On the Parametric Interest of the Option Price from the Black-Scholes Equation

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Abstract— **We have discovered some parameters λ in the Black-Scholes equation which depend on the interest rate r and the volatility σ and later is named the parametric interest. We found that such λ is very important in giving the conditions for the existence of the option price which is the solution of the Black-Scholes equation that may be either weak or strong solutions.**

Keywords: Black-Scholes equation, parametric interest, option price

1 Introduction

During about 10 years ago, financial mathematics has been improved and developed by many researchers. They interested in doing research in this field so that the new knowledge coming out continuously and these make the financial mathematics progress quickly. Because of all countries are experiencing a recession, financial crisis, as a result the problem of investment occured. For this reason, financial mathematics plays a significant role in the solution of such problems. In such condition, there are a number of investors interested in investing in the stock market. The stock trades in various forms both domestic and international. In reality, the investment in such has a high risk, this could lead to the loss easily. Therefore, investors need to have knowledge of such investment by learning the basics of financial mathematics.

Nowadays the research area in financial mathematics can be separated in two groups of activities. The first group is focusing on the real world applications by using the numerical computation and simulation [1]. The second group is the theoritical research focusing on the new results of properties and theorems. Actually our research activities is the second group and there are many papers appear in various journals [see [2], [3]]. Such papers concerning the spectrum, the eigenvalues, the kernel and the delta hedging of stock market.

This research will study the option price from the Black-Scholes equation which is becoming popular trading. The goal of the research is to study the parametric interest rate option which is new for financial mathematics.

In financial mathematics, the well known equation named the Black-Scholes equation plays an important role in solving the option price of stocks [see [4]]. The Black-Scholes equation is given by

$$\frac{\partial u(s,t)}{\partial t}+\frac{\sigma^2 s^2}{2}\frac{\partial^2 u(s,t)}{\partial s^2}+rs\frac{\partial u(s,t)}{\partial s}-ru(s,t)=0 \quad (1)$$

with the terminal condition

$$u(s,T)=(s_T-p)^+ \quad (2)$$

for $0\le t\le T$ where $u(s,t)$ is the option price at time t, r is the interest rate, s is the price of stock at time t, s_T is the price of stock at the expiration time T, σ is the volatility of stock and p is the strike price.

They obtain the solution of (1) that satisfies (2) of the form

$$\begin{aligned} u(s,t) &= s\Phi\left(\frac{\ln\left(\frac{s}{p}\right)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\ &\quad -pe^{-r(T-t)}\Phi\left(\frac{\ln\left(\frac{s}{p}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \end{aligned} \quad (3)$$

which is called Black-Scholes formula, where

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{\frac{-y^2}{2}}dy$$

see [[5],p.91].

We have already studied the parametric interest of the option price from the Black-Scholes equation with no terminal condition, see [6].

But in this research, we study the solution of (1) together with the terminal condition (2), we obtain the other from which is more generalization than the Black-Scholes formula.

We obtain both weak and strong solutions that are the weak option price and the strong option price subject to the conditions of the parametric interest λ. The weak option price

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does not appear in the real world application. But it exists in the space of tempered distribution. The strong option price really exists in the stock market.

2 Preliminaries

Definition 2.1 Given f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A and if there exists the real constant K, a and M such that $\left| f(t) \leq Ke^{at} \right|$ for $t \geq M$.

Then the Laplace transform of $f(t)$, denoted by $\mathcal{L}f(t)$ is defined by

$$\mathcal{L}f(t) = F(\xi) = \int_0^\infty e^{-\xi t} f(t) dt \tag{4}$$

and inverse Laplace transform of $F(\xi)$ is defined by

$$\mathcal{L}^{-1}F(\xi) = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(\xi)\, e^{\xi t} d\xi.$$

Lemma 2.2 (i) $\mathcal{L}\delta(t) = 1$.

(ii) $\mathcal{L}\delta^{(k)}(t) = \xi^k$ where $\delta^{(k)}$ is the Dirac-delta distribution with $k-$derivative and $\xi > 0$.

(iii) $\mathcal{L}(t^p) = \frac{\Gamma(p+1)}{\xi^{p+1}}$ for $p > -1$ and $\xi > 0$.
If p is a positive number n, then $\mathcal{L}(t^p) = \frac{n!}{\xi^{n+1}}$, $\xi > 0$.

(iv) $\mathcal{L}(t^k f(t)) = (-1)^k F^{(k)}(\xi)$.

(v) $\mathcal{L}\left(f^{(k)}(t)\right) = \xi^k F(\xi)$.

Proof. See [[7], pp.227-228].

3 Main Results

Theorem 3.1 Given the Black-Scholes equation

$$\frac{\partial u(s,t)}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 u(s,t)}{\partial s^2} + rs\frac{\partial u(s,t)}{\partial s} - ru(s,t) = 0 \tag{5}$$

with the terminal condition

$$u(s,T) = (s_{\mathrm{T}} - p)^+ \tag{6}$$

where s_T is the price of stock at the expiration time T, p is the strike price, $u(s,t)$ is the option price at time t with $0 \leq t \leq T$, s is the price of stock at time t, r is the interest rate and σ is the volatility of stock.

Then we obtain

$$u(s,t,\lambda) = Ce^{\lambda t}\mathcal{L}^{-1}(\xi^\alpha) \tag{7}$$

as the solution of (5) where $C = \frac{(s_T - p)^+}{Y(s_T)e^{\lambda T}}$ is constant, $Y(s_T)$ is a function of the price s_T of the stock at time T, λ is the parametric interest and $\mathcal{L}^{-1}(\xi^\alpha)$ is the inverse Laplace transform of ξ^α with

$$\alpha = \frac{(2r - 3\sigma^2) \pm \sqrt{(\sigma^2 + 2r)^2 - 8\sigma^2\lambda}}{2\sigma^2} \tag{8}$$

in particular, if $\alpha = m$ where m is nonnegative integer, then (7) becomes

$$u(s,t,\lambda) = Ce^{\lambda t}\delta^{(m)}(s) \tag{9}$$

where $\delta^m(s)$ is the Dirac-delta distribution with $m-$derivatives, $\delta^0 = \delta$ and by (8) $\lambda = (m+2)r - \frac{(m^2+3m+2)}{2}\sigma^2$. If α is negative real number, that is $\alpha < 0$ then (7) becomes

$$u(s,t,\lambda) = Ce^{\lambda t}\frac{s^{-\alpha-1}}{\Gamma(-\alpha)} \tag{10}$$

where $\Gamma(-\alpha)$ is the Gamma function.

In particular, if α is negative integer suppose $\alpha = -n$ then (9) reduces to

$$u(s,t,\lambda) = (s_T - p)^+ e^{-\lambda(T-t)}\left(\frac{s}{s_T}\right)^{n-1} \tag{11}$$

and by (8), $\lambda = -\frac{\sigma^2}{2}n^2 + \left(\frac{3\sigma^2}{2} - r\right)n + 2r - \sigma^2$.

Proof. By changing the variable $R = \ln s$. Then (5) is transformed to

$$\frac{\partial V(R,t)}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V(R,t)}{\partial R^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial V(R,t)}{\partial R} - rV(R,t) = 0 \tag{12}$$

where $V(R,t) = u(s,t)$ with the terminal condition

$$V(R,T) = \left(e^R - p\right)^+. \tag{13}$$

By the method of separation of variable, let $V(R,t) = X(R)U(t)$. Then $\frac{\partial V}{\partial t} = X(R)U'(t)$, $\frac{\partial V}{\partial R} = X'(R)U(t)$ and $\frac{\partial^2 V}{\partial R^2} = X''(R)U(t)$. Substitute in (12)

$$X(R)U'(t) + \frac{\sigma^2}{2}X''(R)U(t)$$

$$+\left(r - \frac{\sigma^2}{2}\right)X'(R)U(t) - rX(R)U(t) = 0$$

or

$$\frac{U'(t)}{U(t)} + \frac{\sigma^2}{2}\frac{X''(R)}{X(R)} + \left(r - \frac{\sigma^2}{2}\right)\frac{X'(R)}{X(R)} - r = 0.$$

Let

$$\frac{U'(t)}{U(t)} = -\frac{\sigma^2}{2}\frac{X''(R)}{X(R)} - \left(r - \frac{\sigma^2}{2}\right)\frac{X'(R)}{X(R)} + r = \lambda$$

where λ is a parameter.

Now $\frac{U'(t)}{U(t)} = \lambda$, thus $U(t) = Ce^{\lambda t}$. We can compute C, since $V(R,t) = X(R)U(t)$ and $X(R) = X(\ln s)$. Let $Y(s) = X(\ln s)$ then $V(R,t) = Y(s)U(t)$. By the terminal condition in (13)

$$V(\ln s_T, T) = Y(s_T)U(T) = (s_T - p)^+$$

thus $U(T) = \frac{(s_T - p)^+}{Y(s_T)}$. Now $U(T) = Ce^{\lambda T}$ thus

$$C = \frac{(s_T - p)^+}{Y(s_T)e^{\lambda T}} \tag{14}$$

we also have

$$-\frac{\sigma^2}{2}\frac{X''(R)}{X(R)} - \left(r - \frac{\sigma^2}{2}\right)\frac{X'(R)}{X(R)} + r = \lambda$$

or

$$\sigma^2 X''(R) + \left(2r - \sigma^2\right) X'(R) - (2r - 2\lambda) X(R) = 0. \quad (15)$$

Put $R = \ln s$ and let $X(R) = X(\ln s) = Y(s)$. Then

$$X'(R) = \frac{dX(R)}{ds}\frac{ds}{dR} = s\frac{dY(s)}{ds}$$

and

$$X''(R) = s^2\frac{d^2Y(s)}{ds^2} + s\frac{dY(s)}{ds}$$

thus

$$\sigma^2\left(s^2\frac{d^2Y(s)}{ds^2} + s\frac{dY(s)}{ds}\right) + \left(2r - \sigma^2\right)s\frac{dY(s)}{ds} - (2r - 2\lambda)Y(s) = 0$$

or

$$\sigma^2 s^2 Y''(s) + 2rsY'(s) - (2r - 2\lambda)Y(s) = 0. \quad (16)$$

The equation (16) is the Euler's equation of order 2. Now take the Laplace transform to (16) and use (iv) and (v) of Lemma 2.2

$$\sigma^2\frac{d^2[\xi^2 Y(\xi)]}{d\xi^2} + (-1)2r\frac{d[\xi Y(\xi)]}{d\xi} - (2r - 2\lambda)Y(\xi) = 0$$

or

$$\sigma^2\xi^2 Y''(\xi) + (4\sigma^2 - 2r)\xi Y'(\xi) + \left(2\sigma^2 - 4r + 2\lambda\right)Y(\xi) = 0$$

that is the Euler's equation.

Let $Y(\xi) = \xi^\alpha$ and substitute into such equation, we obtain

$$\left[\sigma^2\alpha(\alpha - 1) + (4\sigma^2 - 2r)\alpha + \left(2\sigma^2 - 4r + 2\lambda\right)\right]\xi^\alpha = 0$$

since $\xi^\alpha \neq 0$, thus

$$\sigma^2\alpha(\alpha - 1) + (4\sigma^2 - 2r)\alpha + \left(2\sigma^2 - 4r + 2\lambda\right) = 0$$

or

$$\sigma^2\alpha^2 + (3\sigma^2 - 2r)\alpha + \left(2\sigma^2 - 4r + 2\lambda\right) = 0. \quad (17)$$

Thus

$$\alpha = \frac{\left(2r - 3\sigma^2\right) \pm \sqrt{\left(\sigma^2 + 2r\right)^2 - 8\sigma^2\lambda}}{2\sigma^2} \quad (18)$$

we obtain the solution $Y(\xi) = \xi^\alpha$ or $Y(s) = \mathcal{L}^{-1}(\xi^\alpha)$.

Now $V(R, t) = X(R)U(t)$ or

$$u(s, t) = Y(s)U(t) = Ce^{\lambda t}\mathcal{L}^{-1}(\xi^\alpha)$$

or

$$u(s, t, \lambda) = \frac{(s_T - p)^+}{Y(s_T)e^{\lambda T}}e^{\lambda t}\mathcal{L}^{-1}(\xi^\alpha) \quad (19)$$

is the solution of (5) that is the option price. Now we consider, from (18), the case α is real root. That is $\lambda \leq \frac{\left(\sigma^2 + 2r\right)^2}{8\sigma^2}$ in (18) and consider the following cases.

(i) If $\alpha < 0$, then from (19) and (iii) of Lemma 2.2, we obtain

$$\begin{aligned} u(s, t, \lambda) &= \frac{(s_T - p)^+}{Y(s_T)e^{\lambda T}}e^{\lambda t}\frac{s^{-\alpha-1}}{\Gamma(-\alpha)} \\ &= \frac{(s_T - p)^+}{Y(s_T)}e^{-\lambda(T-t)}\frac{s^{-\alpha-1}}{\Gamma(-\alpha)} \end{aligned} \quad (20)$$

which is a strong solution or classical solution.

(ii) If $\alpha \geq 0$ and α is some integer m. By applying Lemma 2.2 (ii) to (19),

we obtain

$$u(s, t, \lambda) = \frac{(s_T - p)^+}{Y(s_T)}e^{-\lambda(T-t)}\delta^{(m)}(s) \quad (21)$$

where $\delta^{(m)}(s)$ is the Dirac-delta distribution of $m-$derivative and $\delta^{(0)}(s) = \delta(s)$ and for $\alpha = m$ in (17), we obtain

$$\lambda = (m + 2)r - \frac{\left(m^2 + 3m + 2\right)}{2}\sigma^2 \quad (22)$$

for some nonnegative integer m.

We see that the solution $u(s, t, \lambda)$ in (21) is a weak solution of the form of Dirac-delta distribution. Thus the option price $u(s, t, \lambda)$ in (21) does not appear in the real world application subject to the condition of λ in (22). But if $m = 0$ in (22) we obtain $\lambda = 2r - \sigma^2$ and (21) reduces to

$$u(s, t, \lambda) = \frac{(s_T - p)^+}{Y(s_T)}e^{-\lambda(T-t)}\delta(s)$$

which is the impulse function that means the option price has a high peak of jumping up and fall down immediately. This situation does not seem to happen in the real stock market.

Corollary 3.2 The solution $u(s, t, \lambda)$ in (20) of Theorem 3.1 reduces to

$$u(s, t, \lambda) = (s_T - p)^+ e^{-\lambda(T-t)}\left(\frac{s}{s_T}\right)^{n-1} \quad (23)$$

for $\alpha = -n$ where n is some positive integer with the parametric interest

$$\lambda = -\frac{\sigma^2}{2}n^2 + \left(\frac{3\sigma^2}{2} - r\right)n + 2r - \sigma^2.$$

Moreover, the terminal condition for $t = T, u(s_T, T, \lambda) = (s_T - p)^+$. Thus (2) holds.

Proof. Put $\alpha = -n$ in (20), we obtain

$$\begin{aligned} u(s, t, \lambda) &= \frac{(s_T - p)^+}{Y(s_T)}e^{-\lambda(T-t)}\frac{s^{n-1}}{\Gamma(n)} \\ &= (s_T - p)^+ e^{-\lambda(T-t)}\frac{s^{n-1}}{Y(s_T)(n-1)!}. \end{aligned}$$

Now the terminal condition

$$u(s_T, T, \lambda) = (s_T - p)^+ \quad for \quad t = T$$

thus

$$u(s_T, T, \lambda) = (s_T - p)^+\frac{s_T^{n-1}}{Y(s_T)(n-1)!} = (s_T - p)^+.$$

It follows that $Y(s_T) = \frac{s_T^{n-1}}{(n-1)!}$ thus

$$u(s, t, \lambda) = (s_T - p)^+ e^{-\lambda(T-t)}\left(\frac{s}{s_T}\right)^{n-1}$$

that (23) holds and from (17) with $\alpha = -n$, we also obtain the parametric interest

$$\lambda = -\frac{\sigma^2}{2}n^2 + \left(\frac{3\sigma^2}{2} - r\right)n + 2r - \sigma^2.$$

Alternatively, if we put $t = T$ in (23) we obtain the terminal condition $u(s_T, T, \lambda) = (s_T - p)^+$. Moreover from (23), for $s = 0$ we obtain the initial condition

$$u(0, t, \lambda) = 0$$

and $t = 0$ we obtain the initial condition

$$u(s, 0, \lambda) = (s_T - p)^+ e^{-\lambda T}\left(\frac{s}{s_T}\right)^{n-1}$$

where T is the expiration time.

Corollary 3.3 The strong solution of the Black-Scholes equation given by (5) can be solved directly by the method of separation of variable. The solution is

$$u(s, t, \lambda) = (s_T - p)^+ \frac{(c_1 s^{m_1} + c_2 s^{m_2})}{(c_1 {s_T}^{m_1} + c_2 {s_T}^{m_2})} \tag{24}$$

where

$$m_1 = \frac{(\sigma^2 - 2r) + \sqrt{(2r + \sigma^2)^2 - 8\lambda\sigma^2}}{2\sigma^2}$$

and

$$m_2 = \frac{(\sigma^2 - 2r) - \sqrt{(2r + \sigma^2)^2 - 8\lambda\sigma^2}}{2\sigma^2}.$$

and c_1, c_2 are constants.

Proof. From (15) we have

$$\sigma^2 X''(R) + (2r - \sigma^2) X'(R) - (2r - 2\lambda) X(R) = 0.$$

Let $m^2 = \frac{d^2}{dR^2}$ and $m = \frac{d}{dR}$. Then we have

$$\sigma^2 m^2 + (2r - \sigma^2) m - (2r - 2\lambda) = 0$$

so that

$$\begin{aligned} m &= \frac{-(2r - \sigma^2) \pm \sqrt{(2r - \sigma^2)^2 + 4\sigma^2(2r - 2\lambda)}}{2\sigma^2} \\ &= \frac{-(2r - \sigma^2) \pm \sqrt{(2r + \sigma^2)^2 - 8\lambda\sigma^2}}{2\sigma^2}. \end{aligned}$$

Consider the case m is real number subjects to the condition $\lambda \le \frac{(\sigma^2 + 2r)^2}{8\sigma^2}$. Let

$$m_1 = \frac{(\sigma^2 - 2r) + \sqrt{(2r + \sigma^2)^2 - 8\lambda\sigma^2}}{2\sigma^2}$$

and

$$m_2 = \frac{(\sigma^2 - 2r) - \sqrt{(2r + \sigma^2)^2 - 8\lambda\sigma^2}}{2\sigma^2}.$$

Thus we have

$$X(R) = c_1 e^{m_1 R} + c_2 e^{m_2 R}$$

as the solution where c_1 and c_2 are constants. Since $V(R, t) = X(R)U(t)$. Then

$$V(R, t) = \left(c_1 e^{m_1 R} + c_2 e^{m_2 R}\right) C e^{\lambda t}.$$

Now from (13)

$$V(R, t) = \left(e^R - p\right)^+$$

or

$$V(\ln s_T, T) = (s_T - p)^+.$$

Thus

$$\begin{aligned} V(\ln s_T, T) &= X(\ln s_T) U(T) \\ &= (c_1 {s_T}^{m_1} + c_2 {s_T}^{m_2}) C e^{\lambda T} \\ &= (s_T - p)^+. \end{aligned}$$

It follows that

$$C = \frac{(s_T - p)^+}{(c_1 {s_T}^{m_1} + c_2 {s_T}^{m_2}) C e^{\lambda T}}.$$

Thus we have

$$V(R, t) = (s_T - p)^+ \frac{(c_1 s^{m_1} + c_2 s^{m_2})}{(c_1 {s_T}^{m_1} + c_2 {s_T}^{m_2})} e^{-\lambda(T - t)}$$

or

$$u(s, t, \lambda) = (s_T - p)^+ \frac{(c_1 s^{m_1} + c_2 s^{m_2})}{(c_1 {s_T}^{m_1} + c_2 {s_T}^{m_2})} e^{-\lambda(T - t)}$$

as a solution.

Now from (12) we have $u(0, t) = 0$ and

$$u(s, t) = V(R, t) = V(\ln s, t).$$

Thus

$$u(0, t) = V(-\infty, t) = 0.$$

Since we have $V(-\infty, t) = X(-\infty)U(t) = 0$ it follows that

$$X(-\infty) = 0.$$

Thus

$$X(-\infty) = c_1 e^{-m_1 \infty} + c_2 e^{-m_2 \infty} = 0.$$

Suppose $m_1 < 0$ and $m_2 > 0$ thus

$$c_1 e^{-m_1 \infty} + 0 = 0.$$

It follows that $c_1 = 0$. Thus we have $X(R) = c_2 e^{m_2 R}$. We also have

$$\begin{aligned} V(R, t) &= (s_T - p)^+ \frac{(c_2 s^{m_2})}{(c_2 {s_T}^{m_2})} e^{-\lambda(T - t)} \\ &= (s_T - p)^+ e^{-\lambda(T - t)} \left(\frac{s}{s_T}\right)^{m_2}. \end{aligned} \tag{25}$$

Let $m_2 = n - 1$, we have

$$V(\ln s, t) = V(R, t) = (s_T - p)^+ e^{-\lambda(T - t)} \left(\frac{s}{s_T}\right)^{n-1}$$

or

$$u(s, t, \lambda) = (s_T - p)^+ e^{-\lambda(T - t)} \left(\frac{s}{s_T}\right)^{n-1}$$

which is the strong solution in (23) of the Corolary 3.2. We see that (23) is the special case of (25).

4 Conclusion

(i) If the parametric interest $\lambda = (m+2)\,r - \frac{m^2+3m+2}{2}\sigma^2$ for some nonnegative integer m, then we obtain the option price $u(s,t,\lambda) = \frac{(s_T-p)^+}{Y(s_T)} e^{-\lambda(T-t)} \delta^{(m)}(s)$ as the weak solution and will not appear in the real world application.

(ii) If the parametric interest $\lambda = -\frac{\sigma^2}{2}n^2 + \left(\frac{3\sigma^2}{2} - r\right)n + 2r - \sigma^2$ for some positive integer n, then we obtain the option price $u(s,t,\lambda) = (s_T-p)^+ e^{-\lambda(T-t)} \left(\frac{s}{s_T}\right)^{n-1}$ as the strong or classical solution that really appears in the real world application which is the different form given by the Black-Scholes formula.

In particular, if $n = 1$ then $\lambda = r$. Thus we obtain

$$u(s,t,r) = (s_T - p)^+ e^{-r(T-t)}$$

or

$$(s_T - p)^+ = u(s,t,r)\, e^{r(T-t)}$$

that means the payoff at the expiration T is equal to the option price $u(s,t,r)$ that put in the bank with the riskness interest r at the time $T-t$.

For the case $n = 2$ we have $\lambda = 0$ we obtain $u(s,t,0) = (s_T - p)^+ \left(\frac{s}{s_T}\right)$, that means the option price $u(s,t,0)$ is equal to the terminal condition or the payoff times the ratio $\frac{s}{s_T}$ of stock.

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