Convergence Analysis of Some New Preconditioned AOR Iterative Methods for L-matrices

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***Abstract*—In this paper, we present a new preconditioner which generalizes two known preconditioners proposed by Wang et al. (2009) and A. J. Li (2011), and prove that the convergence rate of the AOR method with the new preconditioner is faster than the preconditioners introduced by Wang et al. Moreover, we propose other two new preconditioners and study the convergence rates of the new preconditioned AOR methods for solving linear systems. Comparison results show that the new preconditioned AOR methods are better than those of the preconditioned AOR methods presented by J. H. Yun (2011) and A. J. Li (2012). Finally, numerical experiments are provided to confirm the theoretical results studied in this paper.**

***Index Terms*—preconditioner, preconditioned AOR method, Linear system, AOR method, L-matrices.**

I. Introduction

SOLUTIONS of linear systems arise from the scientific computing and engineering technique, such as solving the steady incompressible Navier-Stokes problem [1], preconditioning techniques for large sparse systems arising in finite element limit analysis [2], multigrid method for linear complementarity problem and its implementation on GPU [3], fourth-order singly diagonally implicit Runge-Kutta method for solving one-dimensional Burgers' Equation [4] and so forth. So, research of methods for solving linear systems has important theoretic significance and practical applications. In this paper, by constructing three new preconditioners and combining with the theories of nonnegative matrices, we will conduct further research in preconditioned AOR iterative methods for solving linear systems. In theory, we prove that the convergence rates of the AOR method with the new preconditioners are faster than the existing ones. To illustrate our results, some numerical experiments are given.

Consider the following linear system

$$Ax = b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ are given and $x \in \mathbb{R}^n$ is unknown.

For simplicity, we let $A = I - L - U$, where I is the identity matrix, L and U are strictly lower and strictly upper triangular matrices, respectively. Then the iteration matrix of the AOR iterative method [5] for solving the linear system (1) is

$$L_{r,w} = (I - rL)^{-1}[(1-w)I + (w-r)L + wU], \tag{2}$$

where w and r are real parameters with $w \neq 0$.

In order to accelerate the convergence of iterative method for solving the linear system (1), the original linear system (1) is transformed into the following preconditioned linear system

$$PAx = Pb, \tag{3}$$

where P is called a preconditioner, is a nonsingular matrix. The preconditioned system (3) with the different preconditioners P have been proposed by many authors [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21].

In 2009, Wang and Li [6] and in 2011, A. J. Li [7] presented preconditioners $\tilde{P} = I + \tilde{S}_{\alpha,\beta}$ and $\hat{P} = I + \hat{S}_{\alpha,\beta}$ for L-matrices, respectively, where

$$\tilde{S}_{\alpha,\beta} = (\tilde{s}_{ij})_{n\times n} = \begin{cases} -\frac{a_{n1}}{\alpha_1} - \beta_1, & i = n, j = 1; \\ 0, & \text{others.} \end{cases},$$

$$\hat{S}_{\alpha,\beta} = (\hat{s}_{ij})_{n\times n} = \begin{cases} -\frac{a_{1n}}{\alpha_2} - \beta_2, & i = 1, j = n; \\ 0, & \text{others.} \end{cases}.$$

In 2011, J. H. Yun [8] obtained a preconditioner $P_1 = I + S_1$ for L-matrices, where

$$S_1 = (s1_{ij})_{n\times n} = \begin{cases} -\alpha_i a_{i,i+1}, & i = 1, \cdots, n-1; \\ 0, & \text{others.} \end{cases}.$$

In 2012, A. J. Li [9] proposed another preconditioner $P_2 = I + S_2$ for L-matrices, where

$$S_2 = (s2_{ij})_{n\times n} = \begin{cases} -\beta_{i+1} a_{i+1,i}, & i = 1, \cdots, n-1; \\ 0, & \text{others.} \end{cases}.$$

Let

$$\tilde{A}x = \tilde{b}, \quad \hat{A}x = \hat{b}, \ A_1 x = b_1, \ A_2 x = b_2, \tag{4}$$

where $\tilde{A} = (I + \tilde{S}_{\alpha,\beta})A$, $\tilde{b} = (I + \tilde{S}_{\alpha,\beta})b$, $\hat{A} = (I + \hat{S}_{\alpha,\beta})A$, $\hat{b} = (I + \hat{S}_{\alpha,\beta})b$, $A_1 = (I + P_1)A$, $b_1 = (I + P_1)b$, $A_2 = (I + P_2)A$, $b_2 = (I + P_2)b$. Let

$$\begin{aligned} &\tilde{A} = \tilde{D} - \tilde{L} - \tilde{U}, \ \hat{A} = \hat{D} - \hat{L} - \hat{U}, \\ &A_i = D_i - L_i - U_i (i = 1, 2), \end{aligned} \tag{5}$$

where $\tilde{D}$, $-\tilde{L}$, $-\tilde{U}$ and $\hat{D}$, $-\hat{L}$, $-\hat{U}$ are diagonal, strictly lower and strictly upper triangular matrices of $\tilde{A}$ and $\hat{A}$, respectively, moreover, D_i, $-L_i$, $-U_i$ $(i = 1, 2)$ are diagonal, strictly lower and strictly upper triangular matrices of A_i $(i = 1, 2)$. $\rho(A)$ denotes the spectral radius of A.

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Applying the AOR method to the preconditioned linear systems (4), we get the corresponding preconditioned AOR (PAOR) iterative methods and their iterative matrices are

$$\tilde{L}_{r,w} = (\tilde{D} - r\tilde{L})^{-1}[(1-w)\tilde{D} + (w-r)\tilde{L} + w\tilde{U}], \quad (6)$$
$$L_{1,r,w} = (D_1 - rL_1)^{-1}[(1-w)D_1 + (w-r)L_1 + wU_1], \quad (7)$$
$$L_{2,r,w} = (D_2 - rL_2)^{-1}[(1-w)D_2 + (w-r)L_2 + wU_2], \quad (8)$$

respectively, where w and r are real parameters with $w \neq 0$.

In this paper, we continue this research on the preconditioned AOR method for linear system and consider the new preconditioners $\bar{P} = I + \bar{S}_{\alpha,\beta} = I + \tilde{S}_{\alpha,\beta} + \hat{S}_{\alpha,\beta}$, $P_1^* = I + S_1^* = I + S_1 + \tilde{S}^*$, $P_2^* = I + S_2^* = I + S_2 + \hat{S}^*$, where

$$\tilde{S}^* = \begin{pmatrix} 0 & \cdots & 0 & -\alpha_n a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \quad \hat{S}^* = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -\beta_1 a_{n1} & 0 & \cdots & 0 \end{pmatrix},$$

and the corresponding preconditioned linear systems are

$$\bar{A}x = \bar{b}, \ A_1^* x = b_1^*, \ A_2^* x = b_2^*, \quad (9)$$

where $\bar{A} = (I + \bar{S}_{\alpha,\beta})A$, $\bar{b} = (I + \bar{S}_{\alpha,\beta})b$, $A_1^* = (I + S_1^*)A$, $b_1^* = (I + S_1^*)b$, $A_2^* = (I + S_2^*)A$ and $b_2^* = (I + S_2^*)b$. Similar to (5), let

$$\bar{A} = \bar{D} - \bar{L} - \bar{U},$$
$$A_1^* = D_1^* - L_1^* - U_1^*, \ A_2^* = D_2^* - L_2^* - U_2^*, \quad (10)$$

where $\bar{D}$, $-\bar{L}$, $-\bar{U}$ are diagonal, strictly lower and strictly upper triangular matrices of $\bar{A}$, respectively. Moreover, D_1^*, $-L_1^*$, $-U_1^*$ and D_2^*, $-L_2^*$, $-U_2^*$ are diagonal, strictly lower and strictly upper triangular matrices of A_1^* and A_2^*, respectively. The iterative matrices of AOR method for solving (10) are

$$\bar{L}_{r,w} = (\bar{D} - r\bar{L})^{-1}[(1-w)\bar{D} + (w-r)\bar{L} + w\bar{U}], \quad (11)$$
$$L_{1,r,w}^* = (D_1^* - rL_1^*)^{-1}[(1-w)D_1^* + (w-r)L_1^* + wU_1^*], \quad (12)$$
$$L_{2,r,w}^* = (D_2^* - rL_2^*)^{-1}[(1-w)D_2^* + (w-r)L_2^* + wU_2^*]. \quad (13)$$

The rest of the paper is organized as follows. In Section II, we collect some needed known concepts and lemmas. In Section III, we prove that, for the PAOR iteration, the new preconditioner $\bar{P} = I + \bar{S}_{\alpha,\beta}$ can make the iteration converge faster than the preconditioner $\tilde{P} = I + \tilde{S}_{\alpha,\beta}$. Furthermore, the new preconditioners $P_1^* = I + S_1^*$ and $P_2^* = I + S_2^*$ make the iteration converge faster than the preconditioners $P_1 = I + S_1$ and $P_2 = I + S_2$, respectively. In Section IV, we give several numerical examples to illustrate the obtained results in Section III. In Section V, we give the conclusions.

II. Preliminaries

We shall use the following lemmas and results.

For a vector $x \in \mathbb{R}^n$, $x \geq 0 (x > 0)$ denotes that all components of x are nonnegative (positive). For two vectors $x, y \in \mathbb{R}^n$, $x \geq y (x > y)$ means that $x - y \geq 0 (x - y > 0)$. These definitions carry immediately over to matrices. A matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is called a Z-matrix if $a_{ij} \leq 0$ for $i \neq j$, an L-matrix if A is a Z-matrix and $a_{ii} > 0$ for $i = 1, 2, \cdots, n$, and a nonsingular M-matrix if A is a Z-matrix and $A^{-1} \geq 0$. A matrix A is called irreducible if the directed graph of A is strongly connected [22].

Lemma 2.1 [22] Let $A \geq 0$ be an irreducible matrix. Then
(a) A has a positive eigenvalue equal to $\rho(A)$.
(b) A has an eigenvector $x > 0$ corresponding to $\rho(A)$.
(c) $\rho(A)$ is a simple eigenvalue of A.

Lemma 2.2 [23] Let $A \geq 0$ be a matrix. Then the following hold.
(a) If $Ax \geq \beta x$ for a vector $x \geq 0$ and $x \neq 0$, then $\rho(A) \geq 0$.
(b) If $Ax \leq \gamma x$ for a vector $x > 0$, then $\rho(A) \leq \gamma$. Moreover, if A is irreducible and if $\beta x \leq Ax \leq \gamma x$, equality excluded, for a vector $x \geq 0$ and $x \neq 0$, then $\beta < \rho(A) < \gamma$ and $x > 0$.
(c) If $\beta x < Ax < \gamma x$ for a vector $x > 0$, then $\beta < \rho(A) < \gamma$.

Lemma 2.3 Let $A = (a_{ij})_{n \times n}$ be an L-matrix with $0 < a_{1n}a_{n1} < \alpha_1 (\alpha_1 > 1)$, $0 < a_{1n}a_{n1} < \alpha_2 (\alpha_2 > 1)$ and $\beta_1 \in (-\frac{a_{n1}}{\alpha_1} + \frac{1}{a_{1n}}, -\frac{a_{n1}}{\alpha_1}) \bigcap ((1 - \frac{1}{\alpha_1})a_{n1}, -\frac{a_{n1}}{\alpha_1})$, $\beta_2 \in (-\frac{a_{1n}}{\alpha_2} + \frac{1}{a_{n1}}, -\frac{a_{1n}}{\alpha_2}) \bigcap ((1 - \frac{1}{\alpha_2})a_{1n}, -\frac{a_{1n}}{\alpha_1})$ and $0 \leq r \leq w \leq 1 (w \neq 0$ and $r \neq 1)$, then the iteration matrix $\bar{L}_{r,w}$ defined by (11) is nonnegative and irreducible.

Proof.

$$\bar{A} = (I + \bar{S}_{\alpha,\beta})A = \begin{pmatrix} g_1 & g_2 & \cdots & g_3 \\ a_{21} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_4 & g_5 & \cdots & g_6 \end{pmatrix}, \quad (14)$$

where $g_1 = 1 - (\frac{a_{1n}}{\alpha_2} + \beta_2)a_{n1}$, $g_2 = a_{12} - (\frac{a_{1n}}{\alpha_2} + \beta_2)a_{n2}$, $g_3 = (1 - \frac{1}{\alpha_2})a_{1n} - \beta_2$, $g_4 = (1 - \frac{1}{\alpha_1})a_{n1} - \beta_1$, $g_5 = a_{n2} - (\frac{a_{n1}}{\alpha_1} + \beta_1)a_{12}$ and $g_6 = 1 - (\frac{a_{n1}}{\alpha_1} + \beta_1)a_{1n}$. Since $\beta_1 < -\frac{a_{n1}}{\alpha_1}$, $\beta_2 < -\frac{a_{1n}}{\alpha_2}$, we have $\beta_1 + \frac{a_{n1}}{\alpha_1} < 0$, $\beta_2 + \frac{a_{1n}}{\alpha_2} < 0$. Thus, since A is a L-matrix, we have

$$(\bar{A})_{1i} = a_{1i} - (\beta_1 + \frac{a_{n1}}{\alpha_1})a_{ni} \leq a_{1i} \leq 0$$
$$(i = 2, 3, \cdots, n-1),$$
$$(\bar{A})_{nj} = a_{nj} - (\beta_2 + \frac{a_{1n}}{\alpha_2})a_{1j} \leq a_{nj} \leq 0$$
$$(j = 2, 3, \cdots, n-1).$$

Moreover, $\beta_1 > (1 - \frac{1}{\alpha_1})a_{n1}$, $\beta_2 > (1 - \frac{1}{\alpha_2})a_{1n}$, so we can get

$$(\bar{A})_{1n} = (1 - \frac{1}{\alpha_2})a_{1n} - \beta_2 < 0,$$
$$(\bar{A})_{n1} = (1 - \frac{1}{\alpha_1})a_{n1} - \beta_1 < 0.$$

By Equation (14), we obtain $(\bar{A})_{ij} = (A)_{ij} (i = 2, 3, \cdots, n-1, j = 1, 2, \cdots, n)$. Since A is irreducible, we infer that $\bar{A}$ is also irreducible. Inasmuch as A is a L-matrix, $\bar{L} \geq 0$, $\bar{U} \geq 0$, and $\beta_1 > -\frac{a_{n1}}{\alpha_1} + \frac{1}{a_{1n}}$, $\beta_2 > -\frac{a_{1n}}{\alpha_2} + \frac{1}{a_{n1}}$, we have $(\frac{a_{n1}}{\alpha_1} + \beta_1)a_{1n} < 1$, $(\frac{a_{1n}}{\alpha_2} + \beta_2)a_{n1} < 1$, which means that $\bar{D}$ is a positive diagonal matrix. From Equation (11), we have

$$\begin{aligned} &\bar{L}_{r,w} \\ &= (\bar{D} - r\bar{L})^{-1}[(1-w)\bar{D} + (w-r)\bar{L} + w\bar{U}] \\ &= (I - r\bar{D}^{-1}\bar{L})^{-1}[(1-w)I + (w-r)\bar{D}^{-1}\bar{L} + w\bar{D}^{-1}\bar{U}] \\ &= (I + r\bar{D}^{-1}\bar{L} + (r\bar{D}^{-1}\bar{L})^2 + \cdots + (r\bar{D}^{-1}\bar{L})^{n-1}) \\ &\quad \times [(1-w)I + (w-r)\bar{D}^{-1}\bar{L} + w\bar{D}^{-1}\bar{U}] \\ &= (1-w)I + w(1-r)\bar{D}^{-1}\bar{L} + w\bar{D}^{-1}\bar{U} + T, \end{aligned} \quad (15)$$

where

$$\begin{aligned} T = {} & r\bar{D}^{-1}\bar{L}[(w-r)\bar{D}^{-1}\bar{L} + w\bar{D}^{-1}\bar{U}] + \\ & [(r\bar{D}^{-1}\bar{L})^2 + \cdots + (r\bar{D}^{-1}\bar{L})^{n-1}] \times \\ & [(1-w)I + (w-r)\bar{D}^{-1}\bar{L} + w\bar{D}^{-1}\bar{U}] \geq 0. \end{aligned}$$

So $\bar{L}_{r,w} \geq 0$. Since $\bar{A}$ is irreducible and $0 < w \leq 1$, $0 \leq r < 1$, $(1-w)I + w(1-r)\bar{D}^{-1}\bar{L} + w\bar{D}^{-1}\bar{U}$ is also irreducible. Thus, $\bar{L}_{r,w}$ is nonnegative and irreducible.

Lemma 2.4 Let $A = (a_{ij})_{n\times n}$ be an L-matrix with $\alpha_i a_{i,i+1}a_{i+1,i} < 1$ and $0 \leq \alpha_i < 1$ for all $1 \leq i \leq n-1$. Assume that $\alpha_i a_{i,i+1} \neq 0$ for some $i < n$ and $a_{1n} \neq 0$, $0 < \alpha_n < \min\{\frac{1-\alpha_1 a_{12}a_{21}}{a_{1n}a_{n1}}, 1 - \frac{\alpha_1 a_{12}a_{2n}}{a_{1n}}\}$. and $0 \leq r \leq w \leq 1 (w \neq 0$ and $r \neq 1)$, then the iteration matrix $L^*_{1,r,w}$ defined by (12) is nonnegative and irreducible.

Proof.

$$A_1^* = (I + S_1^*)A = \begin{pmatrix} h_1 & h_2 & \cdots & h_3 \\ h_4 & h_5 & \cdots & h_6 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 1 \end{pmatrix},$$

where $h_1 = 1 - \alpha_1 a_{12}a_{21} - \alpha_n a_{1n}a_{n1}$, $h_2 = (1-\alpha_1)a_{12} - \alpha_n a_{1n}a_{n2}$, $h_3 = (1-\alpha_n)a_{1n} - \alpha_1 a_{12}a_{2n}$, $h_4 = a_{21} - \alpha_2 a_{23}a_{31}$, $h_5 = 1 - \alpha_2 a_{23}a_{32}$ and $h_6 = a_{2n} - \alpha_2 a_{23}a_{3n}$. Since $\alpha_i a_{i,i+1}a_{i+1,i} < 1$ for all $1 \leq i \leq n-1$, and $0 < \alpha_n < \min\{\frac{1-\alpha_1 a_{12}a_{21}}{a_{1n}a_{n1}}, 1 - \frac{\alpha_1 a_{12}a_{2n}}{a_{1n}}\}$, we have

$$(A_1^*)_{ii} = 1 - \alpha_i a_{i,i+1}a_{i+1,i} > 0,\ (i = 2, 3, \cdots, n-1),$$
$$(A_1^*)_{nn} = 1 > 0,$$
$$(A_1^*)_{11} = 1 - \alpha_1 a_{12}a_{21} - \alpha_n a_{1n}a_{n1} > 0.$$

Thus, D_1^* is a positive diagonal matrix.

Moreover, since $0 < \alpha_n < \min\{\frac{1-\alpha_1 a_{12}a_{21}}{a_{1n}a_{n1}}, 1 - \frac{\alpha_1 a_{12}a_{2n}}{a_{1n}}\}$, $0 \leq \alpha_i < 1$ for all $1 \leq i \leq n-1$ and A is a L-matrix, we have

$$(A_1^*)_{1i} = a_{1j} - \alpha_1 a_{12}a_{2j} - \alpha_n a_{1n}a_{nj} \leq a_{1j} \leq 0$$
$$(i = 2, 3, \cdots, n-1),$$
$$(A_1^*)_{12} = (1-\alpha_1)a_{12} - \alpha_n a_{1n}a_{n2} \leq (1-\alpha_1)a_{12} \leq 0,$$
$$(A_1^*)_{1n} = (1-\alpha_n)a_{1n} - \alpha_1 a_{12}a_{2n} < 0,$$
$$(A_1^*)_{ij} = a_{ij} - \alpha_i a_{i,i+1}a_{i+1,j} \leq a_{ij} \leq 0$$
$$(i = 2, 3, \cdots, n-1; j = 1, 2, \cdots, n; i \neq j, i+1 \neq j),$$
$$(A_1^*)_{ij} = (1-\alpha_i)a_{ij} \leq 0$$
$$(i = 2, 3, \cdots, n-1; j = 1, 2, \cdots, n; i+1 = j),$$
$$(A_1^*)_{nj} = a_{nj} \leq 0\ (j = 1, 2, \cdots, n-1).$$

Since A is irreducible, we deduce that A_1^* is also irreducible. Since A is a L-matrix, $L_1^* \geq 0$, $U_1^* \geq 0$. It follows from Equation (12) that

$$\begin{aligned} & L^*_{1,r,w} \\ &= (D_1^* - rL_1^*)^{-1}[(1-w)D_1^* + (w-r)L_1^* + wU_1^*] \\ &= (I - r(D_1^*)^{-1}L_1^*)^{-1} \times \\ &\quad [(1-w)I + (w-r)(D_1^*)^{-1}L_1^* + w(D_1^*)^{-1}U_1^*] \\ &= (1-w)I + w(1-r)(D_1^*)^{-1}L_1^* + w(D_1^*)^{-1}U_1^* + T_1, \end{aligned} \tag{16}$$

where $T_1 \geq 0$. So $L^*_{1,r,w} \geq 0$. Since A_1^* is irreducible and $0 < w \leq 1$, $0 \leq r < 1$, $(1-w)I + w(1-r)(D_1^*)^{-1}L_1^* + w(D_1^*)^{-1}U_1^*$ is also irreducible. Therefore, $L^*_{1,r,w}$ is nonnegative and irreducible.

Similar to the proof of Lemma 2.4, we obtain the following lemma.

Lemma 2.5 Let $A = (a_{ij})_{n\times n}$ be an L-matrix with $0 < \beta_i a_{i,i-1}a_{i-1,i} < 1$, $\beta_i a_{i,i-1} \neq 0$ and $0 < \beta_i \leq 1$ for $2 \leq i \leq n$. Assume that $a_{n1} \neq 0$, $0 < \beta_1 < \min\{\frac{1-\beta_n a_{n,n-1}a_{n-1,n}}{a_{1n}a_{n1}}, 1 - \frac{\beta_n a_{n,n-1}a_{n-1,1}}{a_{n1}}\}$. and $0 \leq r \leq w \leq 1 (w \neq 0$ and $r \neq 1)$, then the iteration matrix $L^*_{2,r,w}$ defined by (13) is nonnegative and irreducible.

Lemma 2.6 [24] Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be an upper triangular matrix or a lower triangular matrix, then A is a nonsingular M-matrix if and only if A is a L-matrix.

Lemma 2.7 [24] If A is a nonsingular M-matrix, then $A^{-1} \geq 0$.

III. Comparison theorems for preconditioned AOR methods

In this section, we present some theorems to compare the convergence rates of the preconditioned AOR methods proposed in this paper with the methods in [6], [8], [9].

Under the above assumptions, we easily get the following equations.

(E1) $\bar{S}_{\alpha,\beta} = \tilde{S}_{\alpha,\beta} + \hat{S}_{\alpha,\beta}$;

(E2) $\bar{D} = \tilde{D} + \hat{D} - I$, $\bar{L} = \tilde{L}$, $\bar{U} = \hat{U}$, $\hat{L} = L$, $\tilde{U} = U$;

(E3) $\tilde{D} - \tilde{L} = I + \tilde{S}_{\alpha,\beta} - L - \tilde{S}_{\alpha,\beta}U$, $\hat{D} - \hat{U} = I + \hat{S}_{\alpha,\beta} - U - \hat{S}_{\alpha,\beta}L$;

(E4) $\tilde{S}_{\alpha,\beta}L = 0$, $\hat{S}_{\alpha,\beta}U = 0$, $\tilde{S}_{\alpha,\beta}\tilde{L} = 0$, $\hat{S}_{\alpha,\beta}\hat{U} = 0$;

(E5) $\bar{D} - \bar{U} = \tilde{D} + \hat{S}_{\alpha,\beta} - U - \hat{S}_{\alpha,\beta}L$, $\bar{D} - \bar{L} = \hat{D} + \tilde{S}_{\alpha,\beta} - L - \tilde{S}_{\alpha,\beta}U$;

(E6) $U = \hat{U} - \hat{S}_{\alpha,\beta}L + \hat{S}_{\alpha,\beta} + \hat{S}_{\alpha,\beta}S_{11}$, $L = \tilde{L} - \tilde{S}_{\alpha,\beta}U + \tilde{S}_{\alpha,\beta} + \tilde{S}_{\alpha,\beta}S_{22}$, where

$$S_{11} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & 0 & \cdots & 0 \end{pmatrix}, \quad S_{22} = \begin{pmatrix} 0 & 0 & \cdots & -a_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix};$$

(E7) $D_1 = I - \acute{D}_1$, $L_1 = L + \acute{L}_1$, $U_1 = U - S_1 + S_1U$, $S_1L = \acute{D}_1 + \acute{L}_1$, where $\acute{D}_1$ and $\acute{L}_1$ are diagonal, strictly lower triangular matrices of S_1L;

(E8) $D_1^* = I - \acute{D}_1 - \grave{D}_1 = D_1 - \grave{D}_1$, $L_1^* = L + \acute{L}_1 = L_1$, $U_1^* = U - S_1 + S_1U + \grave{U}_1 - \tilde{S}^* = U_1 + \grave{U}_1 - \tilde{S}^*$, $\tilde{S}^*L = \grave{D}_1 + \grave{U}_1$, where $\grave{D}_1$ and $\grave{U}_1$ are diagonal, strictly upper triangular matrices of $\tilde{S}^*L$;

(E9) $D_2 = I - \acute{D}_2$, $L_2 = L - S_2 + S_2L$, $U_2 = U + \acute{U}_2$, $S_2U = \acute{D}_2 + \acute{U}_2$, where $\acute{D}_2$ and $\acute{U}_2$ are diagonal, strictly upper triangular matrices of S_2U;

(E10) $D_2^* = I - \acute{D}_2 - \grave{D}_2 = D_2 - \grave{D}_2$, $L_2^* = L - S_2 + S_2L - \hat{S}^* + \grave{L}_2 = L_2 - \hat{S}^* + \grave{L}_2$, $U_2^* = U + \acute{U}_2 = U_2$, $\hat{S}^*U = \grave{D}_2 + \grave{L}_2$, where $\grave{D}_2$ and $\grave{L}_2$ are diagonal, strictly lower triangular matrices of $\hat{S}^*L$.

We first compare the covergence rate of the preconditioned AOR method defined by Equation (11) with that of the preconditioned AOR method defined by Equation (6).

Lemma 3.1 Let $A = (a_{ij})_{n\times n}$ be an L-matrix with $0 < a_{1n}a_{n1} < \alpha_1(\alpha_1 > 1)$, $\beta_1 \in (-\frac{a_{n1}}{\alpha_1} + \frac{1}{a_{1n}}, -\frac{a_{n1}}{\alpha_1}) \bigcap ((1 - \frac{1}{\alpha_1})a_{n1}, -\frac{a_{n1}}{\alpha_1})$ and $0 \leq r \leq w \leq 1 (w \neq 0, r \neq 1)$. $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)^T > 0$ be the positive Perron vector of $\tilde{L}_{r,w}$ with respect to its spectral radius $\tilde{\lambda} = \rho(\tilde{L}_{r,w})$. Then

$$(w - 1 + \tilde{\lambda})\tilde{x}_1 + w\sum_{j=2}^{n} a_{1j}\tilde{x}_j = 0. \tag{17}$$

Proof. Since $\tilde{L}_{r,w}\tilde{x} = \tilde{\lambda}\tilde{x}$, we have

$$[(1-w)\tilde{D} + (w-r)\tilde{L} + w\tilde{U}]\tilde{x} = \tilde{\lambda}(\tilde{D} - r\tilde{L})\tilde{x}. \tag{18}$$

Consider the first entry of (18), we have $(1-w)\tilde{x}_1 - w\sum_{j=2}^{n} a_{1j}\tilde{x}_j = \tilde{\lambda}\tilde{x}_1$, i.e.,

$$(w-1+\tilde{\lambda})\tilde{x}_1 + w\sum_{j=2}^{n} a_{1j}\tilde{x}_j = 0.$$

Theorem 3.1 Let $\tilde{L}_{r,w}$ and $\bar{L}_{r,w}$ be the iteration matrices of the AOR method given by (6) and (11), respectively. If $0 \le r \le w \le 1 (w \neq 0$ and $r \neq 1)$ and A is an irreducible L-matrix with $0 < a_{1n}a_{n1} < \alpha_1(\alpha_1 > 1)$, $0 < a_{1n}a_{n1} < \alpha_2(\alpha_2 > 1)$ and $\beta_1 \in (-\frac{a_{n1}}{\alpha_1} + \frac{1}{a_{1n}}, -\frac{a_{n1}}{\alpha_1}) \bigcap ((1-\frac{1}{\alpha_1})a_{n1}, -\frac{a_{n1}}{\alpha_1})$, $\beta_2 \in (-\frac{a_{1n}}{\alpha_2} + \frac{1}{a_{n1}}, -\frac{a_{1n}}{\alpha_2}) \bigcap ((1-\frac{1}{\alpha_2})a_{1n}, -\frac{a_{1n}}{\alpha_1})$, then
(i) $\rho(\bar{L}_{r,w}) < \rho(\tilde{L}_{r,w}) < \rho(L_{r,w}) < 1$, if $\rho(L_{r,w}) < 1$;
(ii) $\rho(\bar{L}_{r,w}) = \rho(\tilde{L}_{r,w}) = \rho(L_{r,w}) = 1$, if $\rho(L_{r,w}) = 1$;
(iii) $\rho(\bar{L}_{r,w}) > \rho(\tilde{L}_{r,w}) > \rho(L_{r,w}) > 1$, if $\rho(L_{r,w}) > 1$.

Proof. By Lemma 2.3 and Lemma 3 in [6], it is clear that the iteration matrices $\bar{L}_{r,w}$ and $\tilde{L}_{r,w}$ are nonnegative and irreducible. So there is a positive Perron vector $\tilde{x}$, such that

$$\tilde{L}_{r,w}\tilde{x} = \tilde{\lambda}\tilde{x}, \tag{19}$$

where $\tilde{\lambda} = \rho(\tilde{L}_{r,w})$. By Equations (18) and (E2), we have

$$wU\tilde{x} = w\tilde{U}\tilde{x} = (\tilde{\lambda}-1+w)\tilde{D}\tilde{x} + (r-w-\tilde{\lambda}r)\tilde{L}\tilde{x}. \tag{20}$$

Combining (19) with (20) results in

$$\begin{aligned}
&\bar{L}_{r,w}\tilde{x} - \tilde{\lambda}\tilde{x}\\
&= (\bar{D}-r\bar{L})^{-1}[(1-w)\bar{D} + (w-r)\bar{L} + w\bar{U} - \tilde{\lambda}(\bar{D}-r\bar{L})]\tilde{x}\\
&= (\bar{D}-r\bar{L})^{-1}[(1-\tilde{\lambda})\bar{D} - r(1-\tilde{\lambda})\bar{L} - w(\bar{D}-\bar{U}) + w\bar{L}]\tilde{x}\\
&= (\bar{D}-r\bar{L})^{-1}[(1-\tilde{\lambda})\bar{D} - r(1-\tilde{\lambda})\bar{L} -\\
&\quad w(\tilde{D} + \hat{S}_{\alpha,\beta} - U - \hat{S}_{\alpha,\beta}L) + w\bar{L}]\tilde{x}\\
&= (\bar{D}-r\bar{L})^{-1}[(1-\tilde{\lambda})\bar{D} - r(1-\tilde{\lambda})\bar{L} -\\
&\quad w(\tilde{D} + \hat{S}_{\alpha,\beta} - \hat{S}_{\alpha,\beta}L) + w\bar{L}\\
&\quad +(\tilde{\lambda}-1+w)\tilde{D} + (r-w-\tilde{\lambda}r)\tilde{L}]\tilde{x}\\
&= (\bar{D}-r\bar{L})^{-1}[(1-\tilde{\lambda})\bar{D} +\\
&\quad (\tilde{\lambda}-1)\tilde{D} + (w-r+\tilde{\lambda}r)(\bar{L}-\tilde{L}) - w(\hat{S}_{\alpha,\beta} - \hat{S}_{\alpha,\beta}L)]\tilde{x}\\
&= (\bar{D}-r\bar{L})^{-1}[(1-\tilde{\lambda})(\bar{D}-\tilde{D}) - w(\hat{S}_{\alpha,\beta} - \hat{S}_{\alpha,\beta}L)]\tilde{x}\\
&= (\bar{D}-r\bar{L})^{-1}[(1-\tilde{\lambda})(\bar{D}-\tilde{D}) - w\hat{S}_{\alpha,\beta} +\\
&\quad w\hat{S}_{\alpha,\beta}(\tilde{L} - \tilde{S}_{\alpha,\beta}U + \tilde{S}_{\alpha,\beta} + \tilde{S}_{\alpha,\beta}S_{22})]\tilde{x}.
\end{aligned}$$

According to Equation (18), we derive $\tilde{L}\tilde{x} = \frac{1}{w-r+\tilde{\lambda}r}[(\tilde{\lambda}-1+w)\tilde{D} - w\tilde{U}]\tilde{x}$. Thus

$$\begin{aligned}
&\bar{L}_{r,w}\tilde{x} - \tilde{\lambda}\tilde{x}\\
&= (\bar{D}-r\bar{L})^{-1}[(1-\tilde{\lambda})(\bar{D}-\tilde{D}) - w\hat{S}_{\alpha,\beta}\\
&\quad +w\hat{S}_{\alpha,\beta}(\tilde{L} - \tilde{S}_{\alpha,\beta}U + \tilde{S}_{\alpha,\beta} + \tilde{S}_{\alpha,\beta}S_{22})]\tilde{x}\\
&= (\bar{D}-r\bar{L})^{-1}[(1-\tilde{\lambda})(\bar{D}-\tilde{D}) - w\hat{S}_{\alpha,\beta}\\
&\quad +w\hat{S}_{\alpha,\beta}\tilde{L} + w\hat{S}_{\alpha,\beta}(\tilde{S}_{\alpha,\beta}S_{22} + \tilde{S}_{\alpha,\beta} - \tilde{S}_{\alpha,\beta}U)]\tilde{x}\\
&= (\bar{D}-r\bar{L})^{-1}[(1-\tilde{\lambda})(\bar{D}-\tilde{D}) - w\hat{S}_{\alpha,\beta}\\
&\quad +\frac{w}{w-r+\tilde{\lambda}r}\hat{S}_{\alpha,\beta}[(\tilde{\lambda}-1+w)\tilde{D} - w\tilde{U}]\\
&\quad +w\hat{S}_{\alpha,\beta}(\tilde{S}_{\alpha,\beta}S_{22} + \tilde{S}_{\alpha,\beta} - \tilde{S}_{\alpha,\beta}U)]\tilde{x}\\
&= (\bar{D}-r\bar{L})^{-1}[(1-\tilde{\lambda})(\bar{D}-\tilde{D}) - w\hat{S}_{\alpha,\beta}\\
&\quad +\frac{w(\tilde{\lambda}-1+w)}{w-r+\tilde{\lambda}r}\hat{S}_{\alpha,\beta}\tilde{D}\\
&\quad +w\hat{S}_{\alpha,\beta}(\tilde{S}_{\alpha,\beta}S_{22} + \tilde{S}_{\alpha,\beta} - \tilde{S}_{\alpha,\beta}U)]\tilde{x}\\
&= (\bar{D}-r\bar{L})^{-1}[(1-\tilde{\lambda})(\bar{D}-\tilde{D})\\
&\quad +w\hat{S}_{\alpha,\beta}(\tilde{S}_{\alpha,\beta}S_{22} + \tilde{S}_{\alpha,\beta} - \tilde{S}_{\alpha,\beta}U - I)\\
&\quad +\frac{w(\tilde{\lambda}-1+w)}{w-r+\tilde{\lambda}r}\hat{S}_{\alpha,\beta}\tilde{D}]\tilde{x} = (\bar{D}-r\bar{L})^{-1}C\tilde{x},
\end{aligned} \tag{21}$$

where $C = (1-\tilde{\lambda})(\bar{D}-\tilde{D}) + w\hat{S}_{\alpha,\beta}(\tilde{S}_{\alpha,\beta}S_{22} + \tilde{S}_{\alpha,\beta} - \tilde{S}_{\alpha,\beta}U - I) + \frac{w(\tilde{\lambda}-1+w)}{w-r+\tilde{\lambda}r}\hat{S}_{\alpha,\beta}\tilde{D}$. Let $C = (c_{ij})_{n\times n}$, then the matrix C satisfies the following properties:

$$\begin{aligned}
&c_{11} = w(\frac{a_{1n}}{\alpha_2} + \beta_2)(\frac{a_{n1}}{\alpha_1} + \beta_1) - (1-\tilde{\lambda})(\frac{a_{1n}}{\alpha_2} + \beta_2)a_{n1},\\
&c_{1j} = w(\frac{a_{1n}}{\alpha_2} + \beta_2)(\frac{a_{n1}}{\alpha_1} + \beta_1)a_{1j},\ (j = 2, 3, \cdots, n-1),\\
&c_{1n} = w(\frac{a_{1n}}{\alpha_2} + \beta_2) - \frac{w(\tilde{\lambda}-1+w)}{w-r+\tilde{\lambda}r} \times\\
&\quad (\frac{a_{1n}}{\alpha_2} + \beta_2)[1 - (\frac{a_{n1}}{\alpha_1} + \beta_1)a_{1n}],\\
&c_{ij} = 0,\ (i = 2, 3, \cdots, n,\ j = 1, 2, \cdots, n).
\end{aligned}$$

By Lemma 3.1, we have $(w-1+\tilde{\lambda})\tilde{x}_1 + w\sum_{j=2}^{n} a_{1j}\tilde{x}_j = 0$, so we get

$$\begin{aligned}
&(w-1+\tilde{\lambda})(\frac{a_{1n}}{\alpha_2} + \beta_2)(\frac{a_{n1}}{\alpha_1} + \beta_1)\tilde{x}_1\\
&+ w\sum_{j=2}^{n}(\frac{a_{1n}}{\alpha_2} + \beta_2)(\frac{a_{n1}}{\alpha_1} + \beta_1)a_{1j}\tilde{x}_j = 0.
\end{aligned}$$

Further, we have

$$\begin{aligned}
&(C\tilde{x})_1 = c_{11}\tilde{x}_1 + \sum_{j=2}^{n-1} c_{1j}\tilde{x}_j + c_{1n}\tilde{x}_n\\
&= w(\frac{a_{1n}}{\alpha_2} + \beta_2)(\frac{a_{n1}}{\alpha_1} + \beta_1)\tilde{x}_1 - (1-\tilde{\lambda})(\frac{a_{1n}}{\alpha_2} + \beta_2)a_{n1}\tilde{x}_1\\
&\quad +w\sum_{j=2}^{n-1}(\frac{a_{1n}}{\alpha_2} + \beta_2)(\frac{a_{n1}}{\alpha_1} + \beta_1)a_{1j}\tilde{x}_j + w(\frac{a_{1n}}{\alpha_2} + \beta_2)\tilde{x}_n\\
&\quad -\frac{w(\tilde{\lambda}-1+w)}{w-r+\tilde{\lambda}r}(\frac{a_{1n}}{\alpha_2} + \beta_2)[1 - (\frac{a_{n1}}{\alpha_1} + \beta_1)a_{1n}]\tilde{x}_n\\
&\quad -(w-1+\tilde{\lambda})(\frac{a_{1n}}{\alpha_2} + \beta_2)(\frac{a_{n1}}{\alpha_1} + \beta_1)\tilde{x}_1\\
&\quad -w\sum_{j=2}^{n}(\frac{a_{1n}}{\alpha_2} + \beta_2)(\frac{a_{n1}}{\alpha_1} + \beta_1)a_{1j}\tilde{x}_j\\
&= (\tilde{\lambda}-1)(\frac{a_{1n}}{\alpha_2} + \beta_2)[(1-\frac{1}{\alpha_1})a_{n1} - \beta_1]\tilde{x}_1\\
&\quad +\frac{w(1-\tilde{\lambda})(1-r)}{w-r+\tilde{\lambda}r}(\frac{a_{1n}}{\alpha_2} + \beta_2)[1 - (\frac{a_{n1}}{\alpha_1} + \beta_1)a_{1n}]\tilde{x}_n\\
&= (\tilde{\lambda}-1)\{(\frac{a_{1n}}{\alpha_2} + \beta_2)[(1-\frac{1}{\alpha_1})a_{n1} - \beta_1]\tilde{x}_1\\
&\quad -\frac{w(1-r)}{w-r+\tilde{\lambda}r}(\frac{a_{1n}}{\alpha_2} + \beta_2)[1 - (\frac{a_{n1}}{\alpha_1} + \beta_1)a_{1n}]\tilde{x}_n\}\\
&= (\tilde{\lambda}-1)e,
\end{aligned} \tag{22}$$

where $e = (\frac{a_{1n}}{\alpha_2} + \beta_2)[(1-\frac{1}{\alpha_1})a_{n1} - \beta_1]\tilde{x}_1 - \frac{w(1-r)}{w-r+\tilde{\lambda}r}(\frac{a_{1n}}{\alpha_2} + \beta_2)[1 - (\frac{a_{n1}}{\alpha_1} + \beta_1)a_{1n}]\tilde{x}_n$. Since $\beta_1 \in (-\frac{a_{n1}}{\alpha_1} + \frac{1}{a_{1n}}, -\frac{a_{n1}}{\alpha_1}) \bigcap ((1-\frac{1}{\alpha_1})a_{n1}, -\frac{a_{n1}}{\alpha_1})$, $\beta_2 \in (-\frac{a_{1n}}{\alpha_2} + \frac{1}{a_{n1}}, -\frac{a_{1n}}{\alpha_2}) \bigcap ((1-\frac{1}{\alpha_2})a_{1n}, -\frac{a_{1n}}{\alpha_1})$, we have $\frac{a_{1n}}{\alpha_2} + \beta_2 < 0$, $(1-\frac{1}{\alpha_1})a_{n1} - \beta_1 < 0$, $1 - (\frac{a_{n1}}{\alpha_1} + \beta_1)a_{1n} > 0$.

Thus, by conditions $0 \le r \le w \le 1 (w \neq 0, r \neq 1)$, we have $e > 0$. Moreover, by Lemma 2.3, we can obtain that $\bar{D}$ is a positive diagonal matrix and $\bar{L} \ge 0$, then by Lemma

2.6, we obtain that $\bar{D}-r\bar{L}$ is a nonsingular M-matrix, so by Lemma 2.7 we have $(\bar{D}-r\bar{L})^{-1}\geq 0$ and $((\bar{D}-r\bar{L})^{-1})_{ii}>0$ for $i=1,2,\cdots,n$, so we have

(i) If $0<\lambda=\rho(L_{r,w})<1$, by Theorem 1 in [6], we have $0<\tilde{\lambda}<\lambda<1$, then $(C\tilde{x})_1<0$, so, we have $\bar{L}_{r,w}\tilde{x}-\tilde{\lambda}\tilde{x}\leq 0$ but is not equal to the null vector. Therefore $\bar{L}_{r,w}\tilde{x}\leq\tilde{\lambda}\tilde{x}$. By Lemma 2.2, we get $\rho(\bar{L}_{r,w})<\rho(\tilde{L}_{r,w})<\rho(L_{r,w})<1$;

(ii) If $\lambda=\rho(L_{r,w})=1$, by Theorem 1 in [6], we have $\tilde{\lambda}=\lambda=1$, then $(C\tilde{x})_1=0$, so, we have $\bar{L}_{r,w}\tilde{x}-\tilde{\lambda}\tilde{x}=0$, i.e., $\bar{L}_{r,w}\tilde{x}=\tilde{\lambda}\tilde{x}$. By Lemma 2.2, we get $\rho(\bar{L}_{r,w})=\rho(\tilde{L}_{r,w})=\rho(L_{r,w})=1$;

(iii) If $\lambda=\rho(L_{r,w})>1$, by Theorem 1 in [6], we have $\tilde{\lambda}>\lambda>1$, then $(C\tilde{x})_1>0$, so, we have $\bar{L}_{r,w}\tilde{x}-\tilde{\lambda}\tilde{x}\geq 0$ but is not equal to the null vector. Therefore $\bar{L}_{r,w}\tilde{x}\geq\tilde{\lambda}\tilde{x}$. By Lemma 2.2, we get $\rho(\bar{L}_{r,w})>\rho(\tilde{L}_{r,w})>\rho(L_{r,w})>1$.

From Theorem 3.1, it can be seen that the rate of convergence of the preconditioned AOR method defined by (11) is better than that of the preconditioned AOR method defined by (6) under some conditions whenever the latter one is convergent.

We next compare the covergence rate of the preconditioned AOR method defined by Equation (12) with that of the preconditioned AOR method defined by Equation (7).

Theorem 3.2 Let $A=(a_{ij})_{n\times n}$ be an L-matrix with $\alpha_i a_{i,i+1}a_{i+1,i}<1$ and $0\leq\alpha_i<1$ for all $1\leq i\leq n-1$. Assume that $\alpha_i a_{i,i+1}\neq 0$ for some $i<n$ and $a_{1n}\neq 0$, $0<\alpha_n<\min\{\frac{1-\alpha_1 a_{12}a_{21}}{a_{1n}a_{n1}},1-\frac{\alpha_1 a_{12}a_{2n}}{a_{1n}}\}$. If $0\leq r\leq w\leq 1(w\neq 0$ and $r\neq 1)$, then the following holds.
(a) $\rho(L^*_{1,r,w})<\rho(L_{1,r,w})<\rho(L_{r,w})$, if $\rho(L_{r,w})<1$;
(b) $\rho(L^*_{1,r,w})=\rho(L_{1,r,w})=\rho(L_{r,w})$, if $\rho(L_{r,w})=1$;
(c) $\rho(L^*_{1,r,w})>\rho(L_{1,r,w})>\rho(L_{r,w})$, if $\rho(L_{r,w})>1$.

Proof. By Lemma 2.4 and Theorem 3.4 in [8], it is clear that the matrices $L_{1,r,w}$ and $L^*_{1,r,w}$ are nonnegative and irreducible. So there is a positive Perron vector x_1, such that

$$L_{1,r,w}x_1=\lambda_1 x_1, \tag{23}$$

where $\lambda_1=\rho(L_{1,r,w})$. In view of Equation (23), we have

$$[(1-w)D_1+(w-r)L_1+wU_1]x_1=\lambda_1(D_1-rL_1)x_1, \tag{24}$$

i.e.,

$$[(1-w-\lambda_1)D_1+(w-r+\lambda_1 r)L_1+wU_1]x_1=0. \tag{25}$$

Using Equations (E7), (E8), (12) and (25), we obtain

$$\begin{aligned}
&L^*_{1,r,w}x_1-\lambda_1 x_1\\
&=(D^*_1-rL^*_1)^{-1}[(1-w)D^*_1+(w-r)L^*_1+wU^*_1\\
&\quad-\lambda_1(D^*_1-rL^*_1)]x_1\\
&=(D^*_1-rL^*_1)^{-1}[(1-w-\lambda_1)D^*_1\\
&\quad+(w-r+\lambda_1 r)L^*_1+wU^*_1]x_1\\
&=(D^*_1-rL^*_1)^{-1}[(1-w-\lambda_1)(D_1-\grave{D}_1)\\
&\quad+(w-r+\lambda_1 r)L_1+w(U_1+\grave{U}_1-\tilde{S}^*)]x_1\\
&=(D^*_1-rL^*_1)^{-1}[(\lambda_1-1+w)\grave{D}_1+w(\grave{U}_1-\tilde{S}^*)]x_1\\
&=(D^*_1-rL^*_1)^{-1}[(\lambda_1-1)\grave{D}_1+w(\grave{D}_1+\grave{U}_1-\tilde{S}^*)]x_1\\
&=(D^*_1-rL^*_1)^{-1}[(\lambda_1-1)\grave{D}_1+w(\tilde{S}^*L-\tilde{S}^*)]x_1\\
&=(D^*_1-rL^*_1)^{-1}[(\lambda_1-1)\grave{D}_1+w\tilde{S}^*(L-I)]x_1.
\end{aligned}$$

Notice that $\tilde{S}^*L=\tilde{S}^*L_1,\ \tilde{S}^*=\tilde{S}^*I=\tilde{S}^*D_1,\ \tilde{S}^*U_1=0$, and by Equation (24), we have

$$w(L_1+U_1-D_1)x_1=(\lambda_1-1)(D_1-rL_1)x_1, \tag{26}$$

which implies that

$$\begin{aligned}
w\tilde{S}^*(L-I)x_1&=w\tilde{S}^*(L_1+U_1-D_1)x_1\\
&=(\lambda_1-1)\tilde{S}^*(D_1-rL_1)x_1.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&L^*_{1,r,w}x_1-\lambda_1 x_1\\
&=(D^*_1-rL^*_1)^{-1}[(\lambda_1-1)\grave{D}_1+w\tilde{S}^*(L-I)]x_1\\
&=(D^*_1-rL^*_1)^{-1}[(\lambda_1-1)\grave{D}_1+(\lambda_1-1)\tilde{S}^*(D_1-rL_1)]x_1\\
&=(\lambda_1-1)(D^*_1-rL^*_1)^{-1}[\grave{D}_1+\tilde{S}^*(D_1-rL_1)]x_1.
\end{aligned} \tag{27}$$

By Equation (24) and $0\leq r\leq w\leq 1(w\neq 0,r\neq 1)$, we have

$$\begin{aligned}
&(D_1-rL_1)x_1\\
&=\frac{1}{\lambda_1}[(1-w)D_1+(w-r)L_1+wU_1]x_1\geq 0.
\end{aligned} \tag{28}$$

From Theorem 3.4 in [8], we get A_1 is irreducible, so $\frac{1}{\lambda_1}[(1-w)D_1+(w-r)L_1+wU_1]x_1\neq 0$. If the n-th entry of $\frac{1}{\lambda_1}[(1-w)D_1+(w-r)L_1+wU_1]x_1$ is a positive number, it is easy to get

$$\begin{aligned}
&\tilde{S}^*(D_1-rL_1)x_1\\
&=\frac{1}{\lambda_1}\tilde{S}^*[(1-w)D_1+(w-r)L_1+wU_1]x_1\geq 0,\\
&\tilde{S}^*(D_1-rL_1)x_1\\
&=\frac{1}{\lambda_1}\tilde{S}^*[(1-w)D_1+(w-r)L_1+wU_1]x_1\neq 0.
\end{aligned}$$

If $w=1$, the n-th entry of $(w-r)L_1x_1$ is a positive number because of $r\neq 0$; In addition, if $w=r$, the n-th entry of $(1-w)D_1$ is $1-w>0$ because of $w=r\neq 1$. Under the above discussions and $\grave{D}_1\geq 0$, we obtain that $[\grave{D}_1+\tilde{S}^*(D_1-rL_1)]x_1\geq 0,[\grave{D}_1+\tilde{S}^*(D_1-rL_1)]x_1\neq 0$. By Lemma 2.4, we know that D^*_1 is a positive diagonal matrix and $L^*_1\geq 0$, then by Lemma 2.6 we obtain that $(D^*_1-rL^*_1)$ is a nonsingular M-matrix, so by Lemma 2.7 we have $(D^*_1-rL^*_1)^{-1}\geq 0$ and $((D^*_1-rL^*_1)^{-1})_{11}>0$, then $(D^*_1-rL^*_1)^{-1}[\grave{D}_1+\tilde{S}^*(D_1-rL_1)]x_1\geq 0,(D^*_1-rL^*_1)^{-1}[\grave{D}_1+\tilde{S}^*(D_1-rL_1)]x_1\neq 0$.

(a) If $0<\lambda=\rho(L_{r,w})<1$, by Theorem 3.4 in [8], we have $0<\lambda_1<\lambda<1$, by Equation (27), we have $L^*_{1,r,w}x_1-\lambda_1x_1\leq 0$ but is not equal to the null vector. Therefore $L^*_{1,r,w}x_1\leq\lambda_1 x_1$. By Lemma 2.2, we get $\rho(L^*_{1,r,w})<\rho(L_{1,r,w})<\rho(L_{r,w})<1$;

(b) If $\lambda=\rho(L_{r,w})=1$, by Theorem 3.4 in [8], we have $\lambda_1=\lambda=1$, by Equation (27), we have $L^*_{1,r,w}x_1-\lambda_1x_1=0$, i.e., $L^*_{1,r,w}x_1=\lambda_1x_1$. By Lemma 2.2, we get $\rho(L^*_{1,r,w})=\rho(L_{1,r,w})=\rho(L_{r,w})=1$;

(c) If $\lambda=\rho(L_{r,w})>1$, by Theorem 3.4 in [8], we have $\lambda_1>\lambda>1$, by Equation (27), we have $L^*_{1,r,w}x_1-\lambda_1x_1\geq 0$ but is not equal to the null vector. Therefore $L^*_{1,r,w}x_1\geq\lambda_1x_1$. By Lemma 2.2, we get $\rho(L^*_{1,r,w})>\rho(L_{1,r,w})>\rho(L_{r,w})>1$.

Finally, We prove that the following comparison theorem for the preconditioned AOR method defined by Equation (13) with that defined by Equation (8).

Theorem 3.3 Let $A=(a_{ij})_{n\times n}$ be an L-matrix with $0<\beta_i a_{i,i-1}a_{i-1,i}<1$, $\beta_i a_{i,i-1}\neq 0$ and $0<\beta_i\leq 1$ for $2\leq i\leq n$. Assume that $a_{n1}\neq 0$, $0<\beta_1<\min\{\frac{1-\beta_n a_{n,n-1}a_{n-1,n}}{a_{1n}a_{n1}},1-\frac{\beta_n a_{n,n-1}a_{n-1,1}}{a_{n1}}\}$. If $0\leq r\leq w\leq$

$1(w \neq 0$ and $r \neq 1)$, then the following holds.
(a) $\rho(L^*_{2,r,w}) < \rho(L_{2,r,w}) < \rho(L_{r,w})$, if $\rho(L_{r,w}) < 1$;
(b) $\rho(L^*_{2,r,w}) = \rho(L_{2,r,w}) = \rho(L_{r,w})$, if $\rho(L_{r,w}) = 1$;
(c) $\rho(L^*_{2,r,w}) > \rho(L_{2,r,w}) > \rho(L_{r,w})$, if $\rho(L_{r,w}) > 1$.

Proof. By Lemma 2.5 and Theorem 3.1 in [9], it is clear that the matrices $L_{2,r,w}$ and $L^*_{2,r,w}$ are nonnegative and irreducible. So there is a positive Perron vector x_2, such that

$$L_{2,r,w}x_2 = \lambda_2 x_2, \tag{29}$$

where $\lambda_2 = \rho(L_{2,r,w})$. By Equation (29), we have $[(1-w)D_2 + (w-r)L_2 + wU_2]x_2 = \lambda_2(D_2 - rL_2)x_2$, i.e.,

$$[(1-w-\lambda_2)D_2 + (w-r+\lambda_2 r)L_2 + wU_2]x_2 = 0. \tag{30}$$

Using Equations (E9), (E10), (13) and (30), we obtain

$$\begin{aligned}
& L^*_{1,r,w}x_2 - \lambda_2 x_2 \\
=\ & (D^*_2 - rL^*_2)^{-1}[(1-w)D^*_2 + (w-r)L^*_2 + wU^*_2 \\
& -\lambda_2(D^*_2 - rL^*_2)]x_2 \\
=\ & (D^*_2 - rL^*_2)^{-1}[(1-w-\lambda_2)D^*_2 \\
& +(w-r+\lambda_2 r)L^*_2 + wU^*_2]x_2 \\
=\ & (D^*_2 - rL^*_2)^{-1}[(1-w-\lambda_2)(D_2 - \grave{D}_2) \\
& +(w-r+\lambda_2 r)(L_2 - \hat{S}^* + \grave{L}_2) + wU_2]x_2 \\
=\ & (D^*_2 - rL^*_2)^{-1}[(\lambda_2 - 1 + w)\grave{D}_2 \\
& +(w-r+\lambda_2 r)(\grave{L}_2 - \hat{S}^*)]x_2 \\
=\ & (D^*_2 - rL^*_2)^{-1}[(\lambda_2 - 1)\grave{D}_2 + r(\lambda_2 - 1)(\grave{L}_2 - \hat{S}^*) \\
& +w(\grave{D}_2 + \grave{L}_2 - \hat{S}^*)]x_2 \\
=\ & (D^*_2 - rL^*_2)^{-1}[(\lambda_2 - 1)\grave{D}_2 + r(\lambda_2 - 1)(\grave{L}_2 - \hat{S}^*) \\
& +w(\hat{S}^*U - \hat{S}^*)]x_2 \\
=\ & (D^*_2 - rL^*_2)^{-1}[(\lambda_2 - 1)\grave{D}_2 + r(\lambda_2 - 1)(\grave{L}_2 - \hat{S}^*) \\
& +w\hat{S}^*(U - I)]x_2.
\end{aligned}$$

Notice that $\hat{S}^*U = \hat{S}^*U_2,\ \hat{S}^* = \hat{S}^*I = \hat{S}^*D_2,\ \hat{S}^*L_2 = 0$, and by Equation (30), we have $w(L_2 + U_2 - D_2)x_2 = (\lambda_2 - 1)(D_2 - rL_2)x_2$, which yields that

$$\begin{aligned}
w\hat{S}^*(U-I)x_2 &= w\hat{S}^*(L_2 + U_2 - D_2)x_2 \\
&= (\lambda_2 - 1)\hat{S}^*(D_2 - rL_2)x_2.
\end{aligned}$$

Then, we have

$$\begin{aligned}
& L^*_{2,r,w}x_2 - \lambda_2 x_2 \\
=\ & (D^*_2 - rL^*_2)^{-1}[(\lambda_2 - 1)\grave{D}_2 + r(\lambda_2 - 1)(\grave{L}_2 - \hat{S}^*) \\
& +w\hat{S}^*(U - I)]x_2 \\
=\ & (D^*_2 - rL^*_2)^{-1}[(\lambda_2 - 1)\grave{D}_2 + r(\lambda_2 - 1)(\grave{L}_2 - \hat{S}^*) \\
& +(\lambda_2 - 1)\hat{S}^*(D_2 - rL_2)]x_2 \\
=\ & (\lambda_2 - 1)(D^*_2 - rL^*_2)^{-1}[\grave{D}_2 + r(\grave{L}_2 - \hat{S}^*) \\
& +\hat{S}^*(D_2 - rL_2)]x_2 \\
=\ & (\lambda_2 - 1)(D^*_2 - rL^*_2)^{-1}[\grave{D}_2 + r(\grave{L}_2 - \hat{S}^*) + \hat{S}^*D_2]x_2 \\
=\ & (\lambda_2 - 1)(D^*_2 - rL^*_2)^{-1}[\grave{D}_2 + r(\grave{L}_2 - \hat{S}^*) + \hat{S}^*]x_2 \\
=\ & (\lambda_2 - 1)(D^*_2 - rL^*_2)^{-1}[\grave{D}_2 + r\grave{L}_2 + (1-r)\hat{S}^*]x_2. \qquad (31)
\end{aligned}$$

Since $\hat{S}^* \geq 0$ and $\hat{S}^* \neq 0$, $\grave{D}_2 \geq 0$, $\grave{L}_2 \geq 0$, $0 \leq r \leq w \leq 1(w \neq 0, r \neq 1)$, $[\grave{D}_2 + r\grave{L}_2 + (1-r)\hat{S}^*]x_2 \geq 0, [\grave{D}_2 + r\grave{L}_2 + (1-r)\hat{S}^*]x_2 \neq 0$. By Lemma 2.5 and similar to the proof of Lemma 2.4, we can obtain that D^*_2 is a positive diagonal matrix and $L^*_2 \geq 0$, then by Lemma 2.6, we obtain that $(D^*_2 - rL^*_2)$ is a nonsingular M-matrix, so by Lemma 2.7 we have $(D^*_2 - rL^*_2)^{-1} \geq 0$ and $((D^*_2 - rL^*_2)^{-1})_{ii} > 0$ for $i = 1, 2, \cdots, n$, so we have $(D^*_2 - rL^*_2)^{-1}[\grave{D}_2 + r\grave{L}_2 + (1-r)\hat{S}^*]x_2 \geq 0, (D^*_2 - rL^*_2)^{-1}[\grave{D}_2 + r\grave{L}_2 + (1-r)\hat{S}^*]x_2 \neq 0$.

(a) If $0 < \lambda < 1$, by Theorem 3.1 in [9], we have $\lambda_2 < \lambda < 1$, by Equation (31), we have $L^*_{2,r,w}x_2 - \lambda_2 x_2 \leq 0$ but is not equal to the null vector. Therefore $L^*_{2,r,w}x_2 \leq \lambda_2 x_2$. By Lemma 2.2, we get $\rho(L^*_{2,r,w}) < \rho(L_{2,r,w}) < \rho(L_{r,w}) < 1$;

(b) If $\lambda = \rho(L_{r,w}) = 1$, by Theorem 3.1 in [9], we have $\lambda_2 = \lambda = 1$, by Equation (31), we have $L^*_{2,r,w}x_2 = \lambda_2 x_2$. By Lemma 2.2, we get $\rho(L^*_{2,r,w}) = \rho(L_{2,r,w}) = \rho(L_{r,w}) = 1$;

(c) If $\lambda = \rho(L_{r,w}) > 1$, by Theorem 3.1 in [9], we have $\lambda_2 > \lambda > 1$, by Equation (31), we have $L^*_{2,r,w}x_2 - \lambda_2 x_2 \geq 0$ but is not equal to the null vector. Therefore $L^*_{2,r,w}x_2 \geq \lambda_2 x_2$. By Lemma 2.2, we get $\rho(L^*_{2,r,w}) > \rho(L_{2,r,w}) > \rho(L_{r,w}) > 1$.

From Theorems 3.2-3.3, it can be seen that the rates of convergence of the preconditioned AOR methods defined by (12) and (13) are better than those defined by (7) and (8) under some conditions, respectively, whenever the latter ones are convergent.

IV. Numerical examples

In this section, we provide numerical experiments to illustrate the theoretical results in Section III.

Example 4.1 This example is a numerical example introduced in [6]. The coefficient matrix A of (1) is given by:

$$A = \begin{pmatrix}
1 & -\frac{1}{n\times 1100} & \cdots & -\frac{1}{3\times 1100} & -\frac{1}{22} \\
-\frac{1}{n\times 10+1} & 1 & \cdots & -\frac{1}{(n-1)\times 10+2} & -\frac{1}{n\times 10+2} \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
-\frac{1}{3\times 10+1} & -\frac{1}{(n-2)\times 10+(n-1)} & \cdots & 1 & -\frac{1}{n\times 10+(n-1)} \\
-\frac{100}{7} & -\frac{1}{(n-1)\times 10+n} & \cdots & -\frac{1}{2\times 10+n} & 1
\end{pmatrix}.$$

Table I displays the spectral radii of the corresponding iteration matrices with some randomly chosen parameters w, r, α_1, α_2, β_1, β_2, and parameters α_1, α_2, β_1, β_2 satisfy the conditions of Theorem 3.1, where $\rho = \rho(L_{r,w})$, $\tilde{\rho} = \rho(\tilde{L}_{r,w})$, $\bar{\rho} = \rho(\bar{L}_{r,w})$. All numerical experiments were carried out using Matlab 6.5 with $\alpha_1 = \alpha_2$.

TABLE I
Spectral radii of the AOR and preconditioned AOR iteration matrices

n	w	r	$\alpha_1 = \alpha_2$	β_1	β_2	ρ	$\tilde{\rho}$	$\bar{\rho}$
10	0.9	0.85	100	-14.142857	-0.0449	0.7250	0.1698	0.1681
15	0.9	0.8	100	-14.142857	-0.0449	0.7350	0.1905	0.1891
20	0.95	0.7	50	-13.999999	-0.0444	0.7387	0.1726	0.1711
30	0.95	0.85	200	-14.214285	-0.0451	0.7090	0.1584	0.1579

From Table I, it can be seen that $\bar{\rho} < \tilde{\rho} < \rho$ when $\rho < 1$. These numerical results are in accordance with the theoretical results in Theorem 3.1 (i).

Example 4.2 This example is a numerical example introduced in [7] and [9]. The coefficient matrix A of (1) is given by:

$$A = \begin{pmatrix}
1 & -0.2 & -0.3 & -0.1 & -0.2 \\
-0.1 & 1 & -0.1 & -0.3 & -0.1 \\
-0.2 & -0.1 & 1 & -0.1 & -0.2 \\
-0.2 & -0.1 & -0.1 & 1 & -0.3 \\
-0.1 & -0.2 & -0.2 & -0.1 & 1
\end{pmatrix}.$$

Table II and Table III display the spectral radii of the corresponding iteration matrices with some randomly chosen parameters w, r, where $\rho = \rho(L_{r,w})$, $\rho_2 = \rho(L_{2,r,w})$, $\rho_2^* = \rho(L_{2,r,w}^*)$. All numerical experiments were carried out using Matlab 6.5 with $\beta_1 = 1$, $\beta_2 = 0.1$, $\beta_3 = 0.2$, $\beta_4 = 0.1$, $\beta_5 = 0.3$ in Table II and $\beta_1 = 1.1$, $\beta_2 = 1$, $\beta_3 = 1$, $\beta_4 = 1$, $\beta_5 = 1$ in Table III, respectively.

TABLE II
SPECTRAL RADII OF THE AOR AND PRECONDITIONED AOR ITERATION MATRICES

w	r	ρ	ρ_2	ρ_2^*
1	0	0.6551	0.6480	0.6370
0.2	0.1	0.9288	0.9274	0.9251
0.3	0.2	0.8896	0.8876	0.8839
0.6	0.5	0.7531	0.7495	0.7408
0.6	0.6	0.7423	0.7389	0.7295
0.8	0.7	0.6400	0.6358	0.6223
0.9	0.5	0.6297	0.6242	0.6112

From Tables II-III, it can be seen that $\rho_2^* < \rho_2 < \rho$ when $\rho < 1$. These results are in accordance with the theoretical results in Theorem 3.3 (a).

TABLE III
SPECTRAL RADII OF THE AOR AND PRECONDITIONED AOR ITERATION MATRICES

w	r	ρ	ρ_2	ρ_2^*
1	0	0.6551	0.6205	0.6043
0.2	0.1	0.9288	0.9221	0.9187
0.3	0.2	0.8896	0.8798	0.8746
0.6	0.5	0.7531	0.7395	0.7241
0.6	0.6	0.7423	0.7262	0.7139
0.8	0.7	0.6400	0.6203	0.6030

Example 4.3 [10] The coefficient matrix A of (1) is given by:

$$A = \begin{pmatrix} 1 & q & r & s & q & \cdots \\ s & 1 & q & r & \ddots & q \\ q & s & 1 & q & \ddots & s \\ r & q & s & 1 & \ddots & r \\ s & \ddots & \ddots & \ddots & \ddots & q \\ \cdots & s & r & q & s & 1 \end{pmatrix},$$

where $q = -p/n$, $r = -p/(n+1)$, $s = -p/(n+2)$. For $n = 6$ and $p = 1$, Table IV displays the spectral radii of the corresponding iteration matrix with some randomly chosen parameters w, r, where $\rho = \rho(L_{r,w})$, $\rho_1 = \rho(L_{1,r,w})$, $\rho_1^* = \rho(L_{1,r,w}^*)$. All numerical experiments were carried out using Matlab 6.5 with $\alpha_1 = 0.9$, $\alpha_2 = 0.9$, $\alpha_3 = 0.9$, $\alpha_4 = 0.9$, $\alpha_5 = 1$.

TABLE IV
SPECTRAL RADII OF THE AOR AND PRECONDITIONED AOR ITERATION MATRICES

w	r	ρ	ρ_1	ρ_1^*
0.9	0.8	0.6519	0.5849	0.5773
0.95	0.8	0.6325	0.5618	0.5538
0.8	0.7	0.7083	0.6554	0.6481
0.7	0.65	0.7517	0.7078	0.7012

From Table IV, it can be seen that $\rho_1^* < \rho_1 < \rho$ when $\rho < 1$. These numerical results are in accordance with the theoretical results in Theorem 3.2 (a).

V. CONCLUSIONS

In this paper, we proposed three new preconditioners and studied the convergence rates of the new preconditioned AOR methods for solving linear system (1). Comparison results given in Section III show that the new preconditioned AOR methods are better than those proposed by Wang *et al.* [6], J. H. Yun [8] and A. J. Li [9] whenever these methods are convergent. It can also be seen that all the numerical examples given in Section IV are consistent with the theoretical results given in Section III (Tables I-IV).

It would be nice if we can find optimal values of r and w for which the convergence rate of the new preconditioned AOR methods is best. Moreover, by these numerical examples, we found that the relations among $\bar{\rho}$, ρ_1^*, ρ_2^* are not only related to the matrix, but also the seletion of parameters α_i, β_i for $i = 1, 2, \cdots, n$ in the preconditioners $\bar{P}$, P_1^* and P_2^*. And it is difficult and complicated to find the conditions. Future work will include numerical or theoretical studies for finding the optimal values of r and w and the optimal parameters of α_i, β_i for $i = 1, 2, \cdots, n$.

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