On Nonlinear Functional Spaces Based on Pseudo-additions

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***Abstract*—In this paper, we discuss the nonlinear functional spaces based on pseudo-additions. Particularly, we obtain that the some properties of the upper-closures of the regular subspaces of the nonlinear functional space based on a continuous pseudo-addition. Furthermore, we prove that with respect to a strict pseudo-addition, a subset of the nonlinear functional space is an upper-complete normal subspace if and only if the family of all sets whose characteristic functionals are contained in the given subset of the nonlinear functional space is a σ-algebra.**

***Index Terms*—pseudo-addition, nonlinear functional spaces, upper-closure, regular subspaces, normal subspaces.**

I. Introduction

ORIGINALLY functional analysis could be understood as a unifying abstract treatment of important aspects of linear mathematical models for problems in science, but the latter receded more and more into the background during the intensive theoretical investigations. Numerous questions in physics, chemistry, biology, and economics lead to nonlinear problems. Thus nonlinear functional analysis is an important branch of modern mathematics.

Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers a semiring is taken on a real interval $[a, b] \subset [-\infty, +\infty]$ endowed with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$ (see [8], [20], [21], [22], [31]). Based on this structure there were developed the concepts of $\oplus$-measure (pseudo-additive measure), pseudo-integral, pseudo-convolution, pseudo-Laplace transform, etc. The advantage of the pseudo-analysis is that there are covered with one theory, and so with unified methods, problems (usually nonlinear and under uncertainty) from many different fields (system theory, optimization, decision making, control theory, differential equations, difference equations, etc.). Pseudo-analysis uses many mathematical tools from different field as functional equations, variational calculus, measure theory, functional analysis, optimization theory, semiring theory, etc.

Recently, pseudo-analysis has obtained the rapid development in the field of the mechanical, chemical, biological, medical and computer, and has solved some uncertainty problems of knowledge. Pseudo-analysis theory has important applications in the field of computer image processing [1], [11], [19], [27], for example, it can analyze and grasp the variation range of the image gray value, solve the relationship between the grey value and image color change and take appropriate grey value to achieve better image processing effect. With the development of computer technology, pseudo-analysis will also get more and more widely used in computer science.

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Based on the above these, they also play an important role in theories of non-additive measures [9], [33], [34]. The families of the pseudo-operations based on generated g turn out to be solutions of well-known nonlinear functional equations [28], [29], [32]. In many problems with uncertainty as in the theory of probabilistic metric spaces [5], [7], [30], [35], multi-valued logics [6], [12], [13], [16], general measures [10], [34] often we work with many operations different from the usual addition and multiplication of reals. Some of them are triangular norms, triangular conorms, pseudo-additions, pseudo-multiplications, etc. [23], [25].

In this paper, we will discuss the nonlinear functional spaces based on pseudo-additions. Particularly, we will obtain that the some properties of the upper-closures of the regular subspaces of the nonlinear functional space based on a continuous pseudo-addition. Furthermore, we will prove that with respect to a strict pseudo-addition, a subset of the nonlinear functional space is an upper-complete normal subspace if and only if the family of all sets whose characteristic functionals are contained in the given subset of the nonlinear functional space is a σ-algebra.

II. Preliminaries

Let $[a, b]$ be a closed subinterval of $\mathbb{R}$ (in some cases we will also take semiclosed subintervals). The total order on $[a, b]$ will be denoted by $\preceq$. This can be the usual order of the real line, but it can also be another order. We shall denote by Δ is maximum element on $[a, b]$ (usually Δ is either a or b) with respect to this total order.

Definition 2.1: Let $\{x_n\}_{n\geq 1}$ be a sequence from $[a, b]$.

(1) If $x_m \preceq x_n$ whenever $n > m$, then we say that the sequence $\{x_n\}_{n\geq 1}$ is an increasing sequence;

(2) If $x_m \prec x_n$ whenever $n > m$, then we say that the sequence $\{x_n\}_{n\geq 1}$ is a strict increasing sequence;

(3) If $x_n \preceq x_m$ whenever $n > m$, then we say that the sequence $\{x_n\}_{n\geq 1}$ is a decreasing sequence;

(4) If $x_n \prec x_m$ whenever $n > m$, then we say that the sequence $\{x_n\}_{n\geq 1}$ is a strict decreasing sequence.

Let X be a non-empty set, we shall denote by $\mathscr{S}$ and $\mathcal{B}_X$ are algebra and Borel σ-algebra of subsets of a set X, respectively.

Denote by $\mathscr{F}(X)$ is the set of all functionals from X to $[a, b]$. For each $\lambda \in [a, b]$ the constant functional in $\mathscr{F}(X)$ with value λ will also be denoted by λ. It will be clear from the context which usage is intended. A functional $f \in \mathscr{F}(X)$ is said to be elementary if the set of values $f(X)$ of f is a finite subset of $[a, b]$ and the set of such elementary functionals will be denoted by $\mathscr{E}(X)$.

Let f and h be two functions defined on X and with values in $[a, b]$ and $\star$ be arbitrary binary operation on $[a, b]$. Then, we define for any $x \in X$

$$(f \star h)(x) = f(x) \star h(x),$$

and for any $\lambda \in [a,b]$, $(\lambda \star f)(x) = \lambda \star f(x)$. Let $\mathcal{A}$ be a subset of $\mathscr{F}(X)$. If $f \star h \in \mathcal{A}$ for all $f, h \in \mathcal{A}$, then $\mathcal{A}$ is $\star$-closed. The total order $\preceq$ on $[a,b]$ induces a partial order $\preceq$ on $\mathscr{F}(X)$ defined pointwise by stipulating that $f \preceq h$ if and only if $f(x) \preceq h(x)$ for all $x \in X$. Thus $(\mathscr{F}(X), \preceq)$ is a poset, and whenever we consider $\mathscr{F}(X)$ as a poset then it will always be with respect to this partial order.

Definition 2.2: [15] A binary operation $\oplus : [a,b] \times [a,b] \to [a,b]$ is called a pseudo-addition, if it satisfies the following conditions, for all $x, y, z, w \in [a,b]$:

(1) $\mathbf{0} \oplus x = x$, where $\mathbf{0}$ is a zero element (usually $\mathbf{0}$ is either a or b);(boundary condition)

(2) $x \oplus z \preceq y \oplus w$ whenever $x \preceq y$ and $z \preceq w$; (monotonicity)

(3) $x \oplus y = y \oplus x$; (commutativity)

(4) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$. (associativity)

A pseudo-addition $\oplus$ is said to be continuous if it is a continuous function in $[a,b]^2$; a pseudo-addition $\oplus$ is called strict if $\oplus$ is continuous and strictly monotone. The following are examples of pseudo-additions: $x \vee_{\oplus} y = y$ if and only if $x \preceq y$; $x \oplus y = g^{-1}(g(x) + g(y))$, where $g : [a,b] \to [0,1]$ is a strictly monotone and continuous generator surjective function and $x \preceq y$ if and only if $g(x) \le g(y)$. It is obvious that $\Delta \oplus x = \Delta$ for all $x \in [a,b]$.

Let $[a,b]_+ = \{x | x \in [a,b], \mathbf{0} \preceq x\}$. In this paper, we assume $[a,b] = [a,b]_+$.

Definition 2.3: [15] A binary operation $\odot : [a,b] \times [a,b] \to [a,b]$ is called a pseudo-multiplication, if it satisfies the following conditions, for all $x, y, z, w \in [a,b]$:

(1) $\mathbf{1} \odot x = x$, where $\mathbf{1} \in [a,b]$ is an unit element; (boundary condition)

(2) $x \odot z \preceq y \odot w$ whenever $x \preceq y$ and $z \preceq w$; (monotonicity)

(3) $x \odot y = y \odot x$; (commutativity)

(4) $(x \odot y) \odot z = x \odot (y \odot z)$. (associativity)

A pseudo-multiplication $\odot$ is said to be continuous if it is a continuous function in $[a,b]^2$. The following are examples of pseudo-multiplications: $x \wedge_{\odot} y = x$ if and only if $x \preceq y$; $x \odot_g y = g^{-1}(g(x) \cdot g(y))$, where $g : [a,b] \to [0,1]$ is a strictly monotone and continuous generator surjective function and $x \preceq y$ if and only if $g(x) \le g(y)$. It is obvious that $g(\mathbf{0}) = 0$.

We assume also $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e.,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z).$$

The structure $([a,b], \oplus, \odot)$ is called a real semiring.

Because of the associative property of the pseudo-addition $\oplus$, it can be extended by induction to n-ary operation by setting

$$\mathop{\oplus}_{i=1}^{n} x_i = \left(\mathop{\oplus}_{i=1}^{n-1} x_i \right) \oplus x_n.$$

Due to monotonicity, for each sequence $\{x_i\}_{i \in \mathbb{N}}$ of elements of $[a,b]$, the following limit can be considered:

$$\mathop{\oplus}_{i=1}^{\infty} x_i = \lim_{n \to \infty} \mathop{\oplus}_{i=1}^{n} x_i.$$

For any continuous pseudo-addition $\oplus$ and $x, y \in [a,b]$ with $x \preceq y$, there exists at least one point $z \in [a,b]$ such that $y = x \oplus z$. If pseudo-addition $\oplus$ is strict, then there exists only one point $z \in [a,b]$ such that $y = x \oplus z$ for all $x, y \in [a,b]$ with $x \prec \Delta$. Thus we have the following concepts.

Definition 2.4: For any continuous pseudo-addition $\oplus$ and $x, y \in [a,b]$ with $x \preceq y$, the para-complement set $y -_{\oplus} x$ is a nonempty set of all points z such that $y = x \oplus z$.

Example 2.1: Let total order $\preceq$ on $[0,1]$ be the usual order of the real line and pseudo-addition $\oplus$ be the usual multiplication of the real numbers. It is obvious that zero element is 1. If $x = 0$, then $y = 0$ and $y -_{\oplus} x = [0,1]$. If $x \neq 0$, then for any $0 \le y < x$, we have $y -_{\oplus} x = \{y/x\} \subseteq [0,1]$.

Definition 2.5: For any continuous pseudo-addition $\oplus$, if $f, h \in \mathscr{F}(X)$, then define the para-complement set $|f -_{\oplus} h|$ as the set of all those functionals φ such that

$$\varphi(x) = \begin{cases} f(x) -_{\oplus} h(x), & if \;\; h(x) \preceq f(x), \\ h(x) -_{\oplus} f(x), & if \;\; f(x) \prec h(x), \end{cases}$$

for all $x \in X$.

Definition 2.6: For any strict pseudo-addition $\oplus$ and $x, y \in [a,b]$ with $x \preceq y$, the complement $y -'_{\oplus} x$ is defined as

$$y -'_{\oplus} x = \begin{cases} z \in [a,b], & such\ that\ y = x \oplus z,\ if\ x \prec \Delta, \\ \mathbf{0}, & otherwise. \end{cases}$$

Definition 2.7: For any strict pseudo-addition $\oplus$, if $f, h \in \mathscr{F}(X)$ with $h \preceq f$, then define the complement functional $|f -'_{\oplus} h|$ pointwise as

$$|f -'_{\oplus} h|(x) = \begin{cases} f(x) -'_{\oplus} h(x), & if \;\; h(x) \preceq f(x), \\ h(x) -'_{\oplus} f(x), & if \;\; f(x) \prec h(x), \end{cases}$$

for all $x \in X$.

Definition 2.8: For any pseudo-addition $\oplus$, a non-empty subset $\mathcal{K}$ of $\mathscr{F}(X)$ is said to be a functional space with respect to $\oplus$, denoted by $(\mathcal{K}, \oplus)$, if $(\lambda \odot f) \oplus (\mu \odot h) \in \mathcal{K}$ for all $f, h \in \mathcal{K}$ and $\lambda, \mu \in [a,b]$, where $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$.

It is clear that $(\mathscr{F}(X), \oplus)$ is the greatest functional space with respect to any pseudo-addition $\oplus$. Thus the functional space $(\mathcal{K}, \oplus)$ with $\mathcal{K} \subseteq \mathscr{F}(X)$ is also called a subspace of $(\mathscr{F}(X), \oplus)$. If $(\mathcal{K}, \oplus)$ is a functional space with respect to $\oplus$, then we just write $\mathcal{K}$ instead of $(\mathcal{K}, \oplus)$ whenever $\oplus$ can be determined from the context.

Let $\mathcal{A}$ be a subset of $\mathscr{F}(X)$. The poset $\mathcal{A}$ is said to be upper-complete if $\lim_{n \to \infty} f_n \in \mathcal{A}$ for each increasing sequence $\{f_n\}_{n \ge 1}$ from $\mathcal{A}$; the poset $\mathcal{A}$ is said to be lower-complete if $\lim_{n \to \infty} f_n \in \mathcal{A}$ for each decreasing sequence $\{f_n\}_{n \ge 1}$ from $\mathcal{A}$; the poset $\mathcal{A}$ is said to be complete if $\lim_{n \to \infty} f_n \in \mathcal{A}$ for each sequence $\{f_n\}_{n \ge 1}$ from $\mathcal{A}$, where the limit of the functional sequence $\{f_n\}_{n \ge 1}$ is given by $(\lim_{n \to \infty} f_n)(x) = \lim_{n \to \infty} f_n(x)$ for all $x \in X$.

Definition 2.9: For each subset $\mathcal{A}$ of $\mathscr{F}(X)$ the upper-closure of $\mathcal{A}$, denoted by $\hat{\mathcal{A}}$, is the set of all elements of $\mathscr{F}(X)$ having the form $\lim_{n \to \infty} f_n$ for some increasing sequence $\{f_n\}_{n \ge 1}$ from $\mathcal{A}$.

It follows from Definition 2.9 that $\mathcal{A} \subseteq \hat{\mathcal{A}}$ and $\mathcal{A} = \hat{\mathcal{A}}$ if and only if $\mathcal{A}$ is upper-complete.

Definition 2.10: For any continuous pseudo-addition $\oplus$, a subspace $(\mathcal{K}, \oplus)$ will be called para-complemented if $|f -_{\oplus} h| \subseteq \mathcal{K}$ for all $f, h \in \mathcal{K}$; for any strict pseudo-addition $\oplus$, a subspace $(\mathcal{K}, \oplus)$ will be called complemented if $|f -'_{\oplus} h| \in \mathcal{K}$ for all $f, h \in \mathcal{K}$.

Definition 2.11: For any continuous pseudo-addition $\oplus$, a para-complemented subspace $(\mathcal{K}, \oplus)$ is regular if it contains **1** and is closed under $\vee_\oplus$; for any strict pseudo-addition $\oplus$, a complemented subspace $(\mathcal{K}, \oplus)$ is normal if it contains **1** and is closed under $\vee_\oplus$.

Note that $(f \vee_\oplus h) \oplus (f \wedge_\odot h) = f \oplus h$ for all $f, h \in \mathscr{F}(X)$ and thus a para-complemented subspace of $\mathscr{F}(X)$ is $\wedge_\odot$-closed if and only if it is $\vee_\oplus$-closed. It is obvious that regular and normal are closed under $\wedge_\odot$.

Definition 2.12: [4] The pseudo-characteristic function of a set $E \subseteq X$ is defined with:

$$I_E(x) = \begin{cases} \mathbf{0}, & x \notin E, \\ \mathbf{1}, & x \in E, \end{cases}$$

where **0** is zero element for $\oplus$ and **1** is unit element for $\odot$. It is obvious that $I_E \in \mathscr{E}(X)$, for all $E \subseteq X$.

Definition 2.13: [23] A set function $m : \mathscr{A} \to [a, b]$ (or semiclosed interval) is called a σ-$\oplus$-decomposable measure if it satisfies the following conditions:

(1) $m(\emptyset) = \mathbf{0}$;

(2) $m(E \cup F) = m(E) \oplus m(F)$ for all $E, F \in \mathscr{A}$ and $E \cap F = \emptyset$;

(3) $m(\bigcup_{i=1}^{\infty} E_i) = \bigoplus_{i=1}^{\infty} m(E_i)$ for any sequence $\{E_i\}_{i \geq 1}$ of pairwise disjoint sets from $\mathscr{A}$.

A pair $(X, \mathscr{A})$ consisting of a non-empty set X and a σ-algebra of subsets of X is called a measurable space. A functional $f : X \to [a, b]$ is said to be a measurable function if $f^{-1}(\mathcal{B}_{[a,b]}) \subseteq \mathscr{A}$. Let $\mathcal{M}(\mathscr{A})$ be the set of all measurable mappings from $(X, \mathscr{A})$ to $([a, b], \mathscr{B}_{[a,b]})$, i.e.,

$$\mathcal{M}(\mathscr{A}) = \{f \in \mathscr{F}(X) |\ f^{-1}(\mathscr{B}_{[a,b]}) \subseteq \mathscr{A}\}.$$

Then $\mathscr{E}(\mathscr{S})$ will denote the set of those elements $f \in \mathscr{E}(X)$ for which $f^{-1}(\lambda) = \{x \in X | f(x) = \lambda\} \in \mathscr{S}$ for each $\lambda \in f(X)$. In particular, this means that $\mathscr{E}(\mathscr{A}) = \mathcal{M}(\mathscr{A}) \cap \mathscr{E}(X)$.

III. Main results

In this section we will discuss the properties of the upper-closures of regular subspaces of the nonlinear functional space $\mathscr{F}(X)$ base on a continuous pseudo-addition $\oplus$.

Theorem 3.1: Let $\mathcal{A}$ be a $\vee_\oplus$-closed subset of $\mathscr{F}(X)$ and let $\{f_n\}_{n \geq 1}$ be an increasing sequence from $\mathcal{A}$ with $f = \lim_{n \to \infty} f_n$. Then there exists an increasing sequence $\{h_n\}_{n \geq 1}$ from $\mathcal{A}$ with $h_n \preceq f_n$ for all $n \geq 1$ and $\lim_{n \to \infty} h_n = f$. In particular, this implies $\hat{\mathcal{A}}$ is upper-complete.

Proof. For each f_n there exists an increasing sequence $\{f_{(n,k)}\}_{k \geq 1}$ from $\mathcal{A}$ with $f_n = \lim_{k \to \infty} f_{(n,k)}$, since $\mathcal{A} = \hat{\mathcal{A}}$. Let

$$h_k = f_{(1,k)} \vee_\oplus f_{(2,k)} \vee_\oplus \cdots \vee_\oplus f_{(k,k)}$$

for each $m \geq 1$. Since $\mathcal{A}$ is $\vee_\oplus$-closed and $\vee_\oplus$ is monotone, we have $h_k \in \mathcal{A}$ and

$$\begin{aligned} h_k &= f_{(1,k)} \vee_\oplus f_{(2,k)} \vee_\oplus \cdots \vee_\oplus f_{(k,k)} \\ &\preceq f_{(1,k)} \vee_\oplus f_{(2,k)} \vee_\oplus \cdots \vee_\oplus f_{(k,k)} \vee_\oplus f_{(k+1,k)} \\ &\preceq f_{(1,k+1)} \vee_\oplus \cdots \vee_\oplus f_{(k,k+1)} \vee_\oplus f_{(k+1,k+1)} \\ &= h_{k+1}. \end{aligned}$$

Therefore $\{h_k\}_{k \geq 1}$ is an increasing sequence from $\mathcal{A}$. Let $h = \lim_{k \to \infty} h_k$. Since

$$f_{(n,k)} \preceq f_{(1,k)} \vee_\oplus f_{(2,k)} \vee_\oplus \cdots \vee_\oplus f_{(k,k)} = h_k \preceq h$$

for all $k \geq n \geq 1$, we have $f_n = \lim_{k \to \infty} f_{(n,k)} \preceq h$ for all $n \geq 1$. Thus $f = \lim_{n \to \infty} f_n \preceq h$. But on the other hand,

$$h_k = f_{(1,k)} \vee_\oplus \cdots \vee_\oplus f_{(k,k)} \preceq f_1 \vee_\oplus f_2 \vee_\oplus \cdots \vee_\oplus f_k = f_k.$$

Hence, we have $h = \lim_{k \to \infty} h_k \preceq \lim_{k \to \infty} f_k = f$. Consequently, we obtain that $h = f$. □

Example 3.1: Let total order $\preceq$ on $[0, 1]$ be the usual order of the real line and pseudo-operation $\vee_\oplus$ be the usual sup $= \vee$ of the real numbers. It is obvious that zero element is 1. Let $X = \mathbb{R}$ and $\mathcal{A} = \mathscr{F}(X)$. It is obvious that $\mathcal{A}$ is $\vee_\oplus$-closed. We define the function sequence as

$$f_n(x) = \begin{cases} 1, & x \in \left(\frac{1}{n}, +\infty\right), \\ nx, & x \in \left(0, \frac{1}{n}\right], \\ 0, & x \in (-\infty, 0], \end{cases} \qquad n = 1, 2, \cdots,$$

and the function as

$$f(x) = \begin{cases} 1, & (0, +\infty), \\ 0, & (-\infty, 0]. \end{cases}$$

It is obvious that $\{f_n\}_{n \geq 1}$ is an increasing sequence from $\mathcal{A}$ with $f = \lim_{n \to \infty} f_n$. By Theorem 3.1, we get that there exists an increasing sequence $\{h_n\}_{n \geq 1}$ from $\mathcal{A}$ with $h_n \preceq f_n$ for all $n \geq 1$ and $\lim_{n \to \infty} h_n = f$.

Theorem 3.2: Let $\star$ be an increasing and continuous operation on $[a, b]$ and $\mathcal{A}$ a $\star$-closed subset of $\mathscr{F}(X)$. Then $\hat{\mathcal{A}}$ is $\star$-closed. In particular, $\hat{\mathcal{A}}$ is $\vee_\oplus$-closed.

Proof. Let $f, h \in \hat{\mathcal{A}}$ and $\{f_n\}_{n \geq 1}$, $\{h_n\}_{n \geq 1}$ be increasing sequences from $\mathcal{A}$ with $f = \lim_{n \to \infty} f_n$ and $h = \lim_{n \to \infty} h_n$. Since $\mathcal{A}$ is $\star$-closed and $\star$ is an increasing operation on $[a, b]$, we have $f_n \star h_n \in \mathcal{A}$ $(n \geq 1)$ and $\{f_n \star h_n\}_{n \geq 1}$ is an increasing sequence . In addition, since $\star$ is a continuous operation on $[a, b]$, we get that

$$\begin{aligned} & \lim_{n \to \infty} (f_n \star h_n)(x) \\ =\ & \lim_{n \to \infty} (f_n(x) \star h_n(x)) \\ =\ & (\lim_{n \to \infty} f_n(x)) \star (\lim_{n \to \infty} h_n(x)) \\ =\ & f(x) \star h(x) = (f \star h)(x). \end{aligned}$$

Since $\hat{\mathcal{A}}$ is upper-complete, we have $f \star h \in \hat{\mathcal{A}}$. In particular, since $\vee_\oplus$ is an increasing and continuous operation on $[a, b]$, we obtain that $\hat{\mathcal{A}}$ is $\vee_\oplus$-closed. □

Theorem 3.3: Let $(\mathcal{K}, \oplus)$ be a $\vee_\oplus$-closed subspace with respect to a continuous pseudo-addition $\oplus$. Then $(\hat{\mathcal{K}}, \oplus)$ is a subspace which is upper-complete and $\vee_\oplus$-closed. Moreover, if $\mathcal{K}$ is closed under an increasing and continuous operation $\star$ on $\mathscr{F}(X)$, then so is $\hat{\mathcal{K}}$.

Proof. We only prove $(\hat{\mathcal{K}}, \oplus)$ is a subspace, since by Theorem 3.1, we have $\hat{\mathcal{K}}$ is upper-complete; by Theorem 3.2, we have $\hat{\mathcal{K}}$ is $\vee_\oplus$-closed.

Let $f, h \in \hat{\mathcal{K}}$ and $\{f_n\}_{n \geq 1}$, $\{h_n\}_{n \geq 1}$ be increasing sequences from $\mathcal{K} \subseteq \mathscr{F}(X)$ with $f = \lim_{n \to \infty} f_n$ and $h = \lim_{n \to \infty} h_n$. Since $\oplus$ and $\odot$ are monotone pseudo-operations on $[a, b]$ and $(\mathcal{K}, \oplus)$ is a subspace, we have $\{(\lambda \odot f_n) \oplus (\mu \odot h_n)\}_{n \geq 1}$ is increasing sequence from $\mathcal{K}$, for all $\lambda, \mu \in [a, b]$. In addition, since $\oplus$ and $\odot$ are continuous pseudo-operations on $[a,b]$, we get that

$$\begin{aligned} & \lim_{n\to\infty}((\lambda\odot f_n)\oplus(\mu\odot h_n))(x) \\ = & \lim_{n\to\infty}((\lambda\odot f_n(x))\oplus(\mu\odot h_n(x))) \\ = & (\lim_{n\to\infty}(\lambda\odot f_n(x))\oplus(\lim_{n\to\infty}(\mu\odot h_n(x))) \\ = & (\lambda\odot(\lim_{n\to\infty}f_n(x)))\oplus(\mu\odot(\lim_{n\to\infty}h_n(x))) \\ = & (\lambda\odot f(x))\oplus(\mu\odot h(x)) \\ = & ((\lambda\odot f)\oplus(\mu\odot h))(x). \end{aligned}$$

Hence, $(\lambda\odot f)\oplus(\mu\odot h)\in\hat{\mathcal{K}}$, since $\hat{\mathcal{K}}$ is upper-complete. □

Note also that a subspace always contains the constant mapping $\mathbf{0}$, since $\mathbf{0}=(\mathbf{0}\odot f)\oplus(\mathbf{0}\odot h)$. Thus if a subspace $(\mathcal{K},\oplus)$ contains the constant $\mathbf{1}$ then $\lambda\in\mathcal{K}$ for all $\lambda\in[a,b]$, since $\lambda=(\lambda\odot\mathbf{1})\oplus(\mathbf{0}\odot\mathbf{1})$. Let $(\mathcal{K},\oplus)$ be a regular subspace. Then by Theorem 3.3, $\hat{\mathcal{K}}$ is closed under both $\vee_\oplus$ and $\wedge_\odot$ and of course $\mathbf{1}\in\hat{\mathcal{K}}$. Therefore $(\hat{\mathcal{K}},\oplus)$ is regular if and only if $\hat{\mathcal{K}}$ is para-complemented. However, in general this will fail to be the case, and the best partial results are perhaps the following:

Theorem 3.4: Let $(\mathcal{K},\oplus)$ be a $\vee_\oplus$-closed para-complemented subspace with respect to a continuous pseudo-addition $\oplus$. If $(\mathcal{K},\oplus)$ is upper-complete, then it is also lower-complete.

Proof. Let $\{f_n\}_{n\geq 1}$ be a decreasing sequence from $\mathcal{K}$ with $f=\lim_{n\to\infty}f_n$. For each $n\geq 1$, let $h_n=\bigvee_{\oplus\, \varphi\in|f_1-_\oplus f_n|}\varphi$. Since $\oplus$ is continuous and $f_n\preceq f_1$, i.e.,$f_1=f_n\oplus\varphi$ for all $\varphi\in|f_1-_\oplus f_n|$, we have $f_1=f_n\oplus h_n$. Now we show that $\{h_n\}_{n\geq 1}$ is an increasing sequence from $\mathcal{K}$. If otherwise, then there are $x_0\in X$ and k,n with $k>n\geq 1$ such that $h_k(x_0)\prec h_n(x_0)$. Since $\oplus$ is monotonous, it follows that

$$f_1(x_0)\preceq f_k(x_0)\oplus h_n(x_0)\preceq f_n(x_0)\oplus h_n(x_0)=f_1(x_0),$$

which implies that $f_1(x_0)=f_k(x_0)\oplus h_n(x_0)$. This contradicts the definition of $h_k(x_0)$. Then $\{h_n\}_{n\geq 1}$ is an increasing sequence from $\mathcal{K}$ and thus $h=\lim_{n\to\infty}h_n\in\mathcal{K}$. Thus $f_1=f\oplus h$ and so $f\in|f_1-_\oplus h|\subseteq\mathcal{K}$. □

Theorem 3.5: Let $(\mathcal{K},\oplus)$ be a $\vee_\oplus$-closed para-complemented subspace with respect to a continuous pseudo-addition $\oplus$ and let $h\in\mathcal{K}$ and $f\in\hat{\mathcal{K}}$. Then $\hat{\mathcal{K}}\cap(|f-_\oplus h|)\neq\emptyset$.

Proof. Let $E=\{x\in X|\ h(x)\preceq f(x)\}$ and $F=\{x\in X|\ f(x)\prec h(x)\}$ and let $\{f_n\}_{n\geq 1}$ and $\{h_n\}_{n\geq 1}$ be two increasing sequence from $\mathcal{K}$ with $f=\lim_{n\to\infty}f_n$ and $h=\lim_{n\to\infty}h_n$, respectively. Then $\{f_n\vee_\oplus h\}_{n\geq 1}$ and $\{h_n\vee_\oplus f\}_{n\geq 1}$ are two increasing sequence from $\mathcal{K}$, also with $\lim_{n\to\infty}(f_n\vee_\oplus h)(x)=(f\vee_\oplus h)(x)=f(x)$ for all $x\in E$ and $\lim_{n\to\infty}(h_n\vee_\oplus f)(x)=(h\vee_\oplus f)(x)=h(x)$ for all $x\in F$, respectively. Then there exists $\psi_n\in\mathcal{K}$ with

$$\psi_n(x)=\begin{cases}\left(\bigwedge_{\odot\, \psi\in|(f_n\vee_\oplus h)-_\oplus h|}\psi\right)(x), & if\quad x\in E,\\ \left(\bigwedge_{\odot\, \psi\in|(h_n\vee_\oplus f)-_\oplus f|}\psi\right)(x), & if\quad x\in F,\end{cases}$$

since $\mathcal{K}$ is $\wedge_\odot$-closed if and only if it is $\vee_\oplus$-closed, and

$$\begin{cases}(f_n\vee_\oplus h)(x)=(\psi_n\oplus h)(x), & if\quad x\in E,\\ (h_n\vee_\oplus f)(x)=(\psi_n\oplus f)(x), & if\quad x\in F.\end{cases}$$

Let $\varphi_n=\psi_1\vee_\oplus\psi_2\vee_\oplus\cdots\vee_\oplus\psi_n$, and $\{\varphi_n\}_{n\geq 1}$ is an increasing sequence from $\mathcal{K}$ such that

$$\begin{aligned} & (f_n\vee_\oplus h)(x) \\ = & (f_1\vee_\oplus h)(x)\vee_\oplus(f_2\vee_\oplus h)(x)\cdots\vee_\oplus(f_n\vee_\oplus h)(x) \\ = & (\psi_1\oplus h)(x)\vee_\oplus(\psi_2\oplus h)(x)\vee_\oplus\cdots\vee_\oplus(\psi_n\oplus h)(x) \\ = & h(x)\oplus(\psi_1\vee_\oplus\psi_2\vee_\oplus\cdots\vee_\oplus\psi_n)(x) \\ = & h(x)\oplus\varphi_n(x) \end{aligned}$$

for all $x\in E$. Similarly, we get that

$$(h_n\vee_\oplus f)(x)=f(x)\oplus\varphi_n(x)$$

for all $x\in F$. Let $\varphi=\lim_{n\to\infty}\varphi_n$. Then $\varphi\in\hat{\mathcal{K}}$ and $\varphi\in|f-_\oplus h|$. □

Theorem 3.6: Let $(\mathcal{K},\oplus)$ be a $\vee_\oplus$-closed para-complemented subspace with respect to a continuous pseudo-addition $\oplus$. If $(\hat{\mathcal{K}},\oplus)$ is lower-complete, then $\hat{\mathcal{K}}\cap|f-_\oplus h|\neq\emptyset$ for all $f,h\in\hat{\mathcal{K}}$.

Proof. Suppose $\hat{\mathcal{K}}$ is lower-complete. Let $f,h\in\hat{\mathcal{K}}$ and let $\{f_n\}_{n\geq 1}$ and $\{h_n\}_{n\geq 1}$ be two increasing sequence from $\mathcal{K}$ with $f=\lim_{n\to\infty}f_n$ and $h=\lim_{n\to\infty}h_n$, respectively.

Let $E=\{x\in X|\ h(x)\preceq f(x)\}$ and $F=\{x\in X|\ f(x)\prec h(x)\}$. By Theorem 3.2 and Theorem 3.5 there exists $\varphi_n\in\hat{\mathcal{K}}$ with

$$\varphi_n(x)=\begin{cases}\left(\bigwedge_{\odot\, \psi\in\hat{\mathcal{K}}\cap|f-_\oplus h_n|}\psi\right)(x), & if\quad x\in E,\\ \left(\bigwedge_{\odot\, \psi\in\hat{\mathcal{K}}\cap|h-_\oplus f_n|}\psi\right)(x), & if\quad x\in F,\end{cases}$$

since a para-complemented subspace of $\mathscr{F}(X)$ is $\wedge_\odot$-closed if and only if it is $\vee_\oplus$-closed. Now we show that $\{\varphi_n\}_{n\geq 1}$ is a decreasing sequence from $\mathcal{K}$. If on the contrary, then there are $x_0\in X$ and k,n with $k>n\geq 1$ such that $\varphi_n(x_0)\prec\varphi_k(x_0)$. Suppose $x_0\in E$, for the case $x_0\in F$, we can prove it by a similar proof. Since $\oplus$ is monotonous, it follows that

$$f(x_0)\preceq h_k(x_0)\oplus\varphi_n(x_0)\preceq h_k(x_0)\oplus\varphi_k(x_0)=f(x_0),$$

which implies that $f(x_0)=h_k(x_0)\oplus\varphi_n(x_0)$. This contradicts the definition of $\varphi_k(x_0)$. Then $f(x)=(h_n\oplus\varphi_n)(x)$ for all $x\in E$; $h(x)=(f_n\oplus\varphi_n)(x)$ for all $x\in F$, and $\{\varphi_n\}_{n\geq 1}$ is a decreasing sequence from $\hat{\mathcal{K}}$. Hence, $\varphi=\lim_{n\to\infty}\varphi_n\in\hat{\mathcal{K}}$, since $\hat{\mathcal{K}}$ is lower-complete, and $f(x)=(h\oplus\varphi)(x)$ for all $x\in E$ and $h(x)=(f\oplus\varphi)(x)$ for all $x\in F$, i.e., $\varphi\in|f-_\oplus h|$. □

The conclusions of the above theorems still hold on regular subspace, since regular subspace is special case. And also hold for the upper-closures and the normal subspaces with respect to a strict pseudo-addition, particularly we can get the the following corollary.

Corollary 3.1: Let $(\mathcal{K},\oplus)$ be a normal subspace with respect to a strict pseudo-addition $\oplus$. Then $(\hat{\mathcal{K}},\oplus)$ is normal if and only if it is lower-complete.

Proof. We can obtain a similar result with Theorem 3.4 that an upper-complete normal subspace is also lower-complete. Let $(\mathcal{K},\oplus)$ be a normal subspace and $(\hat{\mathcal{K}},\oplus)$ be lower-complete. By the proof of Theorem 3.6, we get that $|f-'_{\oplus}h|\in\hat{\mathcal{K}}$ for all $f,h\in\hat{\mathcal{K}}$, i.e., $(\hat{\mathcal{K}},\oplus)$ is complemented. In the meantime, by some similar discussions with the notes before Theorem 3.4, we can get that $(\hat{\mathcal{K}},\oplus)$ is normal if and only if it is complemented. Hence, $(\hat{\mathcal{K}},\oplus)$ is normal if and only if it is lower-complete. □

Theorem 3.7: (1) $(\mathscr{E}(X),\oplus)$ is a subspace of $(\mathscr{F}(X),\oplus)$ with respect to any continuous pseudo-addition $\oplus$. Moreover if $\oplus$ is a strict pseudo-addition, then $(\mathscr{E}(X),\oplus)$ is a normal subspace.

(2) If $(\mathcal{K},\oplus)$ is a subspace with $I_A\in\mathcal{K}$ for all $A\subseteq X$, then $(\mathscr{E}(X),\oplus)\subseteq(\mathcal{K},\oplus)$, which means $(\mathscr{E}(X),\oplus)$ is the smallest subspace containing the mappings I_A, $A\subseteq X$.

(3) $(\mathscr{F}(X),\oplus)=(\hat{\mathscr{E}(X)},\oplus)$.

Proof. (1) For any $\lambda,\mu\in[a,b]$ and $f,h\in\mathscr{E}(X)$,

$$((\lambda\odot f)\oplus(\mu\odot h))\subseteq A=\{(\lambda\odot\nu)\oplus(\mu\odot\omega)|\nu\in f,\omega\in h\}$$

and A is a finite subset of $[a,b]$ whenever both $f(X)$ and $h(X)$ are. Thus $\mathscr{E}(X)$ is a subspace. Moreover, if $\oplus$ is strict, then for any $f,h\in\mathscr{E}(X)$, $(|f-'_{\oplus}h|)(X)$ is a subset of the finite set $\{|\nu-'_{\oplus}\omega||\nu\in f(X),\omega\in h(X)\}$. It is easy to see $(f\vee_{\oplus}h)(X)$ is finite sets for all $f,h\in\mathscr{E}(X)$. Finally, it is clear that $\mathbf{1}\in\mathscr{E}(X)$.

(2) If $f\in\mathscr{E}(X)$, then for any $\lambda\in f(X)$ and $x\in X$, we have

$$\left(\lambda\odot I_{f^{-1}(\lambda)}\right)(x)=\begin{cases}\lambda=f(x), & x\in f^{-1}(\lambda),\\ \mathbf{0}, & x\notin f^{-1}(\lambda),\end{cases}$$

where $f^{-1}(\lambda)=\{x\in X|f(x)=\lambda\}$. Thus, we have $f=\underset{\lambda\in f(X)}{\oplus}\left(\lambda\odot I_{f^{-1}(\lambda)}\right)$. Since $I_A\in\mathcal{K}$ for each $A\subseteq X$, we obtain that $f\in\mathcal{K}$. Then $\mathscr{E}(X)\subseteq\mathcal{K}$.

(3) Let $f\in\mathscr{F}(X)$ and for each $n\geq 1$ define $f_n\in\mathscr{E}(X)$ by

$$f_n=\overset{2^n}{\underset{k=1}{\oplus}}\left(\lambda_{k-1,n}\odot I_{A_{k,n}}\right),$$

where $\mathbf{0}=\lambda_{0,n}\preceq\lambda_{1,n}\preceq\cdots\preceq\lambda_{2^n,n}=\Delta$, $\lambda_{2i,n+1}=\lambda_{i,n}(0\leq i\leq 2^n)$, and $A_{k,n}=\{x\in X|\ \lambda_{k-1,n}\prec f(x)\preceq\lambda_{k,n}\}$. Then $f_n\preceq f_{n+1}\preceq f$ for each $n\geq 1$ and $f(x)\preceq f_n(x)\oplus\mu_n$ for all $x\in X$ with $f(x)\preceq\Delta$, where $\mu_n=|\lambda_{k,n}-'_{\oplus}\lambda_{k-1,n}|$ and $\lim\limits_{n\to\infty}\mu_n=\mathbf{0}$. Thus $\{f_n\}_{n\geq 1}$ is an increasing sequence from $\mathscr{E}(X)$ with $\lim\limits_{n\to\infty}f_n=f$. □

The above theorems also hold for the upper-closures and the normal subspaces with respect to a strict pseudo-addition. Furthermore, we can get some deeper results for the functional space based on a strict pseudo-addition.

Theorem 3.8: Let $\oplus$ be a strict pseudo-addition. Then $(\mathscr{E}(\mathscr{S}),\oplus)$ is a normal subspace of $(\mathscr{F}(X),\oplus)$.

Proof. For any $\lambda,\mu\in[a,b]$ and $f,h\in\mathscr{E}(X)$,

$$((\lambda\odot f)\oplus(\mu\odot h))^{-1}(\nu)=\bigcup_{(\lambda\odot\delta)\oplus(\mu\odot\omega)=\nu}\left(f^{-1}(\delta)\cap h^{-1}(\omega)\right),$$

where

$$\nu\in A=\{(\lambda\odot\delta)\oplus(\mu\odot\omega)\mid\delta\in f(X),\ \omega\in h(X)\}$$

and A is a finite subset of $[a,b]$ whenever both $f(X)$ and $h(X)$ are. Thus

$$((\lambda\odot f)\oplus(\mu\odot h))^{-1}(\nu)\in\mathscr{S}$$

for each $\nu\in A$. By (1) of Theorem 3.7, we have $(\lambda\odot\delta)\oplus(\mu\odot\omega)\in\mathscr{E}(X)$, then $(\lambda\odot\delta)\oplus(\mu\odot\omega)\in\mathscr{E}(\mathscr{S})$. Thus $\mathscr{E}(\mathscr{S})$ is a subspace. Similarly, we can get that $(f\vee_{\oplus}h)\in\mathscr{E}(\mathscr{S})$ for all $f,h\in\mathscr{E}(\mathscr{S})$.

Moreover, for any $f,h\in\mathscr{E}(\mathscr{S})$, $(|f-'_{\oplus}h|)(X)$ is a subset of the finite set

$$\{|\nu-'_{\oplus}\omega||\ \nu\in f(X),\ \omega\in h(X)\}.$$

Hence, we get that

$$(|f-'_{\oplus}h|)^{-1}(\nu)=\bigcup_{[\delta,\omega]-'_{\oplus}=\nu}(f^{-1}(\delta)\cap h^{-1}(\omega)),$$

where $\nu\in(|f-'_{\oplus}h|)(X)$. Thus $(|f-'_{\oplus}h|)^{-1}(\nu)\in\mathscr{S}$ for each $\nu\in(|f-'_{\oplus}h|)(X)$. By (1) of Theorem 3.7, we have $|f-'_{\oplus}h|\in\mathscr{E}(X)$, then $|f-'_{\oplus}h|\in\mathscr{E}(\mathscr{S})$. Thus $\mathscr{E}(\mathscr{S})$ is a complemented. Finally, it is clear that $\mathbf{1}\in\mathscr{E}(\mathscr{S})$. □

Theorem 3.9: Let $\oplus$ be a strict pseudo-addition. Then the functional I_A is in $(\mathscr{E}(\mathscr{S}),\oplus)$ for all $A\in\mathscr{S}$, and any subspace $(\mathcal{K},\oplus)$ with $I_A\in\mathcal{K}$ for each $A\in\mathscr{S}$ contains $(\mathscr{E}(\mathscr{S}),\oplus)$. Therefore $(\mathscr{E}(\mathscr{S}),\oplus)$ is the smallest subspace containing the functionals I_A, $A\in\mathscr{S}$.

Proof. If $A\in\mathscr{S}$, then $X-A\in\mathscr{S}$, since $\mathscr{S}$ is an algebra of subsets of X. Hence, we have ${I_A}^{-1}(\mathbf{0})=X-A\in\mathscr{S}$ and ${I_A}^{-1}(\mathbf{1})=A\in\mathscr{S}$. Thus the functional I_A is in $(\mathscr{E}(\mathscr{S}),\oplus)$ for all $A\in\mathscr{S}$, since $I_A\in\mathscr{E}(X)$.

If $f\in\mathscr{E}(\mathscr{S})$, then for any $\lambda\in f(X)$ and $x\in X$, we have

$$\left(\lambda\odot I_{f^{-1}(\lambda)}\right)(x)=\begin{cases}\lambda=f(x), & x\in f^{-1}(\lambda),\\ \mathbf{0}, & x\notin f^{-1}(\lambda),\end{cases}$$

where $f^{-1}(\lambda)=\{x\in X|f(x)=\lambda\}\in\mathscr{S}$. Thus, we have $f=\underset{\lambda\in f(X)}{\oplus}\left(\lambda\odot I_{f^{-1}(\lambda)}\right)$. Since $I_A\in\mathcal{K}$ for each $A\in\mathscr{S}$, we obtain that $f\in\mathcal{K}$. Then $\mathscr{E}(\mathscr{S})\subseteq\mathcal{K}$. □

Theorem 3.10: Let $\mathscr{A}$ be σ-algebra of subsets of a set X and let $\oplus$ be a strict pseudo-addition. Then $(\mathcal{M}(\mathscr{A}),\oplus)=(\hat{\mathscr{E}(\mathscr{A})},\oplus)$.

Proof. Let $f\in\mathcal{M}(\mathscr{A})$ and for each $n\geq 1$ define $f_n\in\mathscr{E}(\mathscr{A})$ by

$$f_n=\overset{2^n}{\underset{k=1}{\oplus}}\left(\lambda_{k-1,n}\odot I_{A_{k,n}}\right),$$

where $\mathbf{0}=\lambda_{0,n}\preceq\lambda_{1,n}\preceq\cdots\preceq\lambda_{2^n,n}=\Delta$, $\lambda_{2i,n+1}=\lambda_{i,n}(0\leq i\leq 2^n)$, and $A_{k,n}=\{x\in X|\lambda_{k-1,n}\prec f(x)\preceq\lambda_{k,n}\}\in\mathscr{A}$ for all $n\geq 1,\ 1\leq k\leq 2^n$. Then $\{f_n\}_{n\geq 1}$ is an increasing sequence from $\mathscr{E}(X)$ with $\lim\limits_{n\to\infty}f_n=f$, and therefore $f_n\in\mathscr{E}(\mathscr{A})$. □

Theorem 3.11: Let $\mathscr{A}$ be σ-algebra of subsets of a set X and let $\oplus$ be a strict pseudo-addition. Then $(\mathcal{M}(\mathscr{A}),\oplus)$ are both upper-complete and lower-complete.

Proof. By Theorem 3.1 and Theorem 3.8, we have $(\mathcal{M}(\mathscr{A}),\oplus)$ is upper-complete, since $(\mathcal{M}(\mathscr{A}),\oplus)=(\hat{\mathscr{E}(\mathscr{A})},\oplus)$. Let $\{f_n\}_{n\geq 1}$ be a decreasing sequence from $\mathcal{M}(\mathscr{A})$ with $\lim\limits_{n\to\infty}f_n=f$. Then for all $\lambda\in[a,b]$,

$$\{x\in X|\ f(x)\preceq\lambda\}=\bigcup_{n\geq 1}\{x\in X|\ f_n(x)\preceq\lambda\}$$

is an element of $\mathscr{A}$ and therefore again $f \in \mathcal{M}(\mathscr{A})$. □

Theorem 3.12: Let $\mathscr{A}$ be σ-algebra of subsets of a set X and let $\oplus$ be a strict pseudo-addition. Then $(\mathcal{M}(\mathscr{A}), \oplus)$ is an upper-complete normal subspace of $(\mathscr{F}(X), \oplus)$.

Proof. By Theorem 3.8, $\mathscr{E}(\mathscr{A})$ is a normal subspace of $\mathscr{F}(X)$, and by Theorem 3.10, $\mathcal{M}(\mathscr{A}) = \hat{\mathscr{E}(\mathscr{A})}$. Moreover, by Theorem 3.11, $\mathcal{M}(\mathscr{A})$ is both upper-complete and lower-complete. Therefore by Corollary 3.1, $\mathcal{M}(\mathscr{A})$ is an upper-complete normal subspace of $\mathscr{F}(X)$. □

Theorem 3.13: Let $\mathscr{A}$ be σ-algebra of subsets of a set X and let $\oplus$ be a strict pseudo-addition and $(\mathcal{K}, \oplus)$ be an upper-complete subspace of $(\mathcal{M}(\mathscr{A}), \oplus)$ with $I_A \in \mathcal{K}$ for each $A \in \mathscr{A}$. Then $(\mathcal{K}, \oplus) = (\mathcal{M}(\mathscr{A}), \oplus)$.

Proof. By Theorem 3.9, $\mathscr{E}(\mathscr{A}) = \mathcal{K}$, since $I_A \in \mathcal{K}$ for each $A \in \mathscr{A}$, and by Theorem 3.10, $\mathcal{M}(\mathscr{A}) = \hat{\mathscr{E}(\mathscr{A})}$, i.e., $\mathcal{M}(\mathscr{A}) \subseteq \mathcal{K}$, since $\mathcal{K}$ is upper-complete. Hence, we obtain that $(\mathcal{K}, \oplus) = (\mathcal{M}(\mathscr{A}), \oplus)$, since $(\mathcal{K}, \oplus)$ be an subspace of $(\mathcal{M}(\mathscr{A}), \oplus)$. □

Theorem 3.14: Let $\oplus$ be a strict pseudo-addition and $(\mathcal{K}, \oplus)$ be an upper-complete normal subspace of $(\mathscr{F}(X), \oplus)$. Then

$$\mathscr{A} = \{A \subseteq X | \ I_A \in \mathcal{K}\}$$

is a σ-algebra and $(\mathcal{K}, \oplus) = (\mathcal{M}(\mathscr{A}), \oplus)$.

Proof. If $A \in \mathscr{A}$, then $I_A \in \mathcal{K}$ and I_{X-A} is the element of $\mathscr{F}(X)$ with $\mathbf{1} = I_A \oplus I_{X-A}$. Thus $I_{X-A} = \mathbf{1} -'_{\oplus} I_A \in \mathcal{K}$, since $\mathcal{K}$ is complemented, i.e., $X - A \in \mathscr{A}$ for all $A \in \mathscr{A}$. Let $A, B \in \mathscr{A}$. Then $I_{A \cup B} = I_A \vee_{\oplus} I_B \in \mathcal{K}$, since $\mathcal{K}$ is $\vee_{\oplus}$-closed. Hence, we have $A \cup B \in \mathscr{A}$. This shows $\mathscr{A}$ is an algebra, since $I_X = \mathbf{1} \in \mathcal{K}$.

Now let $\{A_n\}_{n \geq 1}$ be an increasing sequence from $\mathscr{A}$ and let $A = \bigcup_{n \geq 1} A_n$. Then $\{I_{A_n}\}_{n \geq 1}$ is an increasing sequence from $\mathcal{K}$ with $\lim_{n \to \infty} I_{A_n} = I_A$ and thus $I_A \in \mathcal{K}$. This shows $A \in \mathscr{A}$ and therefore $\mathscr{A}$ is a σ-algebra.

Let $f \in \mathcal{K}$ and $\lambda \in [a, b]$ with $\lambda \prec \Delta$. Define the functional as

$$h(x) = \begin{cases} |f(x) -'_{\oplus} \lambda|, & if \ x \in f^{-1}((\lambda, \Delta]), \\ \mathbf{0}, & otherwise. \end{cases}$$

Then $h \in \mathcal{K}$, since $\mathcal{K}$ is complemented, $\wedge_{\odot}$-closed if and only if it is $\vee_{\oplus}$-closed and $\lambda \in \mathcal{K}$. For each $n \geq 1$, let $f_n = \lambda_n * h$, where $\lambda_1 \preceq \lambda_2 \preceq \cdots \lambda_n \preceq \cdots$, $\mathbf{0} \preceq \lambda_n \preceq \Delta$ for all $n \geq 1$, $\lim_{n \to \infty} \lambda_n = \Delta$ and for any $\mathbf{0} \preceq \lambda \preceq \Delta$, the binary operation $*$ is defined as

$$(\lambda * h)(x) = \begin{cases} \lambda \wedge_{\odot} h(x), & if \ \ \mathbf{0} \prec \lambda \wedge_{\odot} h(x) \prec \mathbf{1}, \\ \mathbf{0}, & h(x) = \mathbf{0} \ or \ \lambda = \mathbf{0}, \\ \mathbf{1}, & otherwise. \end{cases}$$

Then $\{f_n\}_{n \geq 1}$ is an increasing sequence from $\mathcal{K}$ with $\lim_{n \to \infty} f_n = I_{f^{-1}((\lambda, \Delta])}$. Thus $I_{f^{-1}((\lambda, \Delta])} \in \mathcal{K}$, since $\mathcal{K}$ is upper-complete, i.e., $I_{f^{-1}((\lambda, \Delta])} \in \mathscr{A}$ which implies that $f \in \mathcal{M}(\mathscr{A})$. Thus, we get that $\mathcal{K} \subseteq \mathcal{M}(\mathscr{A})$. Hence, by Theorem 3.13, we obtain that $\mathcal{K} = \mathcal{M}(\mathscr{A})$, since $I_A \in \mathcal{K}$ for all $A \in \mathscr{A}$. □

By Theorem 3.12 $\mathcal{M}(\mathscr{A})$ is an upper-complete normal subspace of $\mathscr{F}(X)$ and it is clear that $\mathcal{K} = \{A \subseteq X | \ I_A \in \mathcal{M}(\mathscr{A})\}$. Thus the following result holds.

Theorem 3.15: Let $\oplus$ be a strict pseudo-addition. Then $(\mathcal{K}, \oplus)$ is a upper-complete normal subspace if and only if

$$\mathscr{A} = \{A \subseteq X | \ I_A \in \mathcal{K}\}$$

is a σ-algebra and $(\mathcal{K}, \oplus) = (\mathcal{M}(\mathscr{A}), \oplus)$.

IV. Conclusions

In this paper, we have discussed the some properties of the upper-closures of the regular subspaces of the nonlinear functional space $(\mathscr{F}(X), \oplus)$ based on a continuous pseudo-addition. Furthermore, we have obtained that with respect to a strict pseudo-addition, a subset $\mathcal{K}$ of $(\mathscr{F}(X), \oplus)$ is an upper-complete normal subspace if and only if the family of all sets whose characteristic functional spaces based on pseudo-additions. In addition, because the concepts of pseudo-addition-decomposable measures and pseudo-addition-decomposable integrals [2], [3], [4], [15], [17], [18], [20], [21], [22], [26], [31] are very useful in the theory of nonlinear differential and integral equations [14], [24], [28], [29], [32], the relationships between nonlinear functional spaces based on pseudo-additions and those concepts will also be explored in our future research.

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