Lie Symmetry Analysis and Exact Solutions of the Sharma-Tasso-Olever Equation

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Abstract—**In present paper, the Sharma-Tasso-Olever (STO) equation is considered by the Lie symmetry analysis. All of the geometric vector fields to the STO equation are obtained, then the symmetry reductions and exact solutions of the STO equation are investigated. Our results witness that symmetry analysis was very efficient and powerful technique in finding the solutions of the proposed equation.**

Index Terms—**STO equation, Lie symmetry analysis, power series solution, Hyperbolic function solution, rational function solution.**

I. Introduction

RECENTLY, mathematics and physics field have devoted considerable effort to the study of solutions to partial differential equations (PDEs). Among many powerful methods for solving the equation, Lie symmetry analysis provides an effective procedure for integrability, conservation laws, reducing equation and exact solutions of a wide and general class of differential systems representing real physical problems [1], [2]. In [3], Sinkala et al performed the group classification of a bond-pricing PDE of mathematical finance to discover the combinations of arbitrary parameters that allow the PDE to admit a nontrivial symmetry Lie algebra, and computed the admitted Lie point symmetries, identify the corresponding symmetry Lie algebra and solve the PDE. Under the condition of the symmetry group of the PDE is nontrivial, it contains a standard integral transform of the fundamental solutions for PDEs, and fundamental solutions can be reduced to inverting a Laplace transform or some other classical transform in [4]. In [5], by using the direct construction method, all of the first-order multipliers of the generalized nonlinear second-order equation are obtained, and the corresponding complete conservation laws of such equation are provided. Furthermore, Lie symmetry analysis helps to study their group theoretical properties, and effectively assists to derive several mathematical characteristics related with their complete integrability [6]. Also, Lie symmetry analysis and dynamical system method is a feasible approach to dealing with exact explicit solutions to nonlinear PDEs and systems, (see, e.g., [7]–[12]). Liu et al derived the symmetries, bifurcations and exact explicit solutions to the KdV equation by using Lie symmetry analysis and the dynamical system method [13]. The STO equation is a KdV-like equation and has been applied to describe a wide range of physics phenomena of the evolution and interaction to nonlinear waves, such as fluid dynamics, aerodynamics, continuum mechanics, solitons and turbulence et al, it possesses an infinitely many symmetries and the bi-Hamiltonian formulation. Note that the exact solutions for it with different forms can describe different nonlinear waves. In present paper, we will investigate the vector fields, symmetry reductions and exact solutions to the STO equation [14]–[19]

$$u_t + 3au_x^2 + 3au^2u_x + 3auu_{xx} + au_{xxx} = 0, \qquad (1)$$

where $a \neq 0$ is a constant, $u = u(x,t)$ is a field variable, the subscripts denote the partial differentiation of the function u with respect to the parameter x or t, x is the spatial coordinate in the propagation direction and t is the temporal coordinates, which occur in different contexts in mathematical physics. The dissipative u_{xxx} term provides damping at small scales, and the non-linear term u^2u_x stabilizes by transferring energy between large and small scales.

The rest of this paper is organized as follows: In Section II, the vector fields of Eq. (1) are presented by using Lie symmetry analysis method. Based on the optimal dynamical system method, all the similarity reductions to Eq. (1) are obtained. In Section III, the exact analytic solutions to the equation are investigated by means of the power series method et al. respectively. Finally, the conclusions will be given in Section IV.

II. Lie symmetry analysis and similarity reductions

Recall that the geometric vector field of a PDE equation is as follows

$$V = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \eta(x,t,u)\partial_u, \qquad (2)$$

where the coefficient functions $\xi(x,t,u)$, $\tau(x,t,u)$, $\eta(x,t,u)$ of the vector field are to be determined later.

If the vector field (2) generates a symmetry of the equation (1), then V must satisfy the Lie symmetry condition

$$\mathrm{Pr}V(\Delta)|_{\Delta=0} = 0,$$

where $\mathrm{Pr}V$ denotes the 3-th prolongation of V, and $\Delta = u_t + 3au_x^2 + 3au^2u_x + 3auu_{xx} + au_{xxx}$. Moreover, the prolongation $\mathrm{Pr}V$ depends on the equation

$$\mathrm{Pr}V = \eta\partial_u + \eta^x\partial_{u_x} + \eta^{2x}\partial_{u_{2x}} + \eta^{3x}\partial_{u_{3x}},$$

where the coefficient functions $\eta^{kx}(k = 1,2,3)$ are given as

$$\eta^{kx} = D_x^k(\eta - \tau u_t - \xi u_x) + \tau u_{kxt} + \xi u_{(k+1)x}, \quad k = 1,2,3,$$

here symbol D_x denotes the total differentiation operator and is defined as

$$D_x = \partial_x + u_x\partial_u + u_{tx}\partial_{u_t} + u_{2x}\partial_{u_x} + \dots.$$

Manuscript received May 3, 2015; revised June 25, 2015

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Then in terms of the Lie symmetry analysis method, we obtain that all of the geometric vector fields of Eq. (1) are as follows

$$V_1 = x\partial_x + 3t\partial_t - u\partial_u, \quad V_2 = \partial_x, \quad V_3 = \partial_t.$$

Moreover, it is necessary to show that the vector fields of Eq. (1) are closed under the Lie bracket, we have

$$\begin{aligned}&[V_1, V_1] = [V_2, V_2] = [V_3, V_3] = 0,\\ &[V_1, V_2] = -[V_2, V_1] = V_2,\\ &[V_1, V_3] = -[V_3, V_1] = 3V_3,\\ &[V_2, V_3] = -[V_3, V_2] = 0.\end{aligned}$$

Based on the adjoint representations of the vector fields, we obtain the optimal systems of the four STO equation as follows

$$\{V_1, V_2, V_3, V_3 + vV_2\},$$

where $v \neq 0$ is constant.

In the preceding section, we obtained the vector fields and the optimal systems of Eq. (1). Now we deal with the symmetry reductions and exact solutions to the equation, to consider the similarity reductions and group-invariant solutions based on the optimal dynamical system method. From an optimal system of group-invariant solutions to an equation, every other such solutions to the equation can be derived.

For the generator V_1, it yields

$$u = t^{-\frac{1}{3}} f(z), \tag{3}$$

where $z = xt^{-\frac{1}{3}}$. Substituting (3) into Eq. (1), we reduce it into the following ODE

$$-\frac{1}{3}f - \frac{1}{3}zf' + 3af'^2 + 3af^2f' + 3aff'' + af''' = 0, \tag{4}$$

where $f' = \frac{df}{dz}$.

For the generator V_2, we get the trivial solution to Eq. (1) is $u(x,t) = c$, where c is an arbitrary constant.

For the generator V_3, we have

$$u = f(z), \tag{5}$$

where $z = x$. Substituting (5) into Eq. (1), we obtain the following ODE

$$3f'^2 + 3f^2f' + 3ff'' + f''' = 0, \tag{6}$$

where $f' = \frac{df}{dz}$.

For the generator $V_3 + vV_2$, we have

$$u = f(z), \tag{7}$$

where $z = x - vt$, $v > 0$ is regarded as the wave velocity. Substituting (7) into Eq. (1), we have

$$-vf' + 3af'^2 + 3af^2f' + 3aff'' + af''' = 0, \tag{8}$$

where $f' = \frac{df}{dz}$.

III. Exact solutions

By seeking for exact solutions of the PDEs, we mean those that can be obtained from some ODEs or, in general, from PDEs of lower order than the original PDE [20]–[22]. In terms of this definition, the exact solutions to Eq. (1) are obtained actually in both of the preceding Sections II. In spite of this, we still want to detect the explicit solutions expressed in terms of elementary or, at least, known functions of mathematical physics, in terms of quadratures, and so on.

3.1 Exact power series solution to Eq. (4)

We know that the power series can be used to solve differential equation, including many complicated differential equations [23], [24], and so we consider the exact analytic solutions to the reduced equation by using power series method. Once we get the exact analytic solutions of the reduced ODEs, the exact power series solutions to Eq. (1) are obtained. In view of (4), we seek a solution in a power series of the form

$$f(z) = \sum_{n=0}^{\infty} c_n z^n. \tag{9}$$

Substituting (9) into (4), and comparing coefficients, then we obtain the following recursion formula:

$$\begin{aligned} c_{n+3} = &-\frac{1}{(n+1)(n+2)(n+3)}\Bigg(-\frac{c_n}{3a} - \frac{nc_n}{3a}\\ &+ 3\sum_{k=0}^{n}(n+1-k)(k+1)c_{k+1}c_{n+1-k}\\ &+ 3\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)c_i c_{k-i} c_{n+1-k}\\ &+ 3\sum_{k=0}^{n}(n+1-k)(n+2-k)c_k c_{n+2-k}\Bigg), \end{aligned} \tag{10}$$

for all $n = 0, 1, 2, \ldots$.

Thus, for arbitrarily chosen constants c_i $(i = 0, 1, 2)$, we obtain

$$c_3 = -c_0c_2 - \frac{c_0^2 c_1}{2} - \frac{c_1^2}{2} + \frac{c_0}{18a}.$$

Furthermore, (10) yield

$$c_4 = -\frac{1}{12}\big(9c_0c_3 + 9c_1c_2 + 3c_0c_1^2 + 3c_0^2c_2 - \frac{c_1}{3a}\big), \tag{11}$$

$$\begin{aligned} c_5 = &-\frac{1}{60}\big(36c_0c_4 + 36c_1c_3 + 18c_2^2 + 9c_0^2c_3\\ &+ 18c_0c_1c_2 + 3c_1^3 - \frac{c_2}{a}\big), \end{aligned} \tag{12}$$

and so on.

Thus for arbitrary chosen constant numbers c_i $(i = 0, 1, 2)$, the other terms of the sequence $\{c_n\}_{n=0}^{\infty}$ can be determined successively from (11) and (12) in a unique manner. This implies that for Eq. (4), there exists a power series solution (9) with the coefficients given by (11) and (12). Furthermore, it is easy to prove the convergence of the power series (9) with the coefficients given by (11) and (12). Therefore, this power series solution (9) to Eq. (4) is an exact analytical solution.

Hence, the power series solution of Eq. (4) can be written as

$$\begin{aligned}f(z) &= c_0 + c_1 z + c_2 z^2 + \sum_{n=0}^{\infty} c_{n+3} z^{n+3}\\ &= c_0 + c_1 z + c_2 z^2\\ &\quad + \big(-c_0c_2 - \frac{c_0^2c_1}{2} - \frac{c_1^2}{2} + \frac{c_0}{18a}\big)z^3\\ &\quad - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)}\Big(-\frac{c_n}{3a} - \frac{nc_n}{3a}\\ &\quad + 3\sum_{k=0}^{n}(n+1-k)(k+1)c_{k+1}c_{n+1-k}\\ &\quad + 3\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)c_ic_{k-i}c_{n+1-k}\\ &\quad + 3\sum_{k=0}^{n}(n+1-k)(n+2-k)c_kc_{n+2-k}\Big)z^{n+3}.\end{aligned}$$

Thus, the exact power series solution of Eq. (1) is

$$\begin{aligned}u(x,t) &= c_0 t^{-\frac{1}{3}} + c_1 x t^{-\frac{2}{3}} + c_2 x^2 t^{-1}\\ &\quad + \big(-c_0c_2 - \frac{c_0^2c_1}{2} - \frac{c_1^2}{2} + \frac{c_0}{18a}\big)x^3 t^{-\frac{4}{3}}\\ &\quad - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)}\Big(-\frac{c_n}{3a} - \frac{nc_n}{3a}\\ &\quad + 3\sum_{k=0}^{n}(n+1-k)(k+1)c_{k+1}c_{n+1-k}\\ &\quad + 3\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)c_ic_{k-i}c_{n+1-k}\\ &\quad + 3\sum_{k=0}^{n}(n+1-k)(n+2-k)c_kc_{n+2-k}\Big)\\ &\quad \times x^{n+3} t^{-\frac{n+4}{3}}.\end{aligned} \tag{13}$$

In mathematical and physical applications, it will be convenient to write the solution of Eq. (1) in the approximate form

$$\begin{aligned}u(x,t) &= c_0 t^{-\frac{1}{3}} + c_1 x t^{-\frac{2}{3}} + c_2 x^2 t^{-1}\\ &\quad + \big(-c_0c_2 - \frac{c_0^2c_1}{2} - \frac{c_1^2}{2} + \frac{c_0}{18a}\big)x^3 t^{-\frac{4}{3}} - \frac{1}{12}\\ &\quad \times \big(9c_0c_3 + 9c_1c_2 + 3c_0c_1^2 + 3c_0^2c_2 - \frac{c_1}{3a}\big)x^4 t^{-\frac{5}{3}}\\ &\quad - \frac{1}{60}\big(36c_0c_4 + 36c_1c_3 + 18c_2^2 + 9c_0^2c_3\\ &\quad + 18c_0c_1c_2 + 3c_1^3 - \frac{c_2}{a}\big)x^5 t^{-2} + \cdots.\end{aligned} \tag{14}$$

Moreover, we can show that the convergence of the power series solution (9) to Eq. (1). In fact, from (10), we have

$$\begin{aligned}|c_{n+3}| &\le M\Big(2|c_n| + \sum_{k=0}^{n}|c_{k+1}||c_{n+1-k}|\\ &\quad + \sum_{k=0}^{n}\sum_{i=0}^{k}|c_i||c_{k-i}||c_{n+1-k}| + \sum_{k=0}^{n}|c_k||c_{n+2-k}|\Big),\end{aligned}$$

where $M = \max\{\frac{1}{3|a|}, 3\}$. We introduce a power series $A(z) = \sum_{n=0}^{\infty} a_n z^n$, set

$$a_i = |c_i|, \quad i = 0, 1, 2,$$

and

$$\begin{aligned}a_{n+3} &:= M\Big(2a_n + \sum_{k=0}^{n} a_{k+1}a_{n+1-k}\\ &\quad + \sum_{k=0}^{n}\sum_{i=0}^{k} a_ia_{k-i}a_{n+1-k} + \sum_{k=0}^{n} a_ka_{n+2-k}\Big),\\ &\quad n = 0, 1, \ldots.\end{aligned}$$

It is easy to see that

$$|c_n| \le a_n, \quad n = 0, 1, \ldots.$$

In other words, the series $A(z) = \sum_{n=0}^{\infty} a_n z^n$ is majorant series of (9). Further, we show that the series $A(z)$ has positive radius of convergence. Indeed, note that by formal calculation, it yields

$$\begin{aligned}A(z) &= a_0 + a_1 z + a_2 z^2 + M\Big(2\sum_{n=0}^{\infty} a_n z^{n+3}\\ &\quad + \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{k+1}a_{n+1-k}z^{n+3}\\ &\quad + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k} a_ia_{k-i}a_{n+1-k}z^{n+3}\\ &\quad + \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_ka_{n+2-k}z^{n+3}\Big)\\ &= a_0 + a_1 z + a_2 z^2 + M\big(z^2A^3(z) + (2z - a_0z^2)A^2(z)\\ &\quad + (2z^3 - 3a_0z - a_1z^2)A(z) + a_0^2 z\big).\end{aligned}$$

Consider now the implicit functional system with respect to the independent variable z

$$\begin{aligned}\mathcal{A}(z, A) &= A - a_0 - a_1 z - a_2 z^2 - M\big(z^2A^3(z)\\ &\quad + (2z - a_0z^2)A^2(z) + (2z^3 - 3a_0z - a_1z^2)A(z)\\ &\quad + a_0^2 z\big).\end{aligned}$$

Obviously, there exists a $\delta > 0$, the function $\mathcal{A}$ is analytic in the neighborhood $U((0, a_0); \delta)$ of the point $(0, a_0)$, and $\mathcal{A}(0, a_0) = 0$. Furthermore,

$$\frac{\partial}{\partial A}\mathcal{A}(0, a_0) \ne 0.$$

By the implicit function theorem [25], we see that $A(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in a neighborhood of the point 0 and with the positive radius. This implies that the power series (9) converge in a neighborhood $U(0; \delta)$.

We would like to reiterate that the power series solutions which have been obtained in this section are exact analytic solutions to the equation. Moreover, we can see that these power series solutions converge for the chosen constants c_i $(i = 0, 1, 2)$ of (14), it is actual value for mathematical and physical applications.

3.2 Exact rational function solution to Eq. (6)

Integrating Eq. (6), we have

$$f'' + 3ff' + f^3 = 0, \tag{15}$$

taking the integration constant zero. Introduce the transformation

$$f(z) = (\ln \phi(z))'. \tag{16}$$

Substituting (16) into (15), we obtain the following linear equation

$$\phi''' = 0.$$

Solve this equation, we have

$$\phi(z) = \frac{1}{2}c_1 z^2 + c_2 z + c_3, \quad c_1^2 + c_2^2 + c_3^2 \neq 0. \tag{17}$$

Substituting (17) into (16), then the exact rational function solution to Eq. (1) is obtained as

$$u(x,t) = \frac{c_1 x + c_2}{\frac{1}{2}c_1 x^2 + c_2 x + c_3}.$$

3.3 Exact travelling wave solutions to Eq. (8)

Consider the exact analytic solutions to the reduced equation by using trial equation method, the exact travelling wave solutions to Eq. (1) are obtained. By integrating Eq. (8), keeping the integration constant zero, it yields

$$af'' + 3aff' + af^3 - vf = 0. \tag{18}$$

We assume a solution of Eq. (18) in the form [26]

$$f(z) = \frac{\lambda \sinh z}{\alpha + \beta \cosh z}. \tag{19}$$

Substituting for u, u' and u'' in Eq. (18), we obtain an algebraic equation in powers of $\cosh z$ given as

$$\begin{aligned} &- v(\alpha + \beta \cosh z)^2 + a\lambda^2(\cosh^2 z - 1) \\ &\quad + 3a(\lambda\alpha \cosh z + \lambda\beta) - a(2\beta^2 - \alpha^2 + \alpha\beta \cosh z) = 0. \end{aligned}$$

Equating the coefficients of different powers of $\cosh z$ to zero, it yields

$$-v\alpha^2 - a\lambda^2 + 3a\lambda\beta - 2a\beta^2 + a\alpha^2 = 0, \tag{20}$$

$$-2v\alpha\beta + 3a\lambda\alpha - a\alpha\beta = 0, \tag{21}$$

$$-v\beta^2 + a\lambda^2 = 0. \tag{22}$$

In view of Eqs. (21) and (22), we can obtain a constraint relation

$$9av = (a + 2v)^2. \tag{23}$$

From Eq. (22) and (23), Eq. (20) reduces into

$$a\lambda^2 = v\alpha^2. \tag{24}$$

In view of (24), the wave velocity $v > 0$ yields $a > 0$. By Eqs. (22) and (24) one can conclude that $\alpha = \beta$. Thus the solution (19) can be read as

$$f(z) = \pm\sqrt{\frac{v}{a}}\frac{\sinh z}{1 + \cosh z}. \tag{25}$$

Therefore the solution of Eq. (1) can be given as

$$u(x,t) = \pm\sqrt{\frac{v}{a}}\frac{\sinh(x - vt)}{1 + \cosh(x - vt)}, \tag{26}$$

which is a kink-wave [27].

Now we also assume a solution of Eq. (18) in the form

$$f(z) = \frac{\lambda \cosh z}{\alpha + \beta \sinh z}.$$

Substituting for u, u' and u'' in Eq. (18), we obtain an algebraic equation in powers of $\sinh z$ given as

$$\begin{aligned} &- v(\alpha + \beta \sinh z)^2 + a\lambda^2(1 + \sinh^2 z) \\ &\quad + 3a(\lambda\alpha \sinh z - \lambda\beta) + a(2\beta^2 + \alpha^2 - \alpha\beta \cosh z) = 0. \end{aligned}$$

Equating the coefficients of different powers of $\cosh z$ to zero, it yields

$$- v\alpha^2 + a\lambda^2 - 3a\lambda\beta + 2a\beta^2 + a\alpha^2 = 0, \tag{27}$$

$$- 2v\alpha\beta + 3a\lambda\alpha - a\alpha\beta = 0, \tag{28}$$

$$- v\beta^2 + a\lambda^2 = 0. \tag{29}$$

In view of Eqs. (28) and (29), we can obtain a constraint relation

$$9av = (a + 2v)^2. \tag{30}$$

Note that $v > 0$ yields $a > 0$. From Eq. (29) and (30), Eq. (27) reduces into

$$a\lambda^2 = -v\alpha^2. \tag{31}$$

By Eqs. (29) and (31) one can conclude that $\beta = \pm i\alpha$. Thus the solution (19) can be read as

$$f(z) = \pm\sqrt{\frac{v}{a}}\frac{i \cosh z}{1 \pm i \sinh z}. \tag{32}$$

Therefore the solution of Eq. (1) can be given as

$$u(x,t) = \pm\sqrt{\frac{v}{a}}\frac{i \cosh(x - vt)}{1 \pm \sinh(x - vt)}. \tag{33}$$

Eqs. (25), (26), (32) and (33) imply

$$av > 0.$$

This shows that the coefficient of the first nonlinear term and the speed of the wave must carry the same sign.

Remark 1: Similar to the solving Eq. (15), we can also employ the transformation (16) to Eq. (18), we can obtain the following linear equation

$$a\phi''' - v\phi' = 0.$$

Solve this equation, we have

$$\begin{aligned} &\phi_1(z) = c_1, \quad \phi_2(z) = \exp\big(\sqrt{\frac{v}{a}}z\big), \\ &\phi_3(z) = \exp\big(-\sqrt{\frac{v}{a}}z\big), \quad a > 0, v > 0, \end{aligned} \tag{34}$$

and

$$\begin{aligned} &\phi_4(z) = c_2, \quad \phi_5(z) = \exp\big(\sqrt{\frac{v}{-a}}iz\big), \\ &\phi_3(z) = \exp\big(-\sqrt{\frac{v}{-a}}zi\big), \quad a < 0, v > 0, \end{aligned} \tag{35}$$

where c_1, c_2 are arbitrary constants. Substituting (34) and (35) into (16), respectively, then the trivial solutions to Eq. (1) is obtained as

$$\begin{aligned} &u_1(x,t) = 0, \quad u_2(x,t) = \sqrt{\frac{v}{a}}, \quad u_3(x,t) = -\sqrt{\frac{v}{a}}, \\ &\quad a > 0, v > 0, \end{aligned}$$

and

$$u_4(x,t) = 0, \quad u_5(x,t) = \sqrt{\frac{v}{-a}}i, \quad u_6(x,t) = -\sqrt{\frac{v}{-a}}i,$$
$$a < 0, v > 0.$$

However, the solutions u_5 and u_6 yield the real analytical solution Eq. (1)

$$u_{51}(x,t) = -\sqrt{\frac{v}{-a}}\tan\big(\sqrt{\frac{v}{-a}}(x - vt)\big),$$
$$u_{61}(x,t) = \sqrt{\frac{v}{-a}}\cot\big(\sqrt{\frac{v}{-a}}(x - vt)\big), \quad a < 0, v > 0.$$

Remark 2: Under the conditions of the generator V_3 and $V_3 + vV_2$, we can also obtain the power series form solutions to Eqs. (6) and (8), respectively, the details are omitted.

IV. Summary and discussion

In this paper, we have obtained the symmetries and similarity reductions of the STO equation by using Lie symmetry analysis method. All the group-invariant solutions to STO equation (1) are considered based on the optimal system method, then the symmetry reductions and exact solutions of the STO equation are investigated. Being concise and powerful, we note that this approach can also be applied to solve other nonlinear PDEs.

Acknowledgements

The author would like to thank the anonymous reviewers very much for their valuable suggestions on improving this paper.

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