A KN-like Hierarchy with Variable Coefficients and Its Hamiltonian Structure and Exact Solutions

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Abstract—**Starting from the isospectral problem equipped with a loop algebra $\widetilde{A}_1$, we first derive a new and more general KN-like hierarchy with two coefficient functions $\alpha(x)$ and $\beta(t)$. We then establish a Hamiltonian structure of the KN-like hierarchy under the condition that $\beta(t)$ is a non-zero constant by the use of Tu's scheme. Finally, we obtain some exact solutions of the first two equations of the KN-like hierarchy and give an open problem. This paper shows that the KN-like hierarchy is not only Lax integrable but also conditional Liouville integrable.**

Index Terms—**KN-like hierarchy with variable coefficients, isospectral problem, loop algebra, Hamiltonian structure, exact solution.**

I. Introduction

RECENTLY, the study of integrable systems is relatively active because of searching for as many as integrable systems and studying their properties are of both theoretical and practical value [1]. Some meaningful integrable systems have been obtained, such as the ones in [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], which include Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy, Kaup–Newell (KN) hierarchy, Giachetti–Johnson (GJ) hierarchy, and so on. In [2], Tu employed a subalgebra of the loop algebra $\widetilde{A}_1$ to construct some integrable hierarchies and established their corresponding Hamiltonian structures by the trace identity. It is shown that Tu's scheme [2] provides a powerful tool for constructing the Hamiltonian structure of integrable systems.

Since the variable-coefficient systems could describe more realistic physical phenomena than their constant-coefficient counterparts when the inhomogeneities of media and nonuniformities of boundaries are taken into account [25], we shall derive in this paper a KN-like hierarchy with variable coefficients:

$$u_t = \begin{pmatrix} q \\ r \end{pmatrix}_t = JL^{n-1}\begin{pmatrix} \dfrac{\beta(t)}{\alpha(x)}\omega r \\ \dfrac{\beta(t)}{\alpha(x)}\omega q \end{pmatrix} - \frac{\beta'(t)}{\beta(t)}\begin{pmatrix} q \\ r \end{pmatrix}, \tag{1}$$

where $n = 1, 2, \cdots$, $\alpha(x)$ and $\beta(t)$ are two non-zero and smooth functions of x and t respectively, ω is a non-zero constant, the symmetric operator and the recursive operator are employed as:

$$J = \frac{1}{\beta(t)}\begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \quad L = \frac{1}{2\alpha(x)}\begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} \tag{2}$$

with

$$l_{11} = -\partial - \beta^2(t)r\partial^{-1}\frac{q}{\alpha(x)}\partial,$$

$$l_{12} = -\beta^2(t)r\partial^{-1}\frac{r}{\alpha(x)}\partial, \quad l_{21} = -\beta^2(t)q\partial^{-1}\frac{q}{\alpha(x)}\partial,$$

$$l_{22} = \partial - \beta^2(t)q\partial^{-1}\frac{r}{\alpha(x)}\partial.$$

In particular, if we set $\alpha(x) = 1$ and $\beta(t) = 1$, Eq. (1) becomes the known constant-coefficient KN-like hierarchy [7]. So, the KN-like hierarchy (1) to be constructed is more general than the one in [7]. When $n = 1$, the KN-like hierarchy (1) gives a system of new and more general variable-coefficient linear equations:

$$q_t = \frac{\omega}{\alpha(x)}q_x - \frac{\alpha'(x)}{\alpha^2(x)}\omega q - \frac{\beta'(t)}{\beta(t)}q, \tag{3}$$

$$r_t = \frac{\omega}{\alpha(x)}r_x - \frac{\alpha'(x)}{\alpha^2(x)}\omega r - \frac{\beta'(t)}{\beta(t)}r. \tag{4}$$

When we set $n = 2$, the KN-like hierarchy (1) generates the following new and more general variable-coefficient nonlinear equations:

$$\begin{aligned} q_t = \frac{\omega}{2\alpha^4(x)}[&\alpha^2(x)q_{xx} - \alpha(x)\alpha''(x)q - 3\alpha'(x)\alpha(x)q_x \\ &+ 3\alpha'^2(x)q - 2\alpha(x)\beta^2(t)qrq_x - \alpha(x)\beta^2(t)q^2r_x \\ &+3\alpha'(x)\beta^2(t)q^2r] - \frac{\beta'(t)}{\beta(t)}q, \end{aligned} \tag{5}$$

$$\begin{aligned} r_t = \frac{\omega}{2\alpha^4(x)}[&-\alpha^2(x)r_{xx} + \alpha(x)\alpha''(x)r \\ &+ 3\alpha'(x)\alpha(x)r_x - 3\alpha'^2(x)r - 2\alpha(x)\beta^2(t)qrr_x \\ &- \alpha(x)\beta^2(t)r^2q_x + 3\alpha'(x)\beta^2(t)qr^2] - \frac{\beta'(t)}{\beta(t)}r. \end{aligned} \tag{6}$$

Since the soliton phenomena were first observed in 1834 and the Korteweg de–Vries (KdV) equation was solved by the inverse scattering method [26], solving nonlinear partial differential equations (PDEs) has gradually developed into a significant direction in nonlinear science and many solutions were obtained like those in [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41]. More recently, there are some interesting results in solving fractional differential equations and stochastic evolution equations, for example the work in [42], [43], [44]. However, to the best of our knowledge, there is no work on solving the KN-like hierarchy [7] and Eqs. (3)–(6). In the present paper, some exact solutions of Eqs. (3)–(6) are obtained.

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The rest of the paper is organized as follows. In Section 2, we derive the KN-like hierarchy (1) by introducing a new and more general isospectral problem equipped with a loop algebra $\widetilde{A}_1$. In Section 3, we use Tu's scheme [2] to establish a Hamiltonian structure of the KN-like hierarchy (1) in the condition that $\beta(t)$ is a non-zero constant. In Section 4, we give some exact solutions of Eqs. (3)–(6). In Section 5, we conclude this paper.

II. A KN-like hierarchy with variable coefficients

Firstly, we select a set of bases of loop algebra $\widetilde{A}_1$ [7]:

$$h(n)=\begin{pmatrix}\lambda^{2n} & 0\\ 0 & -\lambda^{2n}\end{pmatrix}, \tag{7}$$

$$e(n)=\begin{pmatrix}0 & \lambda^{2n+1}\\ 0 & 0\end{pmatrix},\quad f(n)=\begin{pmatrix}0 & 0\\ \lambda^{2n+1} & 0\end{pmatrix}, \tag{8}$$

$$[h(m),e(n)]=2e(m+n), \tag{9}$$

$$[h(m),f(n)]=-2f(m+n), \tag{10}$$

$$[e(m),f(n)]=h(m+n+1), \tag{11}$$

$$\deg h(n)=2n, \tag{12}$$

$$\deg e(n)=\deg f(n)=2n+1,\quad n\in Z. \tag{13}$$

Secondly, with above preparation we consider the following isospectral problem:

$$\phi_x=U\phi,\quad \lambda_t=0, \tag{14}$$

$$U=\begin{pmatrix}\alpha(x)\lambda^2 & \beta(t)q\lambda\\ \beta(t)r\lambda & -\alpha(x)\lambda^2\end{pmatrix}$$
$$=\alpha(x)h(1)+\beta(t)qe(0)+\beta(t)rf(0), \tag{15}$$

where λ is the spectral parameter.

Setting

$$V=\sum_{m\geq 0}[a_m h(-m)+b_m e(-m)+c_m f(-m)], \tag{16}$$

and solving the adjoint representation of Eqs. (15) and (16):

$$V_x=[U,V], \tag{17}$$

we obtain the following recursive relations:

$$a_{mx}=\beta(t)qc_{m+1}-\beta(t)rb_{m+1}, \tag{18}$$

$$b_{mx}=2\alpha(x)b_{m+1}-2\beta(t)qa_m, \tag{19}$$

$$c_{mx}=-2\alpha(x)c_{m+1}+2\beta(t)ra_m. \tag{20}$$

To determine a_m, b_m and c_m of Eqs. (18)–(20), in this paper we select the initial values

$$a_0=\omega,\quad b_0=0,\quad c_0=0. \tag{21}$$

For example, we have

$$a_1=-\frac{\beta^2(t)}{2\alpha^2(x)}\omega qr,\quad b_1=\frac{\beta(t)}{\alpha(x)}q\omega,\quad c_1=\frac{\beta(t)}{\alpha(x)}r\omega. \tag{22}$$

Thirdly, we introduce the notations

$$V_+^{(n)}=(\lambda^{2n}V)_+=\sum_{m=0}^{n}[a_m h(n-m)$$
$$+b_m e(n-m)+c_m f(n-m)], \tag{23}$$

$$V_-^{(n)}=\lambda^{2n}V-V_+^{(n)}. \tag{24}$$

Then Eq. (17) gives

$$-V_{+x}^{(n)}+[U,V_+^{(n)}]=V_{-x}^{(n)}-[U,V_-^{(n)}]. \tag{25}$$

Note that the terms in the left-hand side of (25) are of degree ≥ 0, while the terms of the right-hand side are of degree ≤ 1. Therefore, both sides of (25) are of degree 0 and 1. In other words, we have

$$-V_{+x}^{(n)}+[U,V_+^{(n)}]=2\alpha(x)c_{n+1}f(0)-2\alpha(x)b_{n+1}e(0)$$
$$+[\beta(t)rb_{n+1}-\beta(t)qc_{n+1}]h(0). \tag{26}$$

Letting $V^{(n)}=V_+^{(n)}+\Delta_n$, $\Delta_n=-a_nh(0)$, and then solving the zero curvature equation $U_t-V_x(n)+[U,V^{(n)}]=0$ yields a Lax integrable system:

$$u_t=\begin{pmatrix}q\\ r\end{pmatrix}_t=\frac{1}{\beta(t)}\begin{pmatrix}2\alpha(x)b_{n+1}-2\beta(t)qa_n\\ -2\alpha(x)c_{n+1}+2\beta(t)ra_n\end{pmatrix}$$
$$-\frac{\beta'(t)}{\beta(t)}\begin{pmatrix}q\\ r\end{pmatrix}=J\begin{pmatrix}c_n\\ b_n\end{pmatrix}-\frac{\beta'(t)}{\beta(t)}\begin{pmatrix}q\\ r\end{pmatrix}, \tag{27}$$

where J is a symmetric operator determined in Eq. (2).

With the help of Eqs. (18)–(21), we have

$$\begin{pmatrix}c_{n+1}\\ b_{n+1}\end{pmatrix}=L\begin{pmatrix}c_n\\ b_n\end{pmatrix}, \tag{28}$$

where L is the recursive operator defined in Eq. (2).

Using Eq. (28), we finally rewrite Eq. (27) as following:

$$u_t=\begin{pmatrix}q\\ r\end{pmatrix}_t=J\begin{pmatrix}c_n\\ b_n\end{pmatrix}-\frac{\beta'(t)}{\beta(t)}\begin{pmatrix}q\\ r\end{pmatrix}$$
$$=JL^{n-1}\begin{pmatrix}\frac{\beta(t)}{\alpha(x)}\omega r\\ \frac{\beta(t)}{\alpha(x)}\omega q\end{pmatrix}-\frac{\beta'(t)}{\beta(t)}\begin{pmatrix}q\\ r\end{pmatrix}, \tag{29}$$

which is just the KN-like hierarchy (1).

III. Hamiltonian Structure

To establish a Hamiltonian structure of the KN-like hierarchy (1), we rewrite Eq. (16) as

$$V=ah(0)+be(0)+cf(0)=\begin{pmatrix}a & b\lambda\\ c\lambda & -a\end{pmatrix}, \tag{30}$$

where

$$a=\sum_{m\geq 0}a_m\lambda^{-2m},\quad b=\sum_{m\geq 0}b_m\lambda^{-2m}, \tag{31}$$

$$c=\sum_{m\geq 0}c_m\lambda^{-2m}. \tag{32}$$

A direct computation gives

$$\langle V,\frac{\partial U}{\partial\lambda}\rangle=[4\alpha(x)a+\beta(t)rb+\beta(t)qc]\lambda, \tag{33}$$

$$\langle V,\frac{\partial U}{\partial q}\rangle=\beta(t)c\lambda^2,\quad \langle V,\frac{\partial U}{\partial r}\rangle=\beta(t)b\lambda^2. \tag{34}$$

Substituting Eqs. (33) and (34) into the trace identity [2]:

$$\frac{\delta}{\delta u}\langle V,\frac{\partial U}{\partial\lambda}\rangle=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\left(\lambda^{\gamma}\begin{pmatrix}\langle V,\frac{\partial U}{\partial q}\rangle\\ \langle V,\frac{\partial U}{\partial r}\rangle\end{pmatrix}\right), \tag{35}$$

we have

$$\frac{\delta}{\delta u}\{[4\alpha(x)a+\beta(t)rb+\beta(t)qc]\lambda\} = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix}\beta(t)c\lambda^2\\ \beta(t)b\lambda^2\end{pmatrix}. \quad (36)$$

Comparing the coefficient of λ^{-2n+1} on both sides of Eq. (36) yields

$$\frac{\delta}{\delta u}[4\alpha(x)a_n+\beta(t)rb_n+\beta(t)qc_n] = (-2n+2+\gamma)\beta(t)\begin{pmatrix}c_n\\ b_n\end{pmatrix}. \quad (37)$$

Further setting $n=1$ and $\beta(t)=\mu$(here μ is a non-zero constant), from Eq. (37) we have $\gamma=0$ and hence obtain:

$$\begin{pmatrix}c_n\\ b_n\end{pmatrix}=\frac{\delta H_n}{\delta u}=\begin{pmatrix}\frac{\delta}{\delta q}\\ \frac{\delta}{\delta r}\end{pmatrix}H_n, \quad (38)$$

$$H_n=\frac{4\alpha(x)a_n+\mu rb_n+\mu qc_n}{\mu(-2n+2)}. \quad (39)$$

Therefore, if $\beta(t)=\mu$ we can write the KN-like hierarchy (1) in the Hamiltonian form:

$$u_t=\begin{pmatrix}q\\ r\end{pmatrix}_t=JL^{n-1}\begin{pmatrix}\frac{\mu}{\alpha(x)}\omega r\\ \frac{\mu}{\alpha(x)}\omega q\end{pmatrix} = JL^{n-1}\begin{pmatrix}c_1\\ b_1\end{pmatrix}=J\begin{pmatrix}c_n\\ b_n\end{pmatrix}=J\frac{\delta H_n}{\delta u}. \quad (40)$$

On the other hand, it is easy to see that

$$JL=L^*J=\frac{1}{2\beta(t)}\begin{pmatrix}k_{11} & k_{12}\\ k_{21} & k_{22}\end{pmatrix}, \quad (41)$$

where

$$k_{11}=-\beta^2(t)\partial\frac{q}{\alpha(x)}\partial^{-1}\frac{q}{\alpha(x)}\partial,$$

$$k_{12}=\partial\frac{1}{\alpha(x)}\partial-\beta^2(t)\partial\frac{q}{\alpha(x)}\partial^{-1}\frac{r}{\alpha(x)}\partial,$$

$$k_{21}=-\partial\frac{1}{\alpha(x)}\partial-\beta^2(t)\partial\frac{r}{\alpha(x)}\partial^{-1}\frac{q}{\alpha(x)}\partial,$$

$$k_{22}=-\beta^2(x)\partial\frac{r}{\alpha(x)}\partial^{-1}\frac{r}{\alpha(x)}\partial.$$

Thus, the KN-like hierarchy (1) is also a Liouville integrable system under the condition that $\beta(t)$ is a non-zero constant.

IV. Special Solutions

To give some special solutions of the KN-like hierarchy (1), in this section we consider Eqs. (3)–(6). For Eqs. (3) and (4), we obtain the following solutions:

$$q=r=\frac{\alpha(x)}{\beta(t)}c[t+\frac{1}{\omega}\int^x\alpha(s)\mathrm{d}s], \quad (42)$$

where $c[t+\frac{1}{\omega}\int^x\alpha(s)\mathrm{d}s]$ is an arbitrary differentiable function of the indicated variables.

In order to solve Eqs. (5) and (6), we suppose $\alpha(x)=1$ and $\beta(t)=1$, then Eqs. (5) and (6) become

$$q_t-\frac{1}{2}\omega(q_{xx}-2qrq_x-q^2r_x)=0, \quad (43)$$

$$r_t-\frac{1}{2}\omega(-r_{xx}-2qrr_x-r^2q_x)=0. \quad (44)$$

By using the travelling wave transformation:

$$q=q(\xi),\quad r=r(\xi),\quad \xi=kx+ct,\quad k,c=\text{consts.}, \quad (45)$$

Eqs. (43) and (44) are converted into two ordinary differential equations (ODEs). Integrating these two ODEs with respect to ξ once and setting the integration constants as A and B respectively, we have

$$cq-\frac{1}{2}\omega(k^2q_\xi-kq^2r)+A=0, \quad (46)$$

$$cr-\frac{1}{2}\omega(-k^2r_\xi-kqr^2)+B=0. \quad (47)$$

It follows from Eq. (46) that

$$r=\frac{k^2\omega q_\xi-2(A+cq)}{k\omega q^2}. \quad (48)$$

Substituting Eq. (48) into Eq. (47) we then have

$$\frac{k^4\omega^2(qq_{\xi\xi}-q_\xi^2)+2Bk\omega q^3+4A(cq+A)}{2k\omega q^3}=0. \quad (49)$$

From Eqs. (48) and (49) we finally obtain non-trivial exact solutions of Eqs. (43) and (44) as follows:

Case 1. When $A=0$ and $B=0$

$$q=c_1\mathrm{e}^{kx+ct},\quad r=\frac{k^2\omega-2c}{kc_1\omega}\mathrm{e}^{-kx-ct}, \quad (50)$$

where c_1 is a non-zero constant.

Case 2. When $A=0$ and $B\neq 0$

$$q=-\frac{4Bk^3\omega}{(2Bkx+ak^4\omega^2t+2bB)^2}, \quad (51)$$

$$r=(2Bkx+ak^4\omega^2t+2bB)\times\frac{(2aBk^3\omega x+a^2k^6\omega^3t+2abBk^2\omega+4B^2)}{4B^2k^2\omega}, \quad (52)$$

where a and b are arbitrary constants.

For the other cases, namely $A\neq 0$ and $B=0$, $A\neq 0$ and $B\neq 0$, we can obtain only constant solutions from Eqs. (48) and (49). Such constant solutions are omitted here for simplicity.

In Figs. 1 and 2, two local spatial structures of solutions (51) and (52) are shown by selecting the parameters $a=-1$, $b=-2$, $B=1$, $k=2$ and $\omega=1.5$. It can be seen from Fig. 1 that solution (51) possesses singularities. For the fixed $x=x_0$, there always exists a finite time $t=-2a^{-1}Bk^{-4}\omega^{-2}(b+kx_0)$ at which solution (51) blow up. In view of the physical significance, solution (51) does not exist after blow-up. In the actual experimental physical system, there is no blow-up but a sharp spike. Thus, the finite time blow-up can provide an approximation to the physical phenomenon [45]. Fig. 2 shows that the amplitude of solution (52) decreases with the growth of $|t|$.

If we keep the arbitrariness of $\alpha(x)$ and $\beta(t)$, Eqs. (5) and (6) have the following trivial solutions:

$$q=\frac{c_1}{c_2}\alpha(x)\beta(t),\quad r=c_2\alpha(x)\beta(t),\quad c_2=\text{const.}. \quad (53)$$

We note here that how to construct non-trivial solutions of Eqs. (5) and (6) without $\alpha(x)$ and $\beta(t)$ being constants is an open problem.

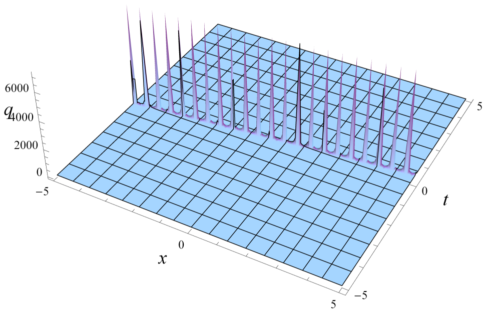

Fig. 1. Spatial structure of solution (51) with parameters $a = -1$, $b = -2$, $B = 1$, $k = 2$ and $\omega = 1.5$.

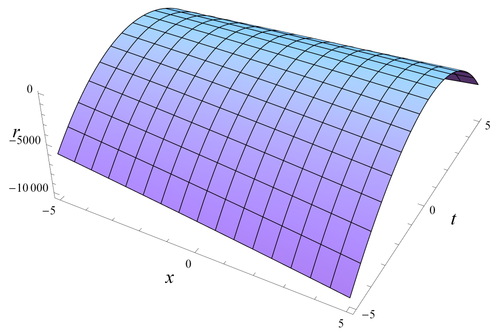

Fig. 2. Spatial structure of solution (52) with parameters $a = -1$, $b = -2$, $B = 1$, $k = 2$ and $\omega = 1.5$.

V. Conclusion

In summary, we have derived a new and more general KN-like hierarchy (1) which includes the known constant-coefficient KN-like hierarchy [7] as a special case. The KN-like hierarchy (1) is a Lax integrable system, and under a certain condition it is Liouville integrable. In special cases, a Hamiltonian structure and some exact solutions of the KN-like hierarchy (1) are obtained. To the best of our knowledge, the KN-like hierarchy (1), the Hamiltonian form (40), and exact solutions (42), (50)–(53) have not been reported in the literature. More importantly, this paper presents a method to derive integrable systems with coefficient functions of spatial-time variables from the related spectral problems. How to construct some other hierarchies with variable coefficients and their Hamiltonian structures and exact solutions are worthy of study. This is our task in the future.

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