$ then $y = \Phi$.

Proof: Let T be an ordered transfield with the absorptive property and let $y \in T$. If $y + x = y$ for all $x \in T$ then $\Phi = y + \Phi = y$ and if $y \times x = y$ for all $x \in T$ then $\Phi = y \times \Phi = y$. ■

Definition 38: Let T and S be ordered transfields. A function $f : T \rightarrow S$ is an isomorphism of ordered transfields if and only if f is bijective and, for all $x, y \in T$, it is the case that:

- i) $f(x + y) = f(x) + f(y)$,
- ii) $f(x \times y) = f(x) \times f(y)$,
- iii) $f(-x) = -f(x)$,
- iv) $f(x^{-1}) = (f(x))^{-1}$,
- v) $f(x) < f(y)$ if and only if $x < y$.

If there is an isomorphism of ordered transfields $f : T \rightarrow S$, we say that T is isomorphic to S .

Theorem 39: If T is an ordered, complete transfield which possess the extremal and absorptive properties then T contains an ordered, complete transfield isomorphic to \mathbb{R}^T .

Proof: Let T be an ordered, complete transfield with the extremal and absorptive properties. Let $F \subset T$ be a field and let $P \subset T$ be the set of positive elements of T .

Denote $P' = F \cap P$ and notice that F is ordered by the set P' . Indeed, given an arbitrary $x \in F$, since $F \subset T$, $x \in T$ and, by Definition 20 item (a), either $x = \Phi$ or $x = \mathbf{0}$ or $x \in P$ or $-x \in P$. Since $\Phi \notin F$ we have that either $x = \mathbf{0}$ or $x \in P$ or $-x \in P$. That is, either $x = \mathbf{0}$ or $x \in P'$ or $-x \in P'$. Now, if $x, y \in P'$ then $x, y \in F$ and $x, y \in P$. Since $x, y \in P$, by Definition 20 item (b), $x + y, x \times y \in P$. Furthermore, since $x, y \in F$, $x + y, x \times y \in F$. Thus $x + y, x \times y \in F \cap P = P'$.

Let \overline{F} be the completion of F . Notice that, since T is complete, $\overline{F} \subset T$. As \overline{F} is an ordered, complete field, there is an isomorphism of ordered fields $h : \mathbb{R} \rightarrow \overline{F}$.

Define $f : \mathbb{R}^T \rightarrow X := \overline{F} \cup \{\Phi, -\infty, \infty\} \subset T$ as

$$f(x) = \begin{cases} h(x), & \text{if } x \in \mathbb{R} \\ \Phi, & \text{if } x = \Phi \\ -\infty, & \text{if } x = -\infty \\ \infty, & \text{if } x = \infty \end{cases} .$$

Since ∞ is the extremal element and Φ is the absorptive element of T and h is an isomorphism of ordered fields, it is not difficult to see that $f(x+y) = f(x) + f(y)$, $f(x \times y) = f(x) \times f(y)$, $f(-x) = -f(x)$ and $f(x^{-1}) = (f(x))^{-1}$ for all $x, y \in \mathbb{R}^T$. Furthermore, from Definition 20, Definition 27, Definition 31 and Proposition 30, it follows that $f(x) < f(y)$ if and only if $x < y$ for all $x, y \in \mathbb{R}^T$. Thus X is an ordered, complete transfield and f is an isomorphism of ordered transfields. ■

Observation 40: By the Theorem 39, every ordered, complete transfield with extremal and absorptive properties contains a copy of \mathbb{R}^T . Thus, unless isomorphism, \mathbb{R}^T is the smallest, ordered, complete transfield which possess the extremal and absorptive properties.

Example 41: Let $a, -a$ be arbitrary, atomic elements and let $T_{41} := \mathbb{R}^T \cup \{a, -a\}$ with the operations $+$, \times , $-$ and $^{-1}$ defined as follows. For each $x, y \in T_{41}$, if $x, y \in \mathbb{R}^T$ then $x + y, x \times y, -x$ and, if $x \neq 0, x^{-1}$ are defined in the usual way. Further:

$$-(a) := -a, \quad -(-a) := a,$$

$$a^{-1} := 0, \quad (-a)^{-1} := 0,$$

$$a + x := x + a := \begin{cases} a & , \text{if } x \in \mathbb{R} \cup \{a\} \\ \Phi & , \text{if } x \in \{-a, \Phi\} \\ x & , \text{if } x \in \{-\infty, \infty\} \end{cases} \quad \text{and}$$

$$-a + x := x + (-a) := -(a - x),$$

$$a \times x := x \times a := \begin{cases} \Phi & , \text{if } x \in \{0, \Phi\} \\ -a & , \text{if } x \in \{-a\} \cup \mathbb{R}^- \\ a & , \text{if } x \in \{a\} \cup \mathbb{R}^+ \\ -\infty & , \text{if } x = -\infty \\ \infty & , \text{if } x = \infty \end{cases} \quad \text{and}$$

$$-a \times x := x \times (-a) := -(a \times x).$$

Define $P := \mathbb{R}^+ \cup \{a, \infty\}$. Now notice that T_{41} is an ordered, complete transfield with the extremal and absorptive properties.

Observe that T_{41} is an example of an ordered, complete transfield with extremal and absorptive properties strictly “larger” than \mathbb{R}^T , since $\mathbb{R}^T \subsetneq T_{41}$. It shows that the condition “contains” cannot be replaced by “is” in Theorem 39 and the condition “smallest” cannot be replaced by “unique” in Observation 40.

As is known, every ordered field has an ordered subfield isomorphic to \mathbb{Q} . This allows us to have the following.

Proposition 42: If T is an ordered transfield, which possess the extremal and absorptive properties, then T contains an ordered transfield isomorphic to \mathbb{Q}^T . (This means that, up to isomorphism, \mathbb{Q}^T is the smallest, ordered transfield which possess the extremal and absorptive properties.)

Proof: The proof is analogous to the proof given for Theorem 39. ■

V. OTHER EXAMPLES OF ALGEBRAIC STRUCTURES

Inspired by the transreal numbers, the algebraic structure of a field has been extended to a transfield. Other structures have already shown themselves to be very useful in the development of transmathematics. In a forthcoming paper¹ we establish the algebraic structures of trans-Boolean algebra and transvector space. Here we just introduce their definitions and give a few remarks about their applications.

Definition 43: A *trans-Boolean algebra* is a structure $(X, \neg, \vee, \wedge, \perp, \top)$, where X is a set, $\perp, \top \in X$, \neg is a function from X to X and \vee and \wedge are functions from $X \times X$ to X such that the following properties are satisfied: (i) existence of an identity element, (ii) commutativity, (iii) associativity and (iv) distributivity. Thus, for all $x, y, z \in X$:

- a) $x \vee \perp = x$ and $x \wedge \top = x$.
- b) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$.
- c) $x \vee (y \vee z) = (x \vee y) \vee z$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$.
- d) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Definition 44: A nonempty set, V , is called a *transvector space on \mathbb{R}^T* if and only if there are two operations $+$: $V \times V \rightarrow V$ and \cdot : $\mathbb{R}^T \times V \rightarrow V$ (named, respectively, *addition* and *scalar multiplication*), such that the following properties are satisfied: additive commutativity, additive associativity, scalar multiplicative associativity, additive identity and scalar multiplicative identity. Which are, respectively, for any $w, u, v \in V$ and $x, y \in \mathbb{R}^T$:

- a) $w + u = u + w$.
- b) $w + (u + v) = (w + u) + v$.
- c) $x \cdot (y \cdot w) = (xy) \cdot w$.
- d) There is $o \in V$ such that $o + w = w$.
- e) $1 \cdot w = w$.

The elements of V are called *transvectors*. Further $x \cdot w$ is customarily denoted as $w \cdot x, xw$ or wx and o as 0 .

¹W. Gomide, T. S. dos Reis and J. A. D. W. Anderson, “Transreal Proof of the Existence of Universal Possible Worlds”. Submitted for review.

In the submitted paper we show that there are universal possible worlds which have the topological property that they can access every world in sequences of worlds that comes arbitrarily close to every possible world. Our method of proof is to establish the space of all possible worlds as a transvector space and to prove certain algebraic and topological properties of that space. We begin by making \mathbb{R}^T a trans-Boolean logic, by defining suitable functions \neg, \vee and \wedge , and prove that the transreal numbers do model classical, fuzzy and a particular paraconsistent logic by establishing homomorphisms between these logics and trans-Boolean logic. Negative infinity models the classical truth value False and positive infinity models the classical truth value True. The real numbers (in the range from zero to one) model fuzzy values. Negative values are more False than True, positive values are more True than False and zero is equally False and True. Zero models the dialethic value that has equal degrees of both falsehood and truthfulness. Nullity models gap values that are neither False nor True and which, more generally, have no degree of falsehood or truthfulness. Thus we can model the semantic values of many logics with algebraic structures inspired by the transreals.

The idea of logical space is inspired by Wittgenstein's conception that the world's logical form is given by a picture that is a "configuration of objects." See [30][31]. We assume that there is a countable infinitude of atomic propositions. Hence the set of atomic propositions can be written in the form $\{P_1, P_2, P_3, \dots\}$, where $P_i \neq P_j$ whenever $i \neq j$. Intuitively a possible world is a binding of propositions to semantic values. That is, at a given possible world, each atomic proposition takes on a semantic value in \mathbb{R}^T . Thus we can interpret a possible world as a function from $\{P_1, P_2, P_3, \dots\}$ to \mathbb{R}^T . But this forms a sequence of elements from \mathbb{R}^T . So, denoting the set of the sequences of elements from \mathbb{R}^T by $(\mathbb{R}^T)^\mathbb{N}$, we call each element of $(\mathbb{R}^T)^\mathbb{N}$ a possible world. Hence $(\mathbb{R}^T)^\mathbb{N}$ is the set of all possible worlds. We notionally label each axis with a unique atomic proposition so that a co-ordinate, on a labelled axis, is the semantic value of that labeling proposition. Thus points in this space are arrangements of semantic values of atomic propositions. In other words, points in this space are possible worlds. We make $(\mathbb{R}^T)^\mathbb{N}$ a transvector space. We give a distance metric so that we can talk about the distance between any two possible worlds and can give the vector that passes from one world to another. We define accessibility between possible worlds and show that there is a possible world w , which accesses a sequence of possible worlds that converges to u , whatever the possible world u . This means that w accesses any possible world by approximation.

Thus we obtain new results in logic by exploiting algebraic structures inspired by the transreals.

VI. CONCLUSION

The transreal numbers have an arithmetic which is closed over addition, subtraction, multiplication and division. The transreals have proved controversial and have not been widely accepted. In the past, constructive proofs have ended controversies over the validity of new number systems so we construct the set of transreal numbers from the set of real numbers and construct transreal arithmetic from real arithmetic. We show that the transreals contain the reals.

We define an algebraic structure, called a transfield, which contains a field. We give some examples of transfields and show that just as the rationals are the smallest, ordered field and reals are the unique, ordered, complete field, so, under established conditions, transrationals are the smallest, ordered transfield and transreals are the smallest, ordered, complete transfield. Thus we extend algebraic structures from the real domain to the transreal domain.

We hope this does enough to end controversy about the correctness of the transreals, leaving an assessment of their usefulness to future experience.

APPENDIX A PROOF OF THEOREM 11

Although somewhat tedious, it is necessary to prove Theorem 11.

Note that if it is not the case that $[x, y] = [w, z] = [-1, 0]$ nor $[x, y] = [w, z] = [1, 0]$ then $[x, y] + [w, z] = [xz + wy, yz]$, even though $[x, y] = [w, z]$. We recall that the *real sign function* is defined by

$$\text{sgn} : \mathbb{R} \rightarrow \mathbb{R}, \text{sgn}(x) = \begin{cases} -1 & , \text{ if } x < 0 \\ 0 & , \text{ if } x = 0 \\ 1 & , \text{ if } x > 0 \end{cases} .$$

Notice that for all $x, y \in \mathbb{R}$ it is the case that $\text{sgn}(x) \times \text{sgn}(y) = \text{sgn}(xy)$. Moreover $[x, 0] = [\text{sgn}(x), 0]$. We use these observations in what follows.

Proof: Let us denote $x = [x_1, x_2]$, $y = [y_1, y_2]$ and $z = [z_1, z_2]$.

a) If $x = y$ then the result is immediate. Otherwise $x + y = [x_1, x_2] + [y_1, y_2] = [x_1y_2 + y_1x_2, x_2y_2] = [y_1x_2 + x_1y_2, y_2x_2] = [y_1, y_2] + [x_1, x_2] = y + x$.

b) if $y = \Phi$ then $x + (\Phi + z) = x + \Phi = \Phi = \Phi + z = (x + \Phi) + z$.

If $y = -\infty$ then

$$\begin{aligned} \Phi + (-\infty + \Phi) &= (-\infty + \Phi) + \Phi = (\Phi + (-\infty)) + \Phi. \\ \Phi + (-\infty + (-\infty)) &= \Phi + (-\infty) = (\Phi + (-\infty)) + (-\infty). \\ \Phi + (-\infty + \infty) &= \Phi + \Phi = \Phi = \Phi + \infty = (\Phi + (-\infty)) + \infty. \end{aligned}$$

$$\Phi + (-\infty + z) = \Phi + (-\infty) = \Phi = \Phi + z = (\Phi + (-\infty)) + z, \text{ for all } z \in \mathbb{R}.$$

$$\begin{aligned} -\infty + (-\infty + \Phi) &= -\infty + \Phi = (-\infty + (-\infty)) + \Phi. \\ -\infty + (-\infty + (-\infty)) &= -\infty = (-\infty + (-\infty)) + (-\infty). \\ -\infty + (-\infty + \infty) &= -\infty + \Phi = \Phi = -\infty + \infty = (-\infty + (-\infty)) + \infty. \end{aligned}$$

$$-\infty + (-\infty + z) = -\infty + (-\infty) = -\infty = -\infty + z = (-\infty + (-\infty)) + z, \text{ for all } z \in \mathbb{R}.$$

$$\infty + (-\infty + \Phi) = \infty + \Phi = \Phi = \Phi + \Phi = (\infty + (-\infty)) + \Phi.$$

$$\infty + (-\infty + (-\infty)) = \infty + (-\infty) = \Phi = \Phi + (-\infty) = (\infty + (-\infty)) + (-\infty).$$

$$\infty + (-\infty + \infty) = \infty + \Phi = \Phi + \infty = (\infty + (-\infty)) + \infty.$$

$$\infty + (-\infty + z) = \infty + (-\infty) = \Phi = \Phi + z = (\infty + (-\infty)) + z, \text{ for all } z \in \mathbb{R}.$$

$$x + (-\infty + \Phi) = x + \Phi = \Phi = -\infty + \Phi = (x + (-\infty)) + \Phi, \text{ for all } x \in \mathbb{R}.$$

$x + (-\infty + (-\infty)) = x + (-\infty) = -\infty = -\infty + (-\infty) = (x + (-\infty)) + (-\infty)$, for all $x \in \mathbb{R}$.
 $x + (-\infty + \infty) = x + \Phi = \Phi = -\infty + \infty = (x + (-\infty)) + \infty$, for all $x \in \mathbb{R}$.
 $x + (-\infty + z) = x + (-\infty) = -\infty = -\infty + z = (x + (-\infty)) + z$, for all $z \in \mathbb{R}$ and all $x \in \mathbb{R}$.

If $y = \infty$ the result holds analogously.

If $y \in \mathbb{R}$ then

$\Phi + (y + \Phi) = (y + \Phi) + \Phi = (\Phi + y) + \Phi$.
 $\Phi + (y + (-\infty)) = \Phi + (-\infty) = (\Phi + y) + (-\infty)$.
 $\Phi + (y + \infty) = \Phi + \infty = (\Phi + y) + \infty$.
 $\Phi + (y + z) = \Phi + \Phi + z = (\Phi + y) + z$, for all $z \in \mathbb{R}$.
 $-\infty + (y + \Phi) = -\infty + \Phi = (-\infty + y) + \Phi$.
 $-\infty + (y + (-\infty)) = -\infty + (-\infty) = (-\infty + y) + (-\infty)$.
 $-\infty + (y + \infty) = -\infty + \infty = (-\infty + y) + \infty$.
 $-\infty + (y + z) = -\infty = -\infty + z = (-\infty + y) + z$, for all $z \in \mathbb{R}$.
 $\infty + (y + \Phi) = \infty + \Phi = (\infty + y) + \Phi$.
 $\infty + (y + (-\infty)) = \infty + (-\infty) = (\infty + y) + (-\infty)$.
 $\infty + (y + \infty) = \infty + \infty = (\infty + y) + \infty$.
 $\infty + (y + z) = \infty = \infty + z = (\infty + y) + z$, for all $z \in \mathbb{R}$.
 $x + (y + \Phi) = x + \Phi = \Phi = (x + y) + \Phi$, for all $x \in \mathbb{R}$.
 $x + (y + (-\infty)) = x + (-\infty) = -\infty = (x + y) + (-\infty)$, for all $x \in \mathbb{R}$.
 $x + (y + \infty) = x + \infty = \infty = (x + y) + \infty$, for all $x \in \mathbb{R}$.
 $x + (y + z) = (x + y) + z$, for all $z \in \mathbb{R}$ and for all $x \in \mathbb{R}$, from the additive associativity of real numbers.

c) $x + 0 = [x_1, x_2] + [0, 1] = [x_1 \times 1 + 0 \times x_2, x_2 \times 1] = [x_1, x_2] = x$.

d) This case is immediate.

e) $x \times y = [x_1, x_2] \times [y_1, y_2] = [x_1y_1, x_2y_2] = [y_1x_1, y_2x_2] = [y_1, y_2] \times [x_1, x_2] = y \times x$.

f) $(x \times y) \times z = ([x_1, x_2] \times [y_1, y_2]) \times [z_1, z_2] = ([x_1y_1, x_2y_2]) \times [z_1, z_2] = [(x_1y_1)z_1, (x_2y_2)z_2] = [x_1(y_1z_1), x_2(y_2z_2)] = [x_1, x_2] \times [y_1z_1, y_2z_2] = [x_1, x_2] \times ([y_1, y_2] \times [z_1, z_2]) = x \times (y \times z)$.

g) $x \times 1 = [x_1, x_2] \times [1, 1] = [x_1 \times 1, x_2 \times 1] = [x_1, x_2] = x$.

h) This case is immediate.

i) (I) $x \notin \{-\infty, \infty\}$.

Suppose $x = \Phi$. Then $x \times (y + z) = \Phi \times (y + z) = \Phi = \Phi + \Phi = (\Phi \times y) + (\Phi \times z) = (x \times y) + (x \times z)$.
 Suppose $x \in \mathbb{R}$. If $y = z = \infty$ or $y = z = -\infty$ then $x \times (y + z) = x \times y = y = y + y = (x \times y) + (x \times y) = (x \times y) + (x \times z)$. Otherwise $x \times (y + z) = [x, 1] \times ([y_1, y_2] + [z_1, z_2]) = [x, 1] \times [y_1z_2 + z_1y_2, y_2z_2] = [x \times (y_1z_2 + z_1y_2), 1 \times (y_2z_2)] = [xy_1z_2 + xz_1y_2, y_2z_2] = [xy_1, y_2] + [xz_1, z_2] = ([x, 1] \times [y_1, y_2]) + ([x, 1] \times [z_1, z_2]) = (x \times y) + (x \times z)$.

(II) $yz > 0$. Note that $\text{sgn}(y_1) = \text{sgn}(z_1)$.

If $y = z = \infty$ or $y = z = -\infty$ then $x \times (y + z) = [x_1, x_2] \times ([y_1, 0] + [z_1, 0]) = [x_1, x_2] \times ([\text{sgn}(y_1), 0] + [\text{sgn}(z_1), 0]) = [x_1, x_2] \times ([\text{sgn}(y_1), 0] + [\text{sgn}(y_1), 0]) = [x_1, x_2] \times [\text{sgn}(y_1), 0] = [x_1 \text{sgn}(y_1), x_2 \times 0] = [x_1 \text{sgn}(y_1), 0] = [x_1 \text{sgn}(y_1), 0] + [x_1 \text{sgn}(y_1), 0] = [x_1 \text{sgn}(y_1), x_2 \times 0] + [x_1 \text{sgn}(z_1), x_2 \times 0] = ([x_1, x_2] \times [\text{sgn}(y_1), 0]) + ([x_1, x_2] \times [\text{sgn}(z_1), 0]) = ([x_1, x_2] \times [y_1, 0]) + ([x_1, x_2] \times [z_1, 0]) = (x \times y) + (x \times z)$.
 Otherwise we have $x_2 = 0$ or $x_2 > 0$. If $x_2 = 0$ then $x \times (y + z) = [x_1, 0] \times ([y_1, y_2] + [z_1, z_2]) = [x_1, 0] \times [y_1z_2 + z_1y_2, y_2z_2] = [x_1 \times (y_1z_2 + z_1y_2), 0 \times (y_2z_2)] = [x_1(y_1z_2 + z_1y_2), 0] = [\text{sgn}(x_1) \text{sgn}(y_1z_2 + z_1y_2), 0] = [\text{sgn}(x_1) \text{sgn}(y_1), 0]$ and $(x \times y) + (x \times z) = ([x_1, 0] \times [y_1, y_2]) + ([x_1, 0] \times [z_1, z_2]) = [x_1y_1, 0 \times y_2] + [x_1z_1, 0 \times z_2] = [x_1y_1, 0] + [x_1z_1, 0] = [\text{sgn}(x_1) \text{sgn}(y_1), 0] + [\text{sgn}(x_1) \text{sgn}(z_1), 0] = [\text{sgn}(x_1) \text{sgn}(y_1), 0] + [\text{sgn}(x_1) \text{sgn}(z_1), 0] = [\text{sgn}(x_1) \text{sgn}(y_1), 0]$. If $x_2 > 0$ then $x \times (y + z) = [x_1, x_2] \times ([y_1, y_2] + [z_1, z_2]) = [x_1, x_2] \times [y_1z_2 + z_1y_2, y_2z_2] = [x_1 \times (y_1z_2 + z_1y_2), x_2 \times (y_2z_2)] = [x_1(y_1z_2 + z_1y_2), x_2y_2z_2] = [x_2x_1(y_1z_2 + z_1y_2), x_2(x_2y_2z_2)] = [x_1y_1x_2z_2 + x_1z_1x_2y_2, x_2x_2y_2z_2] = [x_1y_1, x_2y_2] + [x_1z_1, x_2z_2] = ([x_1, x_2] \times [y_1, y_2]) + ([x_1, x_2] \times [z_1, z_2]) = (x \times y) + (x \times z)$.

(III) $y + z = 0$.

We have $[y_1z_2 + z_1y_2, y_2z_2] = [y_1, y_2] + [z_1, z_2] = [0, 1]$. Thus $y_2 \neq 0$ and $z_2 \neq 0$, whence $y, z \in \mathbb{R}$ and $z = -y$. Thus $x \times (y + z) = x \times 0 = [x_1, x_2] \times [0, 1] = [x_1 \times 0, x_2 \times 1] = [0, x_2] = [0 \times x_2, x_2x_2] = [0, x_2x_2] = [x_1y_2x_2 - x_1y_2x_2, x_2x_2] = [x_1y, x_2] + [-x_1y, x_2] = [x_1y, x_2 \times 1] + [x_1(-y), x_2 \times 1] = ([x_1, x_2] \times [y, 1]) + ([x_1, x_2] \times [-y, 1]) = ([x_1, x_2] \times [y, 1]) + ([x_1, x_2] \times [z, 1]) = (x \times y) + (x \times z)$.

(IV) $x, y, z \in \{-\infty, \infty\}$.

If $y \neq z$ we can suppose, without loss of generality, that $x = \infty, y = -\infty$ and $z = \infty$, whence $x \times (y + z) = \infty \times (-\infty + \infty) = \infty \times \Phi = \Phi = -\infty + \infty = (\infty \times (-\infty)) + (\infty \times \infty) = (x \times y) + (x \times z)$. Otherwise we can suppose, without loss of generality, that $x = -\infty, y = \infty$ and $z = \infty$, whence $x \times (y + z) = -\infty \times (\infty + \infty) = -\infty \times \infty = -\infty = -\infty + (-\infty) = (-\infty \times \infty) + (-\infty \times \infty) = (x \times y) + (x \times z)$.

The equality $(y + z) \times x = (y \times x) + (z \times x)$ follows from the preceding equality and from multiplicative commutativity.

j) Suppose $x \leq y$. Then $\begin{cases} (a_1) z = \Phi \text{ or} \\ (b_1) z = -\infty \text{ or} \\ (c_1) z = \infty \text{ or} \\ (d_1) z \in \mathbb{R} \end{cases}$ and $\begin{cases} (a_2) x = \Phi \text{ or} \\ (b_2) x = -\infty \text{ or} \\ (c_2) x = \infty \text{ or} \\ (d_2) x \in \mathbb{R} \end{cases}$ and $\begin{cases} (a_3) y = \Phi \text{ or} \\ (b_3) y = -\infty \text{ or} \\ (c_3) y = \infty \text{ or} \\ (d_3) y \in \mathbb{R} \end{cases}$.

Notice that the condition pairs (a_2) and (b_3) , (a_2) and (c_3) , (a_2) and (d_3) , (b_2) and (a_3) , (b_2) and (c_3) , (b_2) and (d_3) , (c_2) and (a_3) , (c_2) and (c_3) , (c_2) and (d_3) , (d_2) and (a_3) , (d_2) and (c_3) , (d_2) and (d_3) .

and (a_3) do not occur because if $x = \Phi$ or $y = \Phi$ then $x = y = \Phi$. The pairs (c_2) and (b_3) , (c_2) and (d_3) do not occur because if $x = \infty$ then $y = \infty$. Furthermore, the pairs (d_2) and (b_3) do not occur because if $y = -\infty$ then $x = -\infty$.

If (a_1) occurs then $x + z = x + \Phi = \Phi = y + \Phi = y + z$. If (b_1) occurs then observe the following. If the pair (a_2) and (a_3) occurs then $x + z = \Phi + (-\infty) = y + z$. If the pair (b_2) and (b_3) occurs then $x + z = -\infty + (-\infty) = y + z$. By hypothesis the pair (b_2) and (c_3) does not occur. If the pair (b_2) and (d_3) occurs then $x + z = -\infty + (-\infty) = -\infty = y + (-\infty) = y + z$. If the pair (c_2) and (c_3) occurs then $x + z = \infty + (-\infty) = y + z$. By hypothesis the pair (d_2) and (c_3) does not occur. If the pair (d_2) and (d_3) occurs then $x + z = x + (-\infty) = -\infty = y + (-\infty) = y + z$.

If (c_1) occurs then the result follows analogously to the previous case.

If (d_1) occurs then observe the following. If the pair (a_2) and (a_3) occurs then $x + z = \Phi + z = y + z$. If the pair (b_2) and (b_3) occurs then $x + z = -\infty + z = y + z$. If the pair (b_2) and (c_3) occurs then $x + z = -\infty + z = -\infty < \infty = \infty + z = y + z$. If the pair (b_2) and (d_3) occurs then $x + z = -\infty + z = -\infty < y + z$. If the pair (c_2) and (c_3) occurs then $x + z = \infty + z = y + z$. If the pair (d_2) and (c_3) occurs then $x + z < \infty = \infty + z = y + z$. If the pair (d_2) and (d_3) occurs then the result follows from the real number order relation and real addition.

k) Suppose $x \leq y$. Then

$$\left\{ \begin{array}{l} (a_1) z = \infty \text{ or} \\ (b_1) z \in \mathbb{R} \text{ with } z \geq 0 \end{array} \right. \text{ and } \left\{ \begin{array}{l} (a_2) x = \Phi \text{ or} \\ (b_2) x = -\infty \text{ or} \\ (c_2) x = \infty \text{ or} \\ (d_2) x \in \mathbb{R} \end{array} \right.$$

$$\text{and } \left\{ \begin{array}{l} (a_3) y = \Phi \text{ or} \\ (b_3) y = -\infty \text{ or} \\ (c_3) y = \infty \text{ or} \\ (d_3) y \in \mathbb{R} \end{array} \right.$$

Condition pairs (a_2) and (b_3) , (a_2) and (c_3) , (a_2) and (d_3) , (b_2) and (a_3) , (c_2) and (a_3) , (d_2) and (a_3) do not occur because if $x = \Phi$ or $y = \Phi$ then $x = y = \Phi$. Also pairs (c_2) and (b_3) , (c_2) and (d_3) do not occur because if $x = \infty$ then $y = \infty$. The pair (d_2) and (b_3) does not occur because if $y = -\infty$ then $x = -\infty$.

If (a_1) occurs then observe the following. If the pair (a_2) and (a_3) occurs then $x \times z = \Phi \times \infty = y \times z$. If the pair (b_2) and (b_3) occurs then $x \times z = -\infty \times \infty = y \times z$. If the pair (b_2) and (c_3) occurs then $x \times z = -\infty \times \infty = -\infty < \infty = \infty \times \infty = y \times z$. If the pair (b_2) and (d_3) occurs then $x \times z = -\infty \times \infty = -\infty < \infty = y \times \infty = y \times z$. If the pair (c_2) and (c_3) occurs then $x \times z = \infty \times \infty = y \times z$. If the pair (d_2) and (c_3) occurs then $x \times z = \infty \times \infty = \infty = \infty \times \infty = y \times z$. If the pair (d_2) and (d_3) occurs then $x \times z = x \times \infty = \infty = y \times \infty = y \times z$.

If (b_1) occurs then observe the following. If the pair (a_2) and (a_3) occurs then $x \times z = \Phi \times z = y \times z$. If the pair (b_2) and (b_3) occurs then $x \times z = -\infty \times z = y \times z$. If the pair (b_2) and (c_3) occurs then $x \times z = -\infty \times z \leq \infty \times z = y \times z$. If the pair (b_2) and (d_3) occurs then,

by hypothesis, $z \neq 0$, whence $x \times z = -\infty \times z = -\infty < y \times z$. If the pair (c_2) and (c_3) occurs then $x \times z = \infty \times z = y \times z$. If the pair (d_2) and (c_3) occurs, notice that in this case, by hypothesis, $z \neq 0$, whence $x \times z < \infty = \infty \times z = y \times z$. If the pair (d_2) and (d_3) occurs then the result follows from the real number order relation and real multiplication.

- 1) If $\infty \notin A$ and A is bounded above, in the real sense, then the result follows from the Supremum Axiom. Otherwise ∞ is the unique upper bound of A , whence $\infty = \sup A$. ■

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