Existence and Exponential Stability of Pseudo Almost Periodic Solutions for Mackey-Glass Equation With Time-Varying Delay

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Abstract—**In this paper, a Mackey-Glass equation with time-varying delay is investigated. By applying Lyapunov functional method and differential inequality techniques, a set of sufficient conditions are obtained for the existence and exponential stability of pseudo almost periodic solutions of the model. Some numerical simulations are carried out to support the theoretical findings. Our results improve and generalize those of the previous studies.**

Index Terms—**Mackey-Glass equation, pseudo almost periodic solution, exponential stability, time-varying delay.**

I. Introduction

SINCE Mackey and Glass [1] proposed the following Mackey-Glass equation with delay

$$x'(t) = -\alpha(t)x(t) + \frac{\beta(t)x(t-\tau)}{1+\gamma(t)x^p(t-\tau)}, \tag{1}$$

as a model of hematopoiesis(blood-cell formation) in 1977, many works on the model (1) have been carried out. For example, Alzabut et al. [2] considered the existence and exponential stability of positive almost periodic solutions for (1.1), Gopalsamy et al. [3] analyzed the global attractivity for the model (1.1), Liz et al. [4] obtained the global stability criterion for a family of delayed population models. In details, one can see [5-9]. Recently, Myslo and Tkachenko [10] investigated the permanence, existence of a positive, asymptotically stable, piecewise-continuous, almost periodic solution of model (1) with pulse action. We know that the pseudo almost periodic functions are the natural generalization of the concept of almost periodicity. So far, no attention has been paid to the conditions for the global exponential stability on positive pseudo almost periodic solution of model (1) in terms of its coefficients. On the other hand, since the exponential convergent rate can be unveiled, the global exponential stability plays a key role in characterizing the behavior of dynamical system (see [11-1330-31]). Thus, it is worthwhile to continue to investigate the existence and global exponential stability of positive pseudo almost periodic solutions of (1).

Considering the variation of the environment in many biological and ecological dynamical systems and for the sake of simplicity(here we let $p = 1$ in model (1.1)), we are concerned with the following system

$$x'(t) = -\alpha(t)x(t) + \frac{\beta(t)x(t-\tau(t))}{1+\gamma(t)x(t-\tau(t))}, \tag{2}$$

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where $x \geq 0$, $\tau(t)$ is positive function, the piecewise-continuous functions $\alpha(t), \beta(t)$ and $\gamma(t)$ are positive definite.

The main aim of this article is to establish some sufficient conditions for the existence, uniqueness and exponential stability of pseudo almost periodic solutions of (2). To the best of our knowledge, it is the first time to focus on the existence, uniqueness and exponential stability of pseudo almost periodic solutions of (2). Our results obtained in this paper improve and generalize those in the previous studies [2-10]. Recently, there are few papers that deal with the pseudo almost periodic solutions of differential equations [14-24].

Throughout this paper, it will be assumed that $\alpha : \mathbb{R} \to (0, +\infty)$ is an almost function, $\tau : \mathbb{R} \to [0, +\infty)$ and $\beta, \gamma : \mathbb{R} \to (0, +\infty)$ are uniformly continuous pseudo almost periodic functions, and

$$\begin{cases} \alpha^- = \inf_{t\in\mathbb{R}} \alpha(t) > 0, \alpha^+ = \sup_{t\in\mathbb{R}} \alpha(t), \\ \beta^- = \inf_{t\in\mathbb{R}} \beta(t) > 0, \beta^+ = \sup_{t\in\mathbb{R}} \beta(t), \\ \gamma^- = \inf_{t\in\mathbb{R}} \alpha(t) > 0, \gamma^+ = \sup_{t\in\mathbb{R}} \alpha(t), \\ \tau = \sup_{t\in\mathbb{R}} \tau(t) > 0. \end{cases} \tag{3}$$

Let $\mathbb{R}^+$ denote nonnegative real number space, let $C = C([-\tau, 0], \mathbb{R})$ be the continuous function space equipped with the usual supremum norm $||.||$, and let $C_+ = C([-\tau, 0], \mathbb{R}_+)$. If $x(t)$ is defined on $[-\tau + t_0, \sigma)$ with $t_0, \sigma \in \mathbb{R}$, then we define $x_t(\theta) \in C$ where $x_t(\theta) = x(t+\theta)$ for all $\theta \in [-\tau, 0]$. Due to the biological interpretation of model (1), only positive solutions are meaningful and therefore admissible. Thus we just consider admissible initial condition.

$$x_{t_0} = \varphi, \varphi \in C_+, \varphi(0) > 0. \tag{4}$$

We write $x_t(t_0, \varphi)(x(t; t_0, \varphi)$ for an admissible solution of the admissible initial value problem (2) and (3). Also, let $[t_0, \eta(\varphi))$ be the maximal right interval of existence of $x_t(t_0, \varphi)$.

The remainder of the paper is organized as follows. In Section 2, we introduce some lemmas and definitions, which can be used to check the existence of almost periodic solutions of system (2). In Section 3, we present some new sufficient conditions for the existence of the continuously differentiable pseudo almost periodic solution of (2). Some sufficient conditions on the global exponential stability of pseudo almost periodic solutions of (2) are established in Section 4. An example and its numerical simulations are given to illustrate the effectiveness of the obtained results in Section 5.

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II. Preliminary results

In this section, we would like to recall some basic definitions and lemmas which are used in what follows. In this paper, $\mathrm{BC}(\mathbb{R},\mathbb{R})$ denotes the set of bounded continued functions from $\mathbb{R}$ to $\mathbb{R}$. Note that $(\mathrm{BC}(\mathbb{R},\mathbb{R})), ||.||_\infty)$ is a Banach space where $||.||$ denotes the sup norm $||.||_\infty := \sup_{t\in\mathbb{R}} ||f(t)||$.

Definition 2.1. (see [25-26]) *Let $u(t) \in BC(\mathbb{R},\mathbb{R}), u(t)$ is said to be almost periodic on $\mathbb{R}$ if, for any $\varepsilon > 0$, the set $T(u,\varepsilon) = \{\delta : ||u(t+\delta) - u(t)|| < \varepsilon$ for all $t \in \mathbb{R}\}$ is relatively dense; that is, for any $\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$; for any interval with length $l(\varepsilon)$, there exists a number $\delta = \delta(\varepsilon)$ in this interval such that $||u(t+\delta) - u(t)|| < \varepsilon$, for all $t \in \mathbb{R}$.*

We denote by $\mathrm{AP}(\mathbb{R},\mathbb{R})$ the set of the almost periodic functions from $\mathbb{R}$ to $\mathbb{R}$. Besides, the concept of pseudo almost periodicity (pap) was introduced by Zhang [25] in the early nineties. It is a natural generalization of the classical almost periodicity. Precisely, define the class of functions $\mathrm{PAP}_0(\mathbb{R},\mathbb{R})$ as follows:

$$\left\{ f \in \mathrm{BC}(\mathbb{R},\mathbb{R}) | \lim_{t\to+\infty} \frac{1}{2T}\int_{-T}^{T} |f(t)|dt = 0 \right\}.$$

A function $f \in \mathrm{BC}(\mathbb{R},\mathbb{R})$ is called pseudo almost periodic if it can be expressed as $f = h + \varphi$, where $h \in \mathrm{AP}(\mathbb{R},\mathbb{R})$ and $\varphi \in \mathrm{PAP}_0(\mathbb{R},\mathbb{R})$. The collection of such functions will be denoted by $\mathrm{PAP}(\mathbb{R},\mathbb{R})$. The functions h and φ in the above definition are, respectively, called the almost periodic component and the ergodic perturbation of the pseudo almost periodic function f. The decomposition given in definition above is unique. Observe that $(\mathrm{PAP}(\mathbb{R},\mathbb{R}), ||.||)$ is a Banach space and $\mathrm{AP}(\mathbb{R},\mathbb{R})$ is a proper subspace of $(\mathrm{PAP}(\mathbb{R},\mathbb{R})$ since the function $\phi(t) = \sin^2 t + \sin^2 \sqrt{5}t + \exp(-t^2 \sin^2 t)$ is pseudo almost periodic function but not almost periodic. It should be mentioned that pseudo almost periodic functions possess many interesting properties; we shall need only a few of them and for the proofs we shall refer to [25].

Lemma 2.2. (see [15]) *Let $x_1(.), \sigma(.) \in AP(\mathbb{R},\mathbb{R})$, $\sigma^{'}(.) \in BC(\mathbb{R},\mathbb{R})$ and $x_2(.) \in PAP_0(\mathbb{R},\mathbb{R})$. Then (1) $x_1(t-\sigma(t)) \in AP(\mathbb{R},\mathbb{R})$; (2) $x_2(t-\sigma(t)) \in PAP_0(\mathbb{R},\mathbb{R})$ if $(1-\sigma^{'}(t))^- = \inf_{t\in\mathbb{R}}(1-\sigma^{'}(t)) > 0$.*

Remark 2.3. (see [15]) *Set $x(.) = x_1(.) + x_2(.)$ with $x_1(.) \in AP(\mathbb{R},\mathbb{R})$ and $x_2(.) \in PAP_0(\mathbb{R},\mathbb{R})$. It follows from Lemma 2.2 that $x(t-\sigma(t)) \in PAP(\mathbb{R},\mathbb{R})$ if $(1-\sigma^{'}(t))^- > 0$ and $\sigma(.) \in AP(\mathbb{R},\mathbb{R}), \sigma^{'}(.) \in BC(\mathbb{R},\mathbb{R})$.*

Lemma 2.4. (see [25, page 140]) *Suppose that both functions f and its derivative $f^{'}$ are in $PAP(\mathbb{R},\mathbb{R})$. That is, $f = g + \varphi$ and $f^{'} = \alpha + \beta$, where $g, \alpha \in AP(\mathbb{R},\mathbb{R})$ and $\varphi, \beta \in PAP_0(\mathbb{R},\mathbb{R})$. Then the functions g and φ are continuous differentiable so that $g^{'} = \alpha, \varphi^{'} = \beta$.*

Definition 2.5. (see [27-28]) *Let $x \in \mathbb{R}^p$ and $Q(t)$ be a $p\times p$ continuous matrix defined on $\mathbb{R}$. The linear system*

$$x^{'}(t) = Q(t)x(t) \tag{5}$$

is said to admit an exponential dichotomy on $\mathbb{R}$ if there exist positive constants k, α and projection P and the fundamental solution matrix $X(t)$ of (5) satisfying

$$||X(t)PX^{-1}(s)|| \le ke^{-\alpha(t-s)}, \text{ for } t \ge s,$$

$$||X(t)(I-P)X^{-1}(s)|| \le ke^{-\alpha(S-T)}, \text{ for } t \le s.$$

Lemma 2.6. (see [27]) *Assume that $Q(t)$ is an almost periodic matrix function and $g(t) \in PAP(\mathbb{R},\mathbb{R}^p)$. If the linear system (5) admits an exponential dichotomy, then pseudo almost periodic system*

$$x^{'}(t) = Q(t)x(t) + g(t) \tag{6}$$

has a unique pseudo almost periodic solution $x(t)$, and

$$\begin{aligned} x(t) = &\int_{-\infty}^{t} X(t)PX^{-1}(s)g(s)ds \\ &- \int_{t}^{+\infty} X(t)(I-P)X^{-1}(s)g(s)ds. \end{aligned} \tag{7}$$

Lemma 2.7. (see [27-28]) *Let $a_i(t)$ be an almost periodic function on $\mathbb{R}$ and*

$$M[c_i] = \lim_{T\to+\infty} \frac{1}{T}\int_{t}^{t+T} c_i(s)ds > 0, i = 1, 2, \cdots, p. \tag{8}$$

Then the linear system

$$x^{'}(t) = diag(-a_1(t), -a_2(t), \cdots, -a_p(t))x(t) \tag{9}$$

admits an exponential dichotomy on $\mathbb{R}$.

Lemma 2.8 *Every solution $x(t; t_0, \varphi)$ of (2) and (4) is positive and bounded on $[t_0, \eta(\varphi))$, and $\eta(\varphi) = +\infty$.*

The proof of Lemma 2.8 is similar as that in Zhang et al. [6]. Here we omit it.

III. Existence of pseudo almost periodic solutions

In this section, we will the establish sufficient conditions on the existence of pseudo almost periodic solutions of (2).

Lemma 3.1. *Suppose that there exist two constants m and M such that*

$$\begin{cases} M > m, \sup_{t\in R}\left\{-\alpha(t)M + \frac{\beta(t)}{\gamma(t)}\right\} < 0, \\ \inf_{t\in R}\left\{-\alpha(t)m + \frac{\beta(t)m}{1+\gamma(t)M}\right\} > 0 \end{cases} \tag{10}$$

Then there exists $t_\varphi > t_0$ such that $m < x(t; t_0, \varphi) < M$, for all $t \ge t_\varphi$.

Proof Let $x(t) = x(t; t_0, \varphi)$. We first claim that there exists a $\bar{t} \in [t_0, +\infty)$ such that

$$x(\bar{t}) < M. \tag{11}$$

Otherwise,

$$x(t) \ge M, \text{ for all } t \in [t_0, +\infty), \tag{12}$$

which, together with (10), implies that

$$\begin{aligned} x^{'}(t) &= -\alpha(t)x(t) + \frac{\beta(t)x(t-\tau(t))}{1+\gamma(t)x(t-\tau(t))}, \\ &\le -\alpha(t)x(t) + \frac{\beta(t)}{\gamma(t)} \le -\alpha(t)M + \frac{\beta(t)}{\gamma(t)} \\ &\le \sup_{t\in R}\left\{-\alpha(t)M + \frac{\beta(t)}{\gamma(t)}\right\}, \\ &< 0, \text{for all } t \ge t_0 + \tau. \end{aligned} \tag{13}$$

Then

$$\begin{aligned} x(t) &= x(t_0+\tau)+\int_{t_0+\tau}^{t} x^{'}(s)ds \\ &\leq x(t_0+\tau)+\sup_{t\in R}\left\{-\alpha(t)M+\frac{\beta(t)}{\gamma(t)}\right\} \\ &\quad \times[t-(t_0+\tau)],\ \text{for all } t\geq t_0+\tau. \end{aligned} \tag{14}$$

Thus

$$\lim_{t\to+\infty} x(t)=-\infty. \tag{15}$$

which contradicts the fact that $x(t)$ is positive and bounded on $[t_0,+\infty)$. Hence, (11) holds. In the sequel, we prove that

$$x(t)<M,\ \text{for all } t\in[\bar{t},+\infty). \tag{16}$$

Suppose, for the sake of contradiction, there exists $t^*\in[\bar{t},+\infty)$ such that

$$x(t^*)=M, x(t)<M,\ \text{for all } t\in[\bar{t},T^*). \tag{17}$$

Calculating the derivative of $x(t)$, together with (10), (2), and (17), implies that

$$\begin{aligned} x^{'}(t^*) &= -\alpha(t^*)x(t^*)+\frac{\beta(t)x(t^*-\tau(t^*))}{1+\gamma(t^*)x(t^*-\tau(t^*))}, \\ &\leq -\alpha(t^*)M+\frac{\beta(t^*)}{\gamma(t^*)}<0, \end{aligned} \tag{18}$$

which is a contradiction and implies that (16) holds. We finally show that $s=\liminf_{t\to\infty}x(t)>m$. By way of contradiction, we assume that $0\leq s\leq m$. By the fluctuation lemma in [29, Lemma A.1.], there exists a sequence $\{t_k\}_{k\geq 1}$ such that

$$\begin{cases} t_k\to+\infty, x(t_k)\to\lim_{t\to+\infty}\inf x(t), \\ x^{'}(t_k)\to 0 \text{ as } k\to+\infty. \end{cases} \tag{19}$$

Since $\{x_k\}$ is bounded and equicontinuous, by the Ascoli-Arzel$\acute{a}$ theorem, there exists a subsequence, still denoted by itself for simplicity of notation, such that

$$x_{t_k}\to\varphi^*(k\to+\infty)\quad \text{for some}\quad \varphi^*\in C_+. \tag{20}$$

Moreover,

$$\varphi^*(0)=s\leq\varphi^*(\theta)\leq M \text{ for } \theta\in[-\tau,0). \tag{21}$$

Without loss of generality, we assume that $\beta(t),\beta(t),\gamma(t)$ and $\tau(t)$ are convergent to β^*,β^*,γ^* and τ^*, respectively. This assumptions are reasonable because $\beta(t),\beta(t),\gamma(t)$ and $\tau(t)$ are all almost periodic. It follows from

$$x'(t_k)=-\alpha(t_k)x(t_k)+\frac{\beta(t_k)x(t_k-\tau(t_k))}{1+\gamma(t_k)x(t_k-\tau(t_k))} \tag{22}$$

that(taking limits)

$$\begin{aligned} 0 &= -\alpha^*s+\frac{\beta^*\varphi^*(-\tau)}{1+\gamma^*\varphi^*(-\tau)} \\ &\geq -\alpha^*s+\frac{\beta^*s}{1+\gamma^*M} \\ &\geq \inf_{t\in R}\left\{-\alpha(t)m+\frac{\beta(t)m}{1+\gamma(t)M}\right\}>0 \end{aligned} \tag{23}$$

which is a contradiction. This proves that $l>m$. Hence, from (3.2), we can choose $t_\varphi>t_0$ such that $m<x(t;t_0,\varphi)<M$. This proof of Lemma 3.1 is complete.

IV. Exponential stability of pseudo almost periodic solutions

In this section, we will obtain the exponential stability of the pseudo almost periodic solution of system (2).

Theorem 4.1. *Suppose that*

$$\begin{cases} \alpha,\tau\in AP(\mathbb{R},\mathbb{R}), \tau^{'}(.)\in(\mathbb{R},\mathbb{R}), \\ \beta\in PAP(\mathbb{R},\mathbb{R}), \inf_{t\in\mathbb{R}}(1-\tau^{'}(t))>0 \end{cases} \tag{24}$$

and there exist two positive constants m and M satisfying (10) and

$$\sup_{t\in\mathbb{R}}\left\{-\alpha(t)+\beta(t)\frac{1}{4m}\right\}<0. \tag{25}$$

Then there exists a unique positive pseudo almost periodic solution of (2) in the region $B^*=\{\varphi|\varphi\in PAP(\mathbb{R},\mathbb{R}), m\leq\varphi(t)\leq M,$ for all $t\in\mathbb{R}\}$.

Proof Consider $\chi:[0,1]\to\mathbb{R}$ defined by

$$\chi(u)=\sup_{t\in\mathbb{R}}\left\{-\alpha(t)+\beta(t)\frac{1}{4m}e^u\right\}, u\in[0,1]. \tag{26}$$

Then

$$\chi(0)=\sup_{t\in\mathbb{R}}\left\{-\alpha(t)+\beta(t)\frac{1}{4m}\right\}<0, u\in[0,1]. \tag{27}$$

It follows from (26) that there exists a constat $\xi\in(0,1]$ such that

$$\chi(\xi)=\sup_{t\in\mathbb{R}}\left\{-\alpha(t)+\beta(t)\frac{1}{4m}e^\xi\right\}<0, u\in[0,1]. \tag{28}$$

For any $\phi\in\mathrm{PAP}(\mathbb{R},\mathbb{R})$, In view of (24), Remark 2.3, and the composition theorem of pseudo almost periodic functions [25], we get

$$\frac{\beta(t)\phi(t-\tau(t))}{1+\gamma(t)\phi(t-\tau(t))}\in\mathrm{PAP}(\mathbb{R},\mathbb{R}). \tag{29}$$

Next we consider the auxiliary equation:

$$x'(t)=-\alpha(t)x(t)+\frac{\beta(t)\phi(t-\tau(t))}{1+\gamma(t)\phi(t-\tau(t))}. \tag{30}$$

Since $M[\alpha]>0$, then by Lemma 2.7, we can conclude that the line equation

$$x'(t)=-\alpha(t)x(t) \tag{31}$$

admits an exponential dichotomy on $\mathbb{R}$. Thus, by Lemma 2.7, we obtain that system (30) has exactly one pseudo almost periodic solution:

$$x_\phi(t)=\int_{-\infty}^{t}e^{-\int_s^t\alpha(u)du}\left[\frac{\beta(s)\phi(s-\tau(s))}{1+\gamma(s)\phi(s-\tau(s))}\right]ds. \tag{32}$$

Define a mapping $T:\mathrm{PAP}(\mathbb{R},\mathbb{R})\to\mathrm{PAP}(\mathbb{R},\mathbb{R})$ by setting

$$T(\phi(t))=x_\phi(t),\ \text{for all } \phi\in\mathrm{PAP}(\mathbb{R},\mathbb{R}) \tag{33}$$

By the definition of B^*, we can easily know that B^* is a closed subset of $\mathrm{PAP}(\mathbb{R},\mathbb{R})$. For any $\phi\in B^*$, by (10), we have

$$x_\phi(t) \le \int_{-\infty}^{t} e^{-\int_s^t \alpha(u)du}\left[\frac{\beta(s)}{\gamma(s)}\right]ds \le \int_{-\infty}^{t} e^{-\int_s^t \alpha(u)du}\alpha(s)Mds \le M, \quad (34)$$

$$x_\phi(t) \ge \int_{-\infty}^{t} e^{-\int_s^t \alpha(u)du}\left[\frac{\beta(s)m}{1+\gamma(s)M}\right]ds \ge \int_{-\infty}^{t} e^{-\int_s^t \alpha(u)du}\alpha(s)mds \ge m \quad (35)$$

for all $t \in \mathbb{R}$. Thus we can conclude that the mapping T is a self-mapping from B^* to B^*. Next we prove that the mapping T is a contraction mapping on B^*. For $\varphi, \psi \in B^*$, we have

$$\begin{aligned}&||T(\varphi) - T(\psi)||_\infty = \sup_{t\in\mathbb{R}} |T(\varphi) - T(\psi)| \\ &= \sup_{t\in\mathbb{R}}\left|\int_{-\infty}^{t} e^{-\int_s^t \alpha(u)du}\beta(s)\left[\frac{\varphi(t-\tau(s))}{1+\gamma(s)\varphi(s-\tau)}\right.\right. \\ &\left.\left.-\frac{\psi(t-\tau(s))}{1+\gamma(s)\psi(s-\tau)}\right]ds\right|. \quad (36)\end{aligned}$$

Notice that

$$\begin{aligned}&\left|\frac{x}{1+\gamma(t)x} - \frac{y}{1+\gamma(t)y}\right| = \frac{1}{(1+\gamma(t)\theta)^2}|x-y| \\ &= \frac{1}{4\gamma(t)\theta}|x-y| \le \frac{1}{4m}|x-y|, \quad (37)\end{aligned}$$

where $x, y \in [m, M]$ and θ lies between x and y, it follows from (28), (34),(35) and (36) that

$$\begin{aligned}&||T(\varphi) - T(\psi)||_\infty \le \sup_{t\in\mathbb{R}}\left|\int_{-\infty}^{t} e^{-\int_s^t \alpha(u)du}\beta(s)\right. \\ &\times\frac{1}{4m}|\varphi(s-\tau(s)) - \psi(s-\tau(s)|ds \\ &\le \sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-\int_s^t \alpha(u)du}a(s)e^{-\xi} \\ &\times|\varphi(s-\tau(s)) - \psi(s-\tau(s)|ds \\ &\le e^{-\xi}||\varphi - \psi||_\infty. \quad (38)\end{aligned}$$

Since $e^{\xi} < 1$, we can easily know that the mapping T is a contraction on B^* Applying Theorem 0.3.1 in [8], we obtain that the mapping T possesses a unique fixed point $\varphi^* \in B^*, T\varphi^* = \varphi^*$. By (30), φ^* satisfies (2). Thus φ^* is a positive pseudo almost periodic solution of (2) in B^*. The proof of Theorem 4.1 is now complete.

Theorem 4.2. *If the assumptions of Theorem 4.1, then (2) has at least one positive pseudo almost periodic solution $x^*(t)$. Moreover, $x^*(t)$ is globally exponentially stable, i.e., there exist constants $C_{\varphi,x^*}, t_{\varphi,x^*}$ and $\lambda > 0$ such that*

$$|x(t;t_0,\varphi) - x^*(t)| < C_{\varphi,x^*}e^{-\lambda t}, \text{ for all } t > t_{\varphi,x^*}. \quad (39)$$

Proof According to Theorem 4.1, we know that (2) has a positive pseudo almost periodic solution $x^*(t)$. It suffices to show that $x^*(t)$ is globally exponentially stable. Define a continuous function $\Pi(u)$ as follows

$$\Pi(u) = \sup_{t\in\mathbb{R}}\left\{-[\alpha(t) - u] + \beta(t)\frac{1}{4m}e^{\tau u}\right\}, u \in [0, 1]. \quad (40)$$

Then

$$\Pi(0) = \sup_{t\in\mathbb{R}}\left\{-\alpha(t) + \beta(t)\frac{1}{4m}\right\} < 0, \quad (41)$$

which implies that there exist two constants $\eta > 0$ and $\lambda \in (0, 1]$ such that

$$\Pi(\lambda) = \sup_{t\in\mathbb{R}}\left\{-[\alpha(t) - \lambda] + \beta(t)\frac{1}{4m}e^{\lambda\tau}\right\} < -\eta < 0. \quad (42)$$

Let $x(t) = x(t; t_0, \varphi)$ and $z(t) = x(t) - x^*(t)$, where $t \in [t_0 - \tau, +\infty)$. Then

$$\begin{aligned}z'(t) &= -\alpha(t)z(t) + \beta(t)\left[\frac{x(t-\tau(t))}{1+\gamma(t)x(t-\tau(t))}\right. \\ &\left.-\frac{x^*(t-\tau(t))}{1+\gamma(t)x^*(t-\tau(t))}\right]. \quad (43)\end{aligned}$$

By Lemma 3.1, we know that there exists $t_{\varphi,x^*} > t_0$ such that

$$m \le x(t), x^*(t) \le M, \text{ for all } t \in [t_{\varphi,x^*} - \tau, +\infty). \quad (44)$$

Now we consider the Lyapunov functional

$$V(t) = |z(t)|e^{\lambda t}. \quad (45)$$

Calculating the upper left derivative of $V(t)$ along the solution $z(t)$ of (43), we obtain

$$\begin{aligned}&D^-(V(t)) \le -\alpha(t)|z(t)|e^{\lambda t} + \beta(t)\left|\frac{x(t-\tau(t))}{1+\gamma(t)x(t-\tau(t))}\right. \\ &\left.-\frac{x^*(t-\tau(t))}{1+\gamma(t)x^*(t-\tau(t))}\right|e^{\lambda t} + \lambda|z(t)|e^{\lambda t} \\ &= \left[-(\alpha(t)-\lambda)|z(t)| + \beta(t)\left|\frac{x(t-\tau(t))}{1+\gamma(t)x(t-\tau(t))}\right.\right. \\ &\left.\left.-\frac{x^*(t-\tau(t))}{1+\gamma(t)x^*(t-\tau(t))}\right|\right]e^{\lambda t}, \quad (46)\end{aligned}$$

for all $t > t_{\varphi,x^*}$. In the sequel, we claim that

$$\begin{aligned}V(t) = |z(t)|e^{\lambda t} &< e^{\lambda t_{\varphi,x^*}} \\ &\times\left(\max_{t\in[t_0-\tau, t_{\varphi,x^*}]}|x(t) - x^*(t)| + 1\right) := C_{\varphi,x^*}, \quad (47)\end{aligned}$$

for all $t > t_{\varphi,x^*}$. Otherwise, there must exists $\underline{t} > t_{\varphi,x^*}$ such that

$$V(\underline{t}) = C_{\varphi,x^*}, V(t) < C_{\varphi,x^*}, \text{ for all } t \in [t_0 - \tau, \underline{t}). \quad (48)$$

In view of (37), (46) and (48), we get

$$\begin{aligned}&D^-(V(\underline{t})) \le \left[-(\alpha(\underline{t}) - \lambda)|z(\underline{t})| + \beta(\underline{t})\right. \\ &\left.\times\left|\frac{x(\underline{t}-\tau(\underline{t}))}{1+\gamma(t)x(\underline{t}-\tau(\underline{t}))} - \frac{x^*(\underline{t}-\tau(\underline{t}))}{1+\gamma(\underline{t})x^*(\underline{t}-\tau(\underline{t}))}\right|\right]e^{\lambda\underline{t}} \\ &\le -(\alpha(\underline{t}) - \lambda)|z(\underline{t})|e^{\lambda\underline{t}} + \beta(\underline{t})\frac{1}{4m}e^{\lambda\tau(\underline{t})}e^{\lambda(\underline{t}-\tau(\underline{t}))} \\ &\times|z(\underline{t}-\tau(\underline{t}))| \\ &\le \left[-(\alpha(\underline{t}) - \lambda) + \beta(\underline{t})\frac{1}{4m}e^{\lambda\tau}\right]C_{\varphi,x^*}. \quad (49)\end{aligned}$$

Thus $-(\alpha(\underline{t}) - \lambda) + \beta(\underline{t})\frac{1}{4m}e^{\lambda\tau} \ge 0$, which contradicts (42). Hence, (47) holds. It follows that

$$z(t)| < C_{\varphi,x^*}e^{-\lambda\tau}, \text{ for all } t > t_{\varphi,x^*}.$$

The proof of Theorem 4.2 is complete.

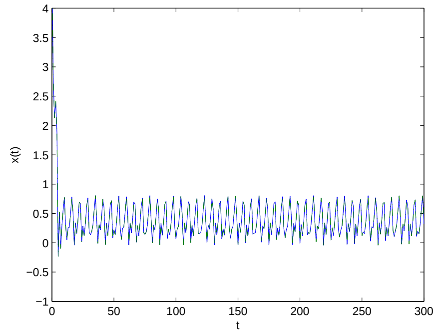

Fig. 1. Time response of state variable x.

V. An example

In this section, we will give an example to illustrate the feasibility and effectiveness of our main results obtained in previous sections. Considering the following Mackey-Glass equation with delay

$$x'(t) = -3x(t) + \left(4.24 + \frac{1}{4}|\sin\sqrt{3}t| + \frac{1}{50}\frac{1}{1+t^2}\right) \times \frac{x(t-e^{0.1\sin t})}{1+\left(4.24+\frac{1}{4}|\sin\sqrt{3}t| + \frac{1}{50}\frac{1}{1+t^2}\right)x(t-e^{0.1\sin t})}. \quad (50)$$

Hence $\alpha^+ = \alpha^- = 3, \beta^+ = 4.51, \beta^- = 3.97, \gamma^+ = 2.125, \gamma^- = 1.865, \tau = e^0.1, 1 - \tau^{'}(t) = 1 - 0.1e^{0.1\sin t}\cos t > 0$. Let $m = 0.3, M = 1$, then

$$-\alpha^- M + \frac{\beta^+}{\gamma^-} = -0.6 < 0,$$

and

$$-\alpha^- m + \frac{\beta^+ m}{1+\gamma^- M} = 0.03 > 0.$$

Then all the conditions in Theorem 4.2 are satisfied, Therefore, (50) has a unique positive pseudo almost periodic solution $x^*(t)$, which is globally exponentially stable. The results are verified by the numerical simulations in Fig.1.

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