Some Inequalities for $p$-Geominimal Surface Area and Related Results

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Abstract—The concepts of $p$-affine and $p$-geominimal surface areas were introduced by Lutwak. In this paper, we establish some Brunn-Minkowski type inequalities of $p$-geominimal surface area combining $L_p$-polar curvature image with various combinations of convex bodies. Moreover, we discuss the equivalence of several inequalities, and also obtain some results similar to $p$-geominimal surface area for the $p$-affine surface area.

Index Terms—convex bodies, $p$-affine surface area, $p$-geominimal surface area, Brunn-Minkowski type inequality.

I. INTRODUCTION

Let $K^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in $n$-dimensional Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors, the set of convex bodies whose centroids lie at the origin and the set of origin-symmetric convex bodies in $\mathbb{R}^n$, we write $K^n_0$, $K^n_e$ and $K^n_o$ respectively. $S^{n-1}_o$ and $S^n_o$ respectively denote the set of star bodies (about the origin) and the set of origin-symmetric star bodies in $\mathbb{R}^n$. Let $S^{n-1}$ denote the unit sphere in $\mathbb{R}^n$, and let $V(K)$ denote the $n$-dimensional volume of a body $K$. For the standard unit ball $B_n$ in $\mathbb{R}^n$, we use $\omega_n = V(B_n)$ to denote its volume.

The study of affine surface area goes back to Blaschke [1] and is about one hundred years old. It was generalized to the $p$-affine surface area by Lutwak in [10]. Since then, considerable attention has been paid to the $p$-affine surface area, which is now at the core of the rapidly developing $L_p$-Brunn-Minkowski theory (see articles [4], [5], [6], [8], [11], [12], [13], [14], [17], [19], [24], [28] or books [7], [22]). In particular, affine isoperimetric inequalities related to the $p$-affine surface area can be found in [10], [29].

Another fundamental concept in convex geometry is geominimal surface area, introduced by Petty [19] more than three decades ago. As Petty explained in [19], the geominimal surface area connects the affine geometry, relative geometry and Minkowski geometry. Hence it receives a lot of attention (see [19], [20], [23]). The geominimal surface area was extended to $p$-geominimal surface area by Lutwak in his seminal paper [10]. The $p$-geominimal surface area shares many properties with the $p$-affine surface area. For instance, both are affine invariant and have the same degree of homogeneity. However, the $p$-geominimal surface area is different from the $p$-affine surface area. For instance, unlike the $p$-affine surface area, $p$-geominimal surface area has no nice integral expression. This leads to a big obstacle on extending the $p$-geominimal surface area. There are many papers on $p$-affine and $p$-geominimal surface areas, see e.g., [16], [18], [25], [26], [27], [28], [29], [30], [32].

Based on the notion of $L_p$-mixed volume, Lutwak introduced the concepts of $p$-affine and $p$-geominimal surface areas, respectively.

For $p \geq 1$ and $K \in K^n_o$, the $p$-affine surface area, $\Omega_p(K)$, was defined in [10] by

$$n \omega_n^p \Omega_p(K) := \inf \{ n V_p(K, Q^n) V(Q^n) : Q \in S^n_o \}.$$

Here $V_p(K, Q^n)$ denotes the $L_p$-mixed volume of $K$ and $Q^n$ (see Section II. A) and $Q^n$ denotes the polar of body $Q$ (see Section II. C).

For $p \geq 1$, Lutwak in [10] defined the $p$-geominimal surface area, $G_p(K)$, of $K \in K^n_o$ by

$$\omega_n^p G_p(K) = \inf \{ n V_p(K, Q^n) V(Q^n) : Q \in S^n_o \}. \quad (1)$$

Further, Lutwak obtained the following inequalities for the $p$-affine and the $p$-geominimal surface areas.

**Lemma 1.1.** (Theorem 4.8 in [10]) Let $K \in K^n_o$ and $p \geq 1$. Then

$$\Omega_p(K)^{n+p} \leq n^{n+p} \omega_n^{n+p} V(K)^{n-p}, \quad (2)$$

with equality if and only if $K$ is an ellipsoid.

**Lemma 1.2.** (Theorem 3.12 in [10]) Let $K \in K^n_o$ and $p \geq 1$. Then

$$G_p(K)^n \leq n^n \omega_n^n V(K)^{n-p}, \quad (3)$$

with equality if and only if $K$ is an ellipsoid.

**Lemma 1.3.** (p. 250) Let $K \in F^n_o$ and $p \geq 1$. Then

$$\Omega_p(K)^{n+p} \leq (\omega_n)^p G_p(K)^n, \quad (4)$$

with equality if and only if $K$ is of $p$-elliptic type.

A convex body $K \in K^n_0$ is said to have a $L_p$-curvature function (see [10]) $f_p(K, \cdot) : S^{n-1} \rightarrow \mathbb{R}$, if its $L_p$-surface area measure $S_p(K, \cdot)$ is absolutely continuous with respect to spherical Lebesgue measure $S$, and

$$\frac{dS_p(K, \cdot)}{dS} = f_p(K, \cdot).$$

Let $F^n_o, F^n_c$ denote the set of all bodies in $K^n_o, K^n_c$ respectively, and both of them have a positive continuous curvature function.

If $K \in S^n_o$, and $p \geq 1$, then define $\Lambda_p^o K \in F^n_o$, the $L_p$-polar curvature image of $K$, by

$$f_p(\Lambda_p^o K, \cdot) = \frac{\omega_n}{V(K)} p(K, \cdot)^{n+p}. \quad (5)$$

When $p = 1$, we write $\Lambda_1^o K = K$, it is just the classical curvature image (see [12], [14]). When $p > 1$, it was defined by Yuan, Zhu, Lv and Leng (see [15], [30], [31]).
The following theorems are our main results: Combining $L_p$-polar curvature image with $p$-geominimal surface area, we establish several Brunn-Minkowski type inequalities of the $p$-geominimal surface area.

**Theorem 1.4.** If $p \geq 1, K, L \in \mathbb{K}^n_p$, and $\lambda, \mu \geq 0$ (not both zero), then

$$G_p(\Lambda^\alpha_p(\lambda \cdot K + \mu \cdot L)) \geq \lambda G_p(\Lambda^\alpha_p(K)) + \mu G_p(\Lambda^\alpha_p(L)),$$

(6)

with equality for $p = 1$ if and only if $K$ and $L$ are homothetic, and for $p > 1$ if and only if $K$ and $L$ are dilates.

Here, $\lambda \cdot K + \mu \cdot L$ denotes the $L_p$-Firey combination of $K$ and $L$ (see (10)).

**Theorem 1.5.** If $1 \leq p \leq n, K, L \in \mathbb{K}^n_p$, and $\lambda, \mu \geq 0$ (not both zero), then

$$G_p(\Lambda^\alpha_p(\lambda \circ K + \mu \circ L)) \leq \lambda G_p(\Lambda^\alpha_p(K)) + \mu G_p(\Lambda^\alpha_p(L)).$$

(7)

The reverse inequality holds when $p > n$. Equality holds in every inequality when $p \neq n$ if and only if $K$ is a dilate of $L$. Here, $\lambda \circ K + \mu \circ L$ denotes the $L_p$-radial combination of $K$ and $L$ (see (13)).

**Theorem 1.6.** If $p \geq 1, K, L \in \mathbb{K}^n_p$, and $\lambda, \mu \geq 0$ (not both zero), then

$$G_p(\Lambda^\alpha_p(\lambda \cdot K + \mu \cdot L)) \geq \lambda G_p(\lambda \cdot L) + \mu G_p(\mu \cdot L),$$

(8)

with equality if and only if $K$ and $L$ are dilates.

Here, $\lambda \cdot K + \mu \cdot L$ denotes the $L_p$-harmonic radial combination of $K$ and $L$ (see (16)).

**Theorem 1.7.** If $n \neq p \geq 1, K, L \in \mathbb{F}^n_p$, and $\lambda, \mu \geq 0$ (not both zero), then

$$G_p(\lambda \cdot K \geq \lambda \cdot L) + G_p(\mu \cdot L) \geq \lambda G_p(\lambda \cdot K) + \mu G_p(\mu \cdot L),$$

(9)

with equality for $p = 1$ if and only if $K$ and $L$ are homothetic, for $p > 1$ if and only if $K$ and $L$ are dilates.

Here, $\lambda \cdot K \geq \lambda \cdot L$ denotes the Blaschke $L_p$-combination of $K$ and $L$ (see (23)).

Please see the next section for above interrelated notations, definitions and their background materials. The proofs of Theorems 1.4-1.7 will be given in Section III of this paper. Moreover, we derive the equivalence of several inequalities in Section IV.

II. PRELIMINARIES

A. $L_p$-Firey Combination and $L_p$-mixed Volume

If $K \in \mathbb{K}^n_p$, then its support function, $h_K = h(K, \cdot) : \mathbb{R}^n \to (-\infty, \infty)$, is defined by (see [22]) $h(K, x) = \max \{x \cdot y : y \in K\}$, $x \in \mathbb{R}^n$, where $x \cdot y$ denotes the standard inner product of $x$ and $y$.

For real $p \geq 1, K, L \in \mathbb{K}^n_p$, and $\alpha, \beta \geq 0$ (not both zero), the $L_p$-Firey combination, $\alpha \cdot K + \beta \cdot L$, is defined by (see [2])

$$h(\alpha \cdot K + \beta \cdot L, \cdot)^p = ah(K, \cdot)^p + bh(L, \cdot)^p.$$  

(10)

For $p \geq 1$, the $L_p$-mixed volume, $V_p(K, L)$, of $K, L \in \mathbb{K}^n_p$, was defined in [9] by

$$n_\cdot V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K + \varepsilon \cdot L) - V(L)}{\varepsilon}.$$  

It was shown in [9] that corresponding to each $K \in \mathbb{K}^n_p$ there is a positive Borel measure $S_p(K, \cdot)$ on $S^{n-1}$ such that

$$V_p(K, Q) = \frac{1}{n} \int_{S^{n-1}} h(Q, u)^p dS_p(K, u)$$

for all $Q \in \mathbb{K}^n_p$. It turns out that the $L_p$-surface area measure $S_p(K, \cdot)$ on $S^{n-1}$ is absolutely continuous with respect to $S(K, \cdot)$, and has the Radon-Nikodym derivative

$$\frac{dS_p(K, \cdot)}{dS(K, \cdot)} = h^{1-p}(K, \cdot).$$

The $L_p$-Brunn-Minkowski inequality was given by Lutwak in [9]: If $K, L \in \mathbb{K}^n_p$, $\lambda, \mu > 0$, and $p \geq 1$, then

$$V(\lambda \cdot K + \mu \cdot L, \cdot)^{\gamma^p} \geq \lambda V(K, \cdot)^{\gamma^p} + \mu V(L, \cdot)^{\gamma^p},$$

(11)

with equality for $p = 1$ if and only if $K$ and $L$ are homothetic, and for $p > 1$ if and only if $K$ and $L$ are dilates.

Taking $\lambda = \mu = \frac{1}{2}$ and $L = -K$ in (10), the $L_p$-difference body, $\Delta_p K$, of $K$ was given by (see [9])

$$\Delta_p K = \frac{1}{2} \cdot K + \frac{1}{2} \cdot (-K).$$

(12)

B. $L_p$-radial Combination and $L_p$-mixed Volume

If $K$ is a compact star-shaped (about the origin) set in $\mathbb{R}^n$, then its radial function, $\rho_K = (\rho_K, \cdot) : \mathbb{R}^n \to [0, \infty)$, is defined by (see [22]) $\rho_K(u) = \max \{\lambda \geq 0 : \lambda u \in K\}$, $u \in S^{n-1}$. If $\rho_K$ is positive and continuous, then $K$ will be called a star body (about the origin). Two star bodies $K$ and $L$ are said to be dilated of one another if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If $K, L \in S^{n-1}_o$ and $\lambda, \mu \geq 0$ (not both zero), then for $p > 0$, the $L_p$-radial combination, $\lambda \cdot K + \mu \cdot L \in S^{n-1}_o$, is defined by (see [3])

$$\rho(\lambda \cdot K + \mu \cdot L, \cdot)^p = \lambda \rho(K, \cdot)^p + \mu \rho(L, \cdot)^p.$$  

(13)

For $p \geq 1$, and $K, L \in S^{n-1}_o$, the $L_p$-dual mixed volume, $V_p(K, L)$, was defined in [3] by

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K + \varepsilon \cdot L) - V(L)}{\varepsilon}.$$  

The following integral representation for the $L_p$-dual mixed volume was obtained in [3]: If $p \geq 1$, and $K, L \in S^{n-1}_o$, then

$$\frac{n}{p} V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n-p} \rho_L(u)^p dS(u),$$

where $S$ is the spherical Lebesgue measure on $S^{n-1}$ (i.e., the $(n-1)$-dimensional Hausdorff measure).

We shall need the following $L_p$-dual Brunn-Minkowski inequality (see [3]): If $K, L \in S^{n-1}_o$ and $0 < p \leq n$, then

$$V(\lambda \cdot K + \mu \cdot L, \cdot)^{\gamma^p} \geq \lambda V(K, \cdot)^{\gamma^p} + \mu V(L, \cdot)^{\gamma^p}.$$  

(14)

The reverse inequality holds when $p \geq n$. Equality holds when $p \neq n$ and only if $K$ is a dilate of $L$.

Taking $\lambda = \mu = \frac{1}{2}$ and $L = -K$ in (13), the $L_p$-radial body, $\Delta_p K$, of $K$ is defined by

$$\Delta_p K = \frac{1}{2} \cdot K + \frac{1}{2} \cdot (-K).$$  

(15)

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C. $L_p$-harmonic Radial Combination and $L_p$-harmonic Mixed Volume

For $K, L \in S^n_0$, $p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the $L_p$-harmonic radial combination, $\lambda * K + \mu * L \in S^n_0$, is defined by (see [10])

$$\rho(\lambda * K + \mu * L, \nu) = \lambda \rho(K, \nu)^p + \mu \rho(L, \nu)^p. \quad (16)$$

If $K \in K^n_0$, the polar set, $K^*$, of $K$ is defined by

$$K^* = \{ x \in \mathbb{R}^n : x \cdot y \leq 1, \text{ for all } y \in K \}. \quad (17)$$

From (17), we can easily have $(K^*)^* = K$, and

$$h_{K^*} = \frac{1}{\rho_K}, \quad \rho_{K^*} = \frac{1}{h_K}, \quad (18)$$

for $K \in K^n_0$.

By (10), (16) and (18), it follows that if $K, L \in K^n_0$ and $\lambda, \mu \geq 0$ (not both zero), then

$$\lambda * K + \mu * L = (\lambda \cdot K^* + \mu \cdot L^*)^*.$$

Define the Santaló product of $K \in K^n_0$ by $V(K)V(K^*)$. The Blaschke-Santaló inequality (see [22]) is one of the fundamental affine isoperimetric inequalities. It states that if $K \in K^n_0$ then

$$V(K)V(K^*) \leq \omega^n_n,$$

with equality if and only if $K$ is an ellipsoid.

For $p \geq 1$ and $K, L \in S^n_0$, the $L_p$-harmonic mixed volume, $V_p(K, L)$, is defined by (see [10])

$$\frac{n}{p} \bar{V}_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K + \varepsilon * L) - V(K)}{\varepsilon}.$$

From the polar coordinate formula, the following integral representation was given in [10]: If $p \geq 1$ and $K, L \in S^n_0$, then

$$\bar{V}_p(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n+p} \rho(L, u)^{-p} dS(u).$$

The Minkowski’s inequality for the $L_p$-harmonic mixed volume can be stated that (see [10]): If $p \geq 1$ and $K, L \in S^n_0$, then

$$\bar{V}_p(K, L)^n \geq V(K)^{n+p}V(L)^{-p}, \quad (19)$$

with equality if and only if $K$ and $L$ are dilates.

The Brunn-Minkowski inequality for the $L_p$-harmonic radial combination can be stated that (see [10]): Suppose $K, L \in S^n_0$, $p \geq 1$ and $\lambda, \mu > 0$, then

$$V(\lambda * K + \mu * L)^{-p/n} \geq \lambda V(K)^{-p/n} + \mu V(L)^{-p/n}, \quad (20)$$

with equality if and only if $K$ and $L$ are dilates each other.

Taking $\lambda = \mu = \frac{1}{2}$ and $L = -K$ in (16), the $L_p$-harmonic radial body, $\hat{\Delta}_p K$, of $K$ is defined by

$$\hat{\Delta}_p K = \frac{1}{2} * K + \frac{1}{2} * (-K). \quad (21)$$

D. $L_p$-affine Surface Area, $L_p$-curvature Image and Blaschke $L_p$-combination

In [10], Lutwak defined the $L_p$-affine surface area as follows: For $K \in F^n_0$ and $p \geq 1$, the $L_p$-affine surface area, $\Omega_p(K)$, of $K$ is defined by

$$\Omega_p(K) = \int_{S^{n-1}} f_p(K, u) \frac{dS(u)}{\omega_n}. \quad (22)$$

Further, Lutwak [10] showed the notion of $L_p$-curvature image as follows: For any $K \in F^n_0$ and $p \geq 1$, define $\lambda_p K \in S^n_0$, the $L_p$-curvature image of $K$, by

$$\rho(\lambda_p K, \nu)^p = \frac{V(\lambda_p K)}{\omega_n} f_p(K, \nu). \quad (23)$$

Note that for $p = 1$, this definition is different from the classical curvature image (see [14]).

The definition of Blaschke $L_p$-combination for convex bodies may be stated that (see [9]) for $K, L \in K^n_0$, $\lambda, \mu \geq 0$ (not both zero) and $n \neq p \geq 1$, the Blaschke $L_p$-combination, $\lambda K + \mu L \in K^n_0$, of $K$ and $L$ is defined by

$$\frac{dS_p(\lambda K + \mu L, \nu)}{\omega_n} = \lambda dS_p(K, \nu) + \mu dS_p(L, \nu). \quad (24)$$

From (22) and (23), Wang and Leng [26] proved the following $L_p$-Brunn-Minkowski inequality: If $K, L \in F^n_0$, $\lambda, \mu > 0$ and $n \neq p \geq 1$, then

$$V(\lambda_p(\lambda K + \mu L)) \geq \lambda V(\lambda_p K) + \mu V(\lambda_p L), \quad (25)$$

with equality for $p = 1$ if and only if $K$ and $L$ are homothetic, for $p > 1$ if and only if $K$ and $L$ are dilates.

III. PROOFS OF THEOREMS

In this section, we prove Theorems 1.4.1-1.7. Taking $L = Q^*$ in Proposition 3.4 of [31], we immediately give:

**Lemma 3.1.** If $p \geq 1$ and $K \in K^n_0$, then for any $Q \in K^n_0$,

$$V_p(\lambda^n Q, K) = \omega_n \bar{V}_p(K, Q^*)/V(K). \quad (26)$$

**Lemma 3.2.** If $p \geq 1$ and $K \in K^n_0$, then

$$G_p(\lambda^n Q, K) = \omega_n \bar{V}_p(K, Q^*)/V(K). \quad (27)$$

**Proof.** By (1), (26) and (27), we have

$$G_p(\lambda^n Q, K) = \omega_n \bar{V}_p(K, Q^*)/V(K). \quad (28)$$

From (1) and (26), it follows that for any $Q \in K^n_0$

$$G_p(\lambda^n Q, K) \leq \omega_n \bar{V}_p(K, Q^*)/V(K). \quad (29)$$

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Since $K \in \mathcal{K}_c^n$, and taking $Q^* = K$, we obtain
\[
G_p(\Lambda_p^c K) \leq n \omega_n \frac{\alpha_p}{\lambda} V(K)^\frac{p}{n}.
\]
Above all, we yield equality (27).

**Proof of Theorem 1.4.** From (27) and (11), it follows that
\[
G_p(\Lambda_p^c K(\lambda \cdot K + \mu \cdot L)) = n \omega_n \frac{\alpha_p}{\lambda} V((\lambda \cdot K + \mu \cdot L))^\frac{p}{n},
\]
\[
\geq \lambda n \omega_n \frac{\alpha_p}{\lambda} V(K)^\frac{p}{n} + \mu n \omega_n \frac{\alpha_p}{\lambda} V(L)^\frac{p}{n} = \lambda G_p(\Lambda_p^c K) + \mu G_p(\Lambda_p^c L).
\]
From the equality condition of inequality (11), we know that equality holds in (6) for $p = 1$ if and only if $K$ and $L$ are homothetic, and for $p > 1$ if and only if $K$ and $L$ are dilates.

According to (6) and (12), we easily get that if $K \in \mathcal{K}_c^n$ and $p \geq 1$, then
\[
G_p(\Lambda_p^c (\Lambda_p^c p K)) = G_p(\Lambda_p^c K).
\]

**Proof of Theorem 1.5.** It follows from (27) and (14) that for $1 \leq p \leq n$,
\[
G_p(\Lambda_p^c(\lambda \circ K + \mu L) = n \omega_n \frac{\alpha_p}{\lambda} V(\lambda \circ K + \mu L))^\frac{p}{n},
\]
\[
\leq \lambda n \omega_n \frac{\alpha_p}{\lambda} V(K)^\frac{p}{n} + \mu n \omega_n \frac{\alpha_p}{\lambda} V(L)^\frac{p}{n} = \lambda G_p(\Lambda_p^c K) + \mu G_p(\Lambda_p^c L).
\]
The reverse inequality holds when $p > n$. From the equality condition of inequality (14), we know that equality holds in (7) when $p \neq n$ if and only if $K$ is a dilate of $L$.

Together (7) with (15), we easily get that if $K \in \mathcal{K}_c^n$ and $p \neq n$, then
\[
G_p(\Lambda_p^c(\Lambda_p^c K)) = G_p(\Lambda_p^c K).
\]

**Proof of Theorem 1.6.** By (27) and (20), we have
\[
G_p(\Lambda_p^c(\lambda \cdot K + \mu \cdot L))^{-1} = (n \omega_n \frac{\alpha_p}{\lambda} V(\lambda \cdot K + \mu \cdot L))^\frac{1}{p},
\]
\[
\geq \lambda(n \omega_n \frac{\alpha_p}{\lambda} V(K))^{-\frac{1}{p}} + \mu(n \omega_n \frac{\alpha_p}{\lambda} V(L))^{-\frac{1}{p}} = \lambda G_p(\Lambda_p^c K)^{-1} + \mu G_p(\Lambda_p^c L)^{-1}.
\]
From the equality condition of inequality (20), we know that equality holds in (8) if and only if $K$ and $L$ are dilates.

An immediate consequence of Theorem 1.6 is:

**Corollary 3.3.** With the same assumptions of Theorem 1, if $\lambda, \mu > 0$, then
\[
4G_p(\Lambda_p^c(\lambda \cdot K + \mu \cdot L)) \leq \frac{1}{\lambda} G_p(\Lambda_p^c K) + \frac{1}{\mu} G_p(\Lambda_p^c L),
\]
with equality if and only if $K$ and $L$ are dilates each other.

**Proof.** Using Cauchy’s inequality and the arithmetic-mean-harmonic-mean inequality in (8), we have
\[
G_p(\Lambda_p^c(\lambda \cdot K + \mu \cdot L)) \leq \frac{1}{\lambda} G_p(\Lambda_p^c K) + \frac{1}{\mu} G_p(\Lambda_p^c L),
\]
\[
\leq \frac{1}{4\lambda} G_p(\Lambda_p^c K) + \frac{1}{4\mu} G_p(\Lambda_p^c L).
\]
This yields the desired inequality.

Combining (8) with (21), we easily get that if $K \in \mathcal{K}_c^n$ and $p \geq 1$, then
\[
G_p(\Lambda_p^c(\Lambda_p^c p K)) = G_p(\Lambda_p^c K).
\]

**Lemma 3.4.** For $n \neq p \geq 1$, the mapping $\Lambda_p : F_c^n \to S_c^n$ is bijective.

**Proof.** For the case $p = 1$, since $\Lambda_n$ is the classical curvature image and $\Lambda : \mathcal{S}_c^n \to F_c^n$ is a bijection (see [14], p.50), $\Lambda_p$ is a bijection. For $n \neq p > 1$, $\Lambda_p : S_c^n \to F_c^n$ was proved in Proposition 3.6 of [31] that it is also a bijection. Thus $\Lambda_p$ is a bijection on the class of origin-symmetric bodies for $n \neq p \geq 1$.

**Proof of Theorem 1.7.** It follows from (5) that $\Lambda_p^c = \Lambda_p^{-1}$ is the inverse image of $\Lambda_p$. By Lemma 3.4, equation (27) and inequality (25), we have
\[
G_p(\lambda K + \mu L) = G_p(\Lambda_p^c(\lambda K + \mu L)^c),
\]
\[
= n \omega_n \frac{\alpha_p}{\lambda} V(\lambda K + \mu L)^\frac{p}{n},
\]
\[
\geq \lambda n \omega_n \frac{\alpha_p}{\lambda} V(K)^\frac{p}{n} + \mu n \omega_n \frac{\alpha_p}{\lambda} V(L)^\frac{p}{n} = \lambda G_p(\Lambda_p^c K) + \mu G_p(\Lambda_p^c L).
\]
From the equality condition of (25), we know that equality holds in (9) for $p = 1$ if and only if $K$ and $L$ are homothetic, and for $p > 1$ if and only if $K$ and $L$ are dilates.

By (9) and (24), we easily get that if $K \in F_c^n$ and $n \neq p \geq 1$, then
\[
G_p(\nabla_p K) = G_p(K).
\]

**IV. THE EQUIVALENCE OF SEVERAL INEQUALITIES**

Define
\[
\mathcal{M}_p^n = \{ K \in F_c^n : there exists a Q \in \mathcal{K}_c^n \text{ with } f_p(K, \cdot) = h(Q, \cdot)^{(n+p)} \},
\]
and call it the $p$-elliptic type if $K \in \mathcal{M}_p^n$ (see [10]).

The following lemma is a direct consequence of Lemma 1.3.

**Lemma 4.1.** Suppose K \in \mathcal{M}_p^n and \ p \geq 1, then
\[
\Omega_p(K)^n + p = (n \omega_n)^p G_p(K)^n.
\]

Let $\mathcal{F}_c^n$ denote the set of all bodies in $\mathcal{K}_c^n$ which has a positive continuous curvature function. Combining inequality (2) with inequality (3), it follows from Lemma 4.1 that

**Theorem 4.2.** Suppose K \in \mathcal{F}_c^n and \ p \geq 1. If K \in \mathcal{M}_p^n, then inequality (3) is equivalent to inequality (2).

Lutwak [10] proved the following Blaschke-Santaló type inequality for p-affine surface area (Theorem 4.10 in [10]):

If $p \geq 1$ and $K \in \mathcal{K}_c^n$, then
\[
\Omega_p(K) \Omega_p(K^*) \leq (n \omega_n)^2,
\]
with equality if and only if $K$ is an ellipsoid.

From (29) and (30), we get the following Blaschke-Santaló type inequality for p-geominimal surface area.
Theorem 4.3. For $p \geq 1$ and $K \in \mathcal{K}_n^0$, if $K \in \mathcal{M}_p^n$, then
\[ G_p(K)G_p(K^*) \leq \left(\frac{n}{n-1}\right)^2, \tag{31} \]
with equality if and only if $K$ is an ellipsoid.
If $p \geq 1$ and $K \in \mathcal{K}_n^0$, then there exists a unique body $T_pK \in \mathcal{K}_n^0$ such that (see Proposition 3.3 in [10])
\[ G_p(T_pK) = nV_p(K) \text{ and } V(T_pK) = \omega_n. \]
A body in $\mathcal{K}_n^0$ will be called $p$-selfminimal if $T_pK$ and $K$ are dilates of each other.
For $K \in \mathcal{K}_n^0$, Lutwak [10] defined the $p$-geominimal area ratio of $K$ by
\[ \left(\frac{G_p(K^n)}{n^nV(K)^{n-p}}\right)^{1/p}, \]
and proved that the $p$-geominimal area ratios are monotone non-decreasing in $p$ (see Theorem 6.3 in [10]): If $K \in \mathcal{K}_n^0$, and $1 \leq p \leq q$, then
\[ \left(\frac{G_p(K^n)}{n^nV(K)^{n-p}}\right)^{1/p} \leq \left(\frac{G_q(K^n)}{n^nV(K)^{n-q}}\right)^{1/q}, \tag{32} \]
with equality if and only if $K$ is $p$-selfminimal.
For $K \in \mathcal{K}_n^0$, Lutwak [10] defined the $p$-affine area ratio of $K$ by
\[ \left(\frac{\Omega_p(K^{n+p})}{n^{n+p}V(K)^{n-p}}\right)^{1/p}, \]
and also obtained that the $p$-affine area ratios are monotone non-decreasing in $p$ (see Proposition 5.13 in [10]): If $K \in \mathcal{F}_n^0$, and $1 \leq p \leq q$, then
\[ \left(\frac{\Omega_p(K^{n+p})}{n^{n+p}V(K)^{n-p}}\right)^{1/p} \leq \left(\frac{\Omega_q(K^{n+q})}{n^{n+q}V(K)^{n-q}}\right)^{1/q}, \tag{33} \]
with equality if and only if $K$ and $\Delta_pK$ are dilates.
The equation (29) implies that if $K \in \mathcal{M}_p^n$ and $p \geq 1$, then
\[ \left(\frac{\Omega_p(K^{n+p})}{n^{n+p}V(K)^{n-p}}\right)^{1/p} = \omega_n \left(\frac{G_p(K^n)}{n^nV(K)^{n-p}}\right)^{1/p}. \tag{34} \]
It is clear from (34) that for $K \in \mathcal{M}_p^n$ inequality (32) and inequality (33) are equivalent.
Lutwak proved the following inequalities (35) and (36) for the $p$-affine area ratio of $K$ and the $p$-geominimal area ratio of $K$. Obviously, they are also equivalent for $K \in \mathcal{M}_p^n$.
If $K \in \mathcal{F}_n^0$ and $p \geq 1$, then (see Proposition 4.7 in [10])
\[ \left(\frac{\Omega_p(K^{n+p})}{n^{n+p}V(K)^{n-p}}\right)^{1/p} \leq V(K)V(K^*), \tag{35} \]
with equality if and only if $K^*$ and $\Delta_pK$ are dilates.
If $K \in \mathcal{K}_n^0$ and $p \geq 1$, then (see Proposition 6.2 in [10])
\[ \left(\frac{G_p(K^n)}{n^nV(K)^{n-p}}\right)^{1/p} \leq V(K)V(K^*)/\omega_n, \tag{36} \]
with equality if and only if $K$ is $p$-selfminimal.
We note that due to equality (29), Theorems 1.4-1.7 have obvious analogs for the $p$-affine surface area.

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