A Condition Based Maintenance for System Subject to Competing Failure due to Degradation and Shock

Wenping Huang, Jinglun Zhou, and Juhong Ning

Abstract—**Degradation and shock are two common mechanisms accounting for system failure. In this paper, we assume that the system experiences both continuous smooth degradation process and shock process, and the dependence of them is that a higher degradation level will lead to a higher probability of traumatic failure, which is caused by shock. After system reliability model is obtained, a condition based maintenance model is developed. The goal of the optimal maintenance scheme is to minimize average long run cost rate by properly choosing the preventive deterioration level and the length of an inspection cycle. A numerical example is provided to illustrate the application of the model, and the sensitivity analysis about system parameters has been discussed .**

Index Terms—**Competing failure, degradation failure, random shock, traumatic failure, condition based maintenance**

I. Introduction

With the development of modern science and manufacturing technology, there are more and more products have the characters of high reliability. As a result, it will cost too much of time or cost to fail a defective product even under highly accelerate environments. In this case, if there exits some quality characteristic that have correlation with reliability, the reliability of a product can be analyzed by the degradation data. There are several methods that can be used to model the degradation data. Some of these models are general degradation path models, Markov models, and continuous-time stochastic processes.

The general degradation path model is a regression model with random or fixed coefficients fitted to the degradation observations. Both linear and nonlinear models are used to model degradation. Lu and Meeker [1] introduced a nonlinear mixed-effect model and used a two-stage method to obtain the system reliability. Yuan and Pandey [2] developed an advanced nonlinear mixed effect degradation model for unbalanced degradation inspection data. This model provided improved degradation prediction by reducing the variance associated with the degradation of each unit. Haghighi and Nikulin [3] employed parametric and non-parametric methods to estimate the survival function and its parameters for a system with multiple conditionally independent failure modes where the degradation path was in form of linear multiplicative function.

Continuous-time models or continuous-time Markov processes are helpful to model the continuous stochastic degradation processes. Gamma processes, compound Poisson processes, and Wiener processes are the typical models of this type model. Lawless and Crowder [4] developed a random effect gamma process with covariates to capture the heterogeneity of the degradation path. Guo and Tan [5] updated the parameter estimates of a gamma process using the Bayesian approach. Wang [6] developed a nonparametric method namely pseudo-likelihood to estimate the unknown parameters of a non-stationary gamma process.

Barker and Newby [7] developed an optimal inspection policy for a multi-component system where the degradation path of each component was modeled by Wiener process. Nicolai et al. [8] used three different stochastic models to estimate the reliability of the organic coating systems protecting steel structures, and compared them by different criteria. Wang [9] modeled the degradation level of bridge beams by a Wiener process with random drift and diffusion parameters. He also used the maximum likelihood estimation to estimate the associated parameters. Van Noortwijk [10] introduced the successful maintenance applications of gamma processes, the statistical properties of the gamma process, methods for estimation, approximation, and simulation of gamma processes were also reviewed.

Besides degradation process, shock is also an important reason accounting for system failure. Many researchers have focused on studying shock models. Esary and Marshall [11] discussed the relation of the continuous life distribution and the discrete failure distribution P_k of not surviving the first shock. Shanthikumar and Sumita [12] extended Poisson shock to a general shock, and the reliability of the system was obtained. Cha and Finkelstein [13] considered some new classes of extreme shocks, and then the models survival probabilities and some corollaries were obtained. Li and Kong [14] generalized δ-shock model, the results of survival function and some asymptotic properties were given. Eryilmaz [15] developed a generalized run-related δ-shock

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Wenping Huang and Jinglun Zhou are with the School of Information System and Management National University of Defense Technology of China, Changsha, Hunan 410073, China (phone: +86-15270807922; fax: +86-15270807922; e-mail: jxnuhwp1@ 163.com, zjl@163.com).

Juhong Ning is with the College of Mathematics and Information Science, Jiangxi Normal University, Nanchang, Jiangxi 330027, China (e-mail: ningjuh@163.com).

model in which the systems failed when the inter-arrival time of k consecutive shocks was less than certain threshold δ.

There has also been a growing interest in considering the maintenance optimization with dependent competing risks in recent years. Klutke and Yang [16] studied the average availability of maintained systems subject to shocks and graceful degradation with hidden failures. Wang and Pham [17] assumed that systems were subject to multiple degradation processes and random shock process, the dependence of different degradation process is fitted by the copula method, and the time-varying technique is used to modulate the relationship between the shock process and the degradation processes. In order to give a more explicit dependent relationship, Chen [18] used the degradation level as a variable of the arrival rate function of the fatal shock, and an inspection/replacement policy was discussed based on the proposed model. Castro [19] developed a dependent relationship for two competing failure models in which the non-maintainable failure number affects the maintainable failure rate. The optimal number of preventive maintenance and the interval between successive preventive maintenance are determined with the objective of minimizing the expected cost rate. Zequeira and Bérenguer [20] studied the imperfect maintenance policies with the consideration of two competing failure modes, where the hazard rate of the maintainable failure mode depended on the hazard rate of the non-maintainable failure mode. Deloux et al. [21] considered a system with two failure mechanisms due to an excessive deterioration level and a shock. The optimal maintenance strategy was studied in an approach that combined statistical process control and condition-based maintenance. Peng et al. [22] presented a preventive maintenance policy for systems subjected to multiple competing failures where the external random shock contributes to the internal degradation. More discussion about condition based maintenance can be reference as [23-25].

The dependence assumption of the above papers mainly concentrates on the shock process increases the degradation process. However, in many practical situations, the shock process is affected by the degradation level. Huynh et al. [26] considered this case on the assumption that the shock arrival times were influenced by the degradation level. That is,

$$\lambda(X(t))=\begin{cases}\lambda_1(t) & X(t)\le M_s,\\ \lambda_2(t) & X(t)>M_s.\end{cases} \tag{1}$$

Where $\lambda(X(t))$ is traumatic failure rates, $\lambda_1(t)$ and $\lambda_2(t)$ denote two continuous and non-decreasing failure rates at time t with $\lambda_1(t)\le \lambda_2(t)$, M_s represents a fixed deterioration level. Then, a reliability model and condition based maintenance model were discussed. Under the condition of (1), $\lambda(X(t))$ was affected by the degradation level only at the point of $X(t)=M_s$. However, in actual situation, the shock arrival times generally are not affected by the degradation level. When the system is subjected to a shock, it is more prone to failure with the increasing of the system degradation level. In this paper, under the assumption of the above case, we develop the competing failure reliability model and discuss the condition based maintenance optimization.

The remainder of this paper is organized as follows. In section II, we briefly introduce the system assumptions and present the system reliability model of the dependent competing failure, where degradation process is assumed the gamma processes, and traumatic failure is adopted by the shock processes. We then provide a condition based maintenance model, and derive the minimal average long run maintenance cost rate in section III. In section IV, an illustrative example is provided to elaborate on the benefits of our model.

II. Reliability Model

A. System Assumption

In this paper, we consider a system subject to two dependent failure processes. The following assumptions are made. The rationale of each assumption is explained in the corresponding sections.

1. We use the gamma process to model the degradation process. When the degradation level exceeds the failure threshold, the system is regarded as failure. We call this kind of failure as degradation failure.
2. Random shock arrival process follows a homogeneous Poisson process with arrival rate λ . The shock, may result in the system fails immediately, We call this kind of failure as traumatic failure, or it has no harm to the system.
3. The degradation process and random shock process are dependent. For an arriving random shock, a higher degradation level will lead to a higher probability of traumatic failure.
4. The system is inspected periodically, in each inspection time point, if the system fails, it will be replaced with a new one. Otherwise, the degradation level is measured, if the degradation level exceeds some threshold M , the system will be replaced by a new one; or the system will be run continuously without any maintenance action until next periodical inspection time.
5. If the system fails, it remains idle and no maintenance actions are taken until next scheduled inspection.
6. The measurement and replacement are assumed to be instantaneous, perfect and non-destructive.

B. Modeling for Degradation Failure

In many practical engineer situations, the system deterioration often possesses the characteristic of gradual damage monotonically accumulating over time in a sequence of tiny increments. In addition, temporal variability must be taken into account during the system degradation process. Gamma process is a stochastic process with independent increments, and it is the best model for monotonic and gradual deterioration process [27-29], so we use the gamma process to model the degradation process.

Let $X(t)$ denote the degradation level of the gamma process at time t with shape parameter α , scale parameter β , which has the following properties:

1) $X(0)=0$ with the probability one;

2) $X(t)$ has independent increment;

3) $X(t)-X(v)\sim Ga(\alpha(t-v),\beta)$ for all $t>v\geq 0$.

Where $Ga(x\mid\alpha(t-v),\beta)$ is the gamma distribution, and the probability density function is given by

$$f_{\alpha(t-v),\beta}(x)=\frac{\beta^{\alpha(t-v)}}{\Gamma(\alpha(t-v))}x^{\alpha(t-v)-1}e^{-\beta x},x\geq 0\,, \tag{2}$$

where $\Gamma(\alpha)=\int_0^\infty u^{\alpha-1}e^{-u}du$ .the expectation of the gamma process is $E(X(t))=\frac{\alpha}{\beta}$, and its variance is $\mathrm{Var}(X(t))=\frac{\alpha}{\beta^2}$.

For $X(t)$ is continuous and monotonically, the system fails if $X(t)$ exceeds the threshold L. Let T_d be the failure time of degradation process, the lifetime distribution only considering with degradation process can then be given by

$$\begin{aligned}F_d(t)&=P(T_d<t)=P(X(t)>L)\\&=\int_L^\infty f_{\alpha t,\beta}(x)dx=\frac{\Gamma(\alpha t,L\beta)}{\Gamma(\alpha t)},\end{aligned} \tag{3}$$

where $\Gamma(\alpha,x)=\int_x^\infty z^{\alpha-1}e^{-z}dz$, denotes the incomplete gamma function for $x\geq 0$ and $\alpha>0$.

The probability density function of the T_d is

$$f_d(t)=\frac{\partial F_d(t)}{\partial t}=\frac{\alpha}{\Gamma(\alpha t)}\int_{L\beta}^\infty\left(\log(z)-\psi(\alpha t)\right)z^{\alpha t-1}e^{-z}dz\,, \tag{4}$$

where $\psi(\alpha)$ is the derivative of the logarithm of the gamma function

$$\psi(\alpha)=\frac{\Gamma'(\alpha)}{\Gamma(\alpha)}=\frac{\partial\log\Gamma(\alpha)}{\partial\alpha}.$$

The reliability function of the system only considering the degradation process is

$$R_d(t)=1-F_d(t)=1-\frac{\Gamma(\alpha t,L\beta)}{\Gamma(\alpha t)}. \tag{5}$$

C. Modeling for Traumatic Failures

In engineering applications, shock is common cause of products failure when the products are under the environments of external shock, which has been extensively studied by many authors. In the literatures, random shock models are mainly classified by six categories [30].

The degradation process and shock process are dependent . However, Most of the proposed models assume that the shock process is independent of the degradation process. In practical situations, with the degradation process, the higher the degradation level, the more the system is vulnerable to the shocks.

In this paper, we assume that the shock arrival process follows a homogeneous Poisson process with arrival rate λ, let $N(t)$ denote the number of random shocks that have arrived by time *t*. According to the stochastic process theory, we have follow expression

$$P(N(t)=n)=\frac{e^{-\lambda t}\left(\lambda t\right)^n}{n!},\quad n=0,1,2,\cdots.$$

According to the assumption 2, when there is the arriving of a random shock, it will make system fail immediately or no effect on system. Let $p(t)$ denote the probability of each random shock causes the system fail at time *t*. it is clearly that $p(t)$ is increasing with the degradation level raising. That is, the $p(t)$ is affected by the degradation level $X(t)$, so we use $p(X(t))$ to denote the traumatic failure probability under the condition of degradation level $X(t)$. We assume that $p(X(t))=1-b\exp(-ax(t))$, where $X(t)$ is the system degradation level at time *t*, a,b are constant coefficients with $a>0$, $0<b<1$, which can be determined by the analysis of the system failure physics or the experiment.

We then let $N_1(t)$ denote the number of random shock that cause the system traumatic failure at time *t*, $N_2(t)$ denote the number of random shock that have no effect on the system at time *t*. Based on the decomposition theory of Poisson process [31], the two processes are independent homogeneous Poisson process, $\lambda p(x(t))$ and $\lambda(1-p(x(t)))$ are their arrival rate respectively.

Let T_t be the traumatic failure time, the reliability function of a system is only subject to the extreme shock process can be derived as

$$\begin{aligned}&R_t(t)\\&=\int_0^\infty P(T_t>t\mid X(t)=x)f_{\alpha t,\beta}(x)dx\\&=\int_0^\infty P(N_1(t)=0X(t)=x)\frac{\beta^{\alpha t}}{\Gamma(\alpha t)}x^{\alpha t-1}e^{-\beta x}dx\\&=\int_0^\infty\exp\left(-\int_0^t\lambda p(x)wdw\right)\frac{\beta^{\alpha t}}{\Gamma(\alpha t)}x^{\alpha t-1}e^{-\beta x}dx\\&=\int_0^\infty\exp\left(-\int_0^t\lambda(1-b\exp(-ax))wdw\right)\frac{\beta^{\alpha t}}{\Gamma(\alpha t)}x^{\alpha t-1}e^{-\beta x}dx\,.\end{aligned} \tag{6}$$

The lifetime distribution function of the system only considering the random shock process is

$$\begin{aligned}&F_t(t)=1-R_t(t)\\&=1-\int_0^\infty\exp\left(-\int_0^t\lambda(1-b\exp(-ax))wdw\right)\frac{\beta^{\alpha t}}{\Gamma(\alpha t)}x^{\alpha t-1}e^{-\beta x}dx\end{aligned}. \tag{7}$$

D. System Reliability Model

When the fatal shock arrives or the degradation level is beyond the threshold L, the system fails immediately. Denote by $T_s=\min\{T_d,T_t\}$ the system failure time. Then the system reliability function is

$$\begin{aligned} R(t) &= P(T_s > t) \\ &= P(T_t > t, T_d > t) \\ &= P(N_1(t) = 0, X(t) < L) \\ &= \int_0^L P(N_1(t) = 0 \mid X(t) = x) f_{\alpha t,\beta}(x)dx . \end{aligned}$$

Applying (2) and (6), we have

$$R(t) = \int_0^L \exp\left(-\int_0^t \lambda(1-b\exp(-ax))wdw\right)\frac{\beta^{\alpha t}}{\Gamma(\alpha(t))}x^{\alpha t-1}e^{-\beta x}dx$$
$$= \frac{\beta^{\alpha t}}{\Gamma(\alpha(t))}\int_0^L \exp\left(-\int_0^t \lambda(1-b\exp(-ax))wdw\right)x^{\alpha t-1}e^{-\beta x}dx . \quad (8)$$

When the system degradation level exceeds the failure threshold L, but no fatal shock happens until time t, the degradation failure happens. In this case, the failure probability distribution is given by

$$\begin{aligned} F_{T_d}(t) &= \int_0^t R_t(u)dF_d(u) \\ &= \int_0^t \left[1-\exp\left(-\int_0^t \lambda(1-b\exp(-ax))wdw\right)\right] \\ &\quad \cdot\frac{\alpha}{\Gamma(\alpha u)}\int_{L\beta}^{\infty}\left(\log(z)-\psi(\alpha u)\right)z^{\alpha u-1}e^{-z}dzdu. \end{aligned} \quad (9)$$

Another special case of the problem under study is when there is a fatal shock, but the system degradation level stays below the failure threshold L until time t. In this case, the failure probability distribution is given by

$$\begin{aligned} F_{T_s}(t) &= \int_0^t R_d(u)dF_t(u) \\ &= \int_0^t \left[1-\frac{\Gamma(\alpha u, L\beta)}{\Gamma(\alpha u)}\right]\cdot \\ &\quad d\left[\int_0^\infty \exp\left(-\int_0^u \lambda(1-b\exp(-ax))wdw\right)\frac{\beta^{\alpha u}}{\Gamma(\alpha u)}x^{\alpha u-1}e^{-\beta x}dx\right] \end{aligned}$$
(10)

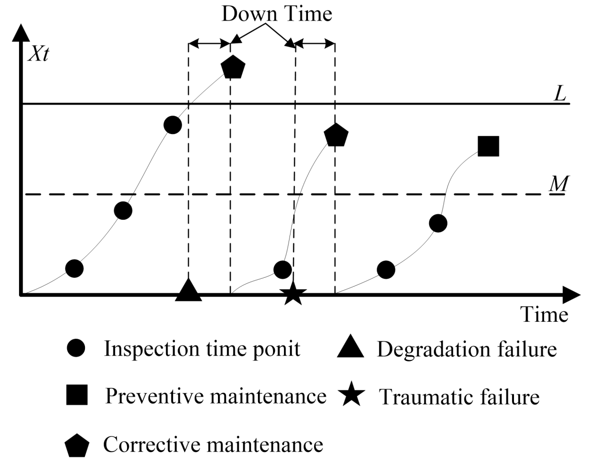

Fig. 1. Condition based maintenance model for the competing failure.

III. MAINTENANCE MODEL AND OPTIMIZATION

Fig. 1 illustrates a sample behavior of the condition based maintenance model. In the figure, the system degradation level exceeds degradation failure threshold L in the fourth inspection cycle, then it is acted by corrective maintenance. The system fails due to the random shock in the sixth inspection cycle. The system is replaced for its degradation level exceeds the preventive maintenance threshold M although it is not exceeds the degradation failure threshold L in the ninth inspection cycle, the rest of other 6 inspection cycle is adopted no any action to the system.

There are many optimal maintenance policies, such as the minimal average long run maintenance cost rate, the maximal system availability and the maximal mean residual life etc [32]. In this paper, we use the average long run maintenance cost rate model.

We define a replacement cycle as the time interval between two consecutive replacements, which is denoted by τ_r. Let $C(t)$ be the cumulative maintenance cost until time t. Based on the renewal theory [33], we have the average long run maintenance cost rate as

$$\lim_{t\to\infty}\frac{C(t)}{t} = \frac{E(TC)}{E(\tau_r)},$$

where $E(TC)$ is expected maintenance cost incurred in a replacement cycle, $E(\tau_r)$ is the expected length of a replacement cycle.

The optimal objective of maintenance model is to minimize the average long run maintenance cost rate, which is determined by the decision variables periodic interval T and the preventive maintenance threshold M.

The maintenance cost in a replacement cycle includes the replacement cost, inspection cost and the cost due to the system idle during the system failure period. In addition, we assume that the cost of preventive replacement and the cost of corrective replacement are different. Then the expected maintenance costs incurred in a cycle can be expressed as

$$E(TC) = C_pP_p + C_cP_c + C_dE[W_r] + C_iE[R], \quad (11)$$

where C_p is the cost of a preventive replacement; C_c is the cost of a corrective replacement; C_d is the unit cost of system inactivity; C_i is the cost of each inspection; P_p is the probability of a preventive replacement in a replacement cycle; P_c is the probability of a corrective replacement in a replacement cycle. $E[W_r]$ is the expected system down time in a replacement cycle; $E[R]$ is the expected inspection number in a replacement cycle.

For the complexity of the derivation $E(TC)$ and $E(\tau_r)$, in the following subsections, we first calculate the preventive maintenance probability P_p and according corrective maintenance probability P_c, expected down time $E[W_r]$ and expected length of a replacement cycle respectively.

A. Preventive Maintenance Probability

Just as shown in Fig 1, the preventive maintenance action can only be taken in scheduled inspection time point *k*T. Let $P_p((k+1)T)$ denote the preventive maintenance probability of the (*k*+1)th (*k*=0,1,2,⋯) inspection cycle,

As the preventive maintenance probability P_p is the sum of the different inspection cycle preventive maintenance probability $P_p((k+1)T)$, so we have

$$P_p = \sum_{k=0}^{\infty} P_p((k+1)T) . \quad (12)$$

Now, we turn to analyze the expression of $P_p((k+1)T)$. We define the event $A((k+1)T)$ as

$$A((k+1)T) = \{X(kT) < M, M < X((k+1)T) < L, N_1((k+1)T) = 0\}$$

Then, we can see that the preventive maintenance action is taken in the (*k*+1)th inspection cycle only and if only the event $A((k+1)T)$ occurs. So can get $P_p((k+1)T)$ as

$$\begin{aligned} &P_p((k+1)T) \\ &= P(A(k+1)T) \\ &= P(X(kT) < M, M < X(k+1)T < L, N_1(k+1)T = 0) \\ &= P(M < X(k+1)T < L, N_1(k+1)T = 0 \mid X(kT) < M) \\ &\quad .P(X(kT) < M). \end{aligned}$$

Applying (2) and (6), we have

$$\begin{aligned} P_p((k+1)T) &= \int_0^M \frac{\beta^{\alpha kT}}{\Gamma(\alpha kT)} u^{\alpha kT-1} e^{-\beta u} du \\ &.\int_{M-u}^{L-u} \frac{\beta^{\alpha T}}{\Gamma(\alpha T)} x^{\alpha T-1} e^{-\beta x} \exp\left(-\lambda \int_0^{(K+1)T} (1-\exp(-ax)w)dw\right) dx . \end{aligned} \quad (13)$$

Then, equation (13) can be derived by the equation (12) directly.

B. Corrective Maintenance Probability

A corrective maintenance is performed at time (*k*+1)*T* as the system fails in (*k*+1)th inspection cycle. That is, the system suffers from fatal shock or the system degradation level first time exceeds the degradation failure threshold *L* in the time intervals [*kT*, (*k*+1)*T*], but there is no fatal shock before time *kT* and the degradation level is below *M* at inspection time *kT*. We can express the event as following two mutually exclusive events

$$B((k+1)T) = \{X(kT) < M, N_1(kT) = 0, N_1((k+1)T) > 0\} ,$$

and

$$C((k+1)T) = \{X(kT) < M, X((k+1)T) > L, N_1((k+1)T) = 0\}$$

.

Let $P_c((k+1)T)$ denote the corrective maintenance probability at time $(k+1)T$ for $k = 0,1,2,\cdots$. Then, we can get the $P_c((k+1)T)$ as

$$\begin{aligned} &P_c((k+1)T) \\ &= P(B((k+1)T) \cup C((k+1)T)) \\ &= P(B((k+1)T)) + P(C((k+1)T)). \end{aligned}$$

To get the expression of the $P_c((k+1)T)$, we firstly derive the following events probabilities.

$$\begin{aligned} &P(N_1(kT) = 0 \mid X(kT) = u) \\ &= \exp\left(-\lambda \int_0^{kT} (1-\exp(-ax)w)dw\right), \end{aligned}$$

which can be derived by (6) directly. Then,

$$\begin{aligned} &P(B((k+1)T)) \\ &= P\{X(kT) < M, N_1(kT) = 0, N_1((k+1)T) > 0\} \\ &= \int_0^M \exp\left(-\lambda \int_0^{kT} (1-\exp(-ax)w)dw\right) \frac{\beta^{\alpha kT}}{\Gamma(\alpha kT)} u^{\alpha kT-1} e^{-\beta u} du \\ &.\int_{L-u}^{\infty} \frac{\beta^{\alpha T}}{\Gamma(\alpha T)} x^{\alpha T-1} e^{-\beta x} \sum_{h=1}^{\infty} \frac{\left(\lambda \int_0^T (1-\exp(-ax)w)dw\right)^h}{h!} \\ &\quad .\exp(-\lambda \int_0^T (1-\exp(-ax)w)dw) \; dx, \end{aligned}$$

and

$$\begin{aligned} &P(C((k+1)T)) \\ &= P\{X(kT) < M, X((k+1)T) > L, N_1((k+1)T) = 0\} \\ &= \int_0^M \frac{\beta^{\alpha kT}}{\Gamma(\alpha kT)} u^{\alpha kT-1} e^{-\beta u} du. \\ &.\int_{L-u}^{\infty} \frac{\beta^{\alpha T}}{\Gamma(\alpha T)} x^{\alpha T-1} e^{-\beta x} \exp\left[\int_0^{(k+1)T} (1-\exp(-ax)w)dw\right] dx. \end{aligned}$$

Then, we can get

$$\begin{aligned} &P_c((k+1)T) \\ &= \int_0^M \exp\left(-\lambda \int_0^{kT} (1-\exp(-ax)w)dw\right) \frac{\beta^{\alpha kT}}{\Gamma(\alpha kT)} u^{\alpha kT-1} e^{-\beta u} du \\ &.\int_{L-u}^{\infty} \frac{\beta^{\alpha T}}{\Gamma(\alpha T)} x^{\alpha T-1} e^{-\beta x} \sum_{h=1}^{\infty} \frac{\left(\lambda \int_0^T (1-\exp(-ax)w)dw\right)^h}{h!} \\ &\quad .\exp(-\lambda \int_0^T (1-\exp(-ax)w)dw) \; dx \\ &+ \int_0^M \frac{\beta^{\alpha kT}}{\Gamma(\alpha kT)} u^{\alpha kT-1} e^{-\beta u} du \\ &.\int_{L-u}^{\infty} \frac{\beta^{\alpha T}}{\Gamma(\alpha T)} x^{\alpha T-1} e^{-\beta x} \exp\left[\int_0^{(k+1)T} (1-\exp(-ax)w)dw\right] dx \end{aligned} \quad (14)$$

Now，the corrective maintenance probability is

$$P_c = \sum_{k=0}^{\infty} P_c((k+1)T) . \quad (15)$$

Where $P_c((k+1)T)$ is given by (14)

C. Expected Down Time in a Replacement Cycle

When the system fails in the time intervals [kT, $(k+1)T$], there are two kinds of failure: degradation failure caused by degradation process and traumatic failure caused by random shock. Let $W_d((k+1)T)$ be the expected system down time in the $(k+1)$th inspection cycle [kT, $(k+1)T$] caused by degradation failure, $W_s((k+1)T)$ be the expected system down time in the $(k+1)$th inspection cycle [kT, $(k+1)T$] caused by traumatic failure.

Then $W_d((k+1)T)$ can be expressed as

$$W_d((k+1)T) = \int_0^M \frac{\beta^{\alpha kT}}{\Gamma(\alpha kT)} u^{\alpha kT-1} e^{-\beta u} du \,.\int_{kT}^{(k+1)T} ((k+1)T-t) dF_{T_d}(t),$$

where $F_{T_d}(t)$ is given by (9).

$W_s((k+1)T)$ can be expressed as

$$W_s((k+1)T) = \int_0^M \frac{\beta^{\alpha kT}}{\Gamma(\alpha kT)} u^{\alpha kT-1} e^{-\beta u} du \,.\int_{kT}^{(k+1)T} ((k+1)T-t) dF_{T_s}(t),$$

where $F_{T_s}(t)$ is given by (10).

Now, we can get the expectation of the system down time as

$$\begin{aligned} E[W_r] &= \sum_{k=0}^{\infty} \left[W_d((k+1)T) + W_s((k+1)T) \right] \\ &= \int_0^M \frac{\beta^{\alpha kT}}{\Gamma(\alpha kT)} u^{\alpha kT-1} e^{-\beta u} du \,.\int_{kT}^{(k+1)T} ((k+1)T-t) dF_{T_d}(t) \\ &+ \int_0^M \frac{\beta^{\alpha kT}}{\Gamma(\alpha kT)} u^{\alpha kT-1} e^{-\beta u} du \,.\int_{kT}^{(k+1)T} ((k+1)T-t) dF_{T_s}(t). \end{aligned} \quad (16)$$

D. Expected Number of Inspections in a Replacement Cycle

If the system is performed preventive maintenance or corrective maintenance in the kth inspection cycle, then the system inspection time length is kT. Therefore, the expected length of a cycle is

$$E(\tau_r) = \sum_{k=1}^{\infty} kT(P_p(kT) + P_c(kT)). \quad (17)$$

Where $P_p(kT)$ and $P_c(kT)$ are given by (7) and (8) respectively.

Considering the expected number of inspections in a replacement cycle $E[R]$ is given by

$$E[R] = \frac{E(\tau_r)}{T}. \quad (18)$$

Let $C(T,M)$ be the average long run maintenance cost rate. Applying (11) and (12), we can get

$$C(T,M) = \frac{C_p P_p + C_c P_c + C_d E[W_r] + C_i E[R]}{E(\tau_r)}. \quad (19)$$

Where $P_p, P_c, E[W_r], E[R]$ and $E(\tau_r)$ are given by (13), (15), (16), (17) and (18) respectively.

The optimization problem for this maintenance scheme is reduced to find the values T and M that minimize the function $C(T,M)$ given by (19), that is

$$C(T_{opt}, M_{opt}) = \inf \left\{ C(T,M), T>0, 0<M<L \right\}. \quad (20)$$

For the complexity of the (19), we cannot get the analytical solution of the T_{opt} and M_{opt}. However, we can obtain the solution by the computing software, such as MATLAB etc.

IV. Illustrative Example

To demonstrate the reliability and maintenance models in the paper, the example of Micro-Electro-Mechanical Systems (MEMS) devices is given in this section. MEMS have been effectively used in many commercial products and critical applications. According to reliability test experimental data conducted by Sandia National Laboratories[34], MEMS experiences degradation failures due to continuous degradation over time and debris from shock loads and traumatic failures due to spring fracture, it satisfies the model assumptions in this paper.

A. Optimal Maintenance Policy

We adopt the model parameter from [9] and [13], some parameter is assumed in the paper. The random shock process follows a homogeneous Poisson process with rate $\lambda = 2.5 \times 10^{-5}$, a=0.0005. The degradation processes is modeled according to a gamma process with $\alpha = 1.02 \times 10^{-4}$, $\beta = 1.2 \times 10^{4}$, the system fails when its degradation level exceeds $L = 1.25 \times 10^{-3}$ um^3. We assume that the sequent of costs is $C_i = 7\$$, $C_d = 34\$$, $C_p = 50\$$, $C_c = 100\$$.

A solution has been numerically found by simulation, the simulation model has been implemented in MATLAB software. A grid of size 500 in the interval (0,106) has been considered for T, similarly, a grid of size 1×10^{-5} in the interval $(0,\ 1.5 \times 10^{-3})$ has been considered for M. As a result, $T_{opt} = 8.47 \times 10^4$, $M_{opt} = 4.65 \times 10^{-4}$ are obtained. The minimal average long run maintenance cost rate is $C(T_{opt}, M_{opt}) = 35.67\$$.

B. Sensitivity Analysis of the System Parameters

This subsection concentrates on analyze the varying of the optimal average long run maintenance cost rate with the different system parameters. The gamma process parameters α, β, the degradation failures threshold L and the arrival rate of random shock λ are considered respectively.

The values of the gamma process parameters are modified according to the following specification:

$$\alpha_{(V_i\%)} = \alpha\left[1+\frac{v_i}{100}\right],\ \beta_{(V_j\%)} = \beta\left[1+\frac{v_j}{100}\right]$$

Where v_i and v_j are respectively the ith and jth position of the vector $v = (-10,-5,-1,0,1,5,10)$. Then, the parameter values for α, β can be simultaneous and independently modified both for increasing and decreasing changes.

Let $C_{\alpha_{(V_i\%)},\beta_{(V_j\%)}}$ be the minimal expected cost rate obtained by varying the gamma process parameters simultaneously. Then, a relative measure is defined as

$$V_{\alpha_{(V_i\%)},\beta_{(V_j\%)}} = \frac{\left|C(T_{opt},M_{opt}) - C_{\alpha_{(V_i\%)},\beta_{(V_j\%)}}\right|}{C(T_{opt},M_{opt})}. \quad (21)$$

Where $C(T_{opt},M_{opt})$ is the minimal expected cost rate, which is previously calculated with the original parameter value α, β.

For fixed i and j, $V_{\alpha_{(V_i\%)},\beta_{(V_j\%)}}$ measures the relative difference between the current optimal cost and the optimal cost that has been calculated by suing the modified parameter values. If this quantity is multiplied by 100, the result is expressed in percentage. The closer to zero, the less influence on the solutions the modified parameter values have.

Table I shows the relative variation percentages. Each cell represents $V_{\alpha_{(V_i\%)},\beta_{(V_j\%)}}$ multiplied by 100. The results also show that the parameter α has greater effects on $V_{\alpha_{(V_i\%)},\beta_{(V_j\%)}}$ than the parameter β.

When L increases from 1.1×10^{-3} um^3 to 1.5×10^{-3} um^3 as shown in .Table II, the optimal average long run maintenance cost rate increases from 33.57$ to 36.797$, and the optimal preventive replacement threshold M increases from 6.85×10^4 to 7.29×10^4. This indicates that a larger failure threshold value L results in a larger preventive replacement threshold M. whereas the optimal inspection cycle T is insensitive to the variation of L.

TABLE I
THE SENSITIVITY ANALYSIS OF THE PARAMETERS α, β

	0.9β	0.95β	0.99β	β	1.01β	1.05β	1.1β
0.9α	1.95	2.31	3.70	4.89	5.29	7.89	11.10
0.95α	2.86	0.95	1.78	2.41	3.31	5.59	8.93
0.99α	5.13	3.32	0.64	0.97	2.11	3.92	6.42
α	6.42	3.23	0.57	0	0.56	2.92	6.11
1.01α	6.76	3.73	1.44	0.37	0.39	2.75	5.72
1.05α	8.43	5.29	3.21	2.41	1.74	1.04	4.73
1.1α	12.35	8.37	5.37	5.03	3.75	1.70	1.53

As shown in Table III, when the random shock rate λ varies from 1.0×10^{-5} to 5.0×10^{-5}, The optimal average long run maintenance cost rate ranges from 27.54$ to 43.87$. In addition, it indicates that the optimal inspection cycle T decreases with the increase of the random shock rate λ, the MEMS should be inspected more frequency, whereas the optimal M is insensitive to the variation of λ.

TABLE II
THE SENSITIVITY ANALYSIS OF THE PARAMETER L

L	1.1×10^{-3}	1.2×10^{-3}	1.3×10^{-3}	1.4×10^{-3}	1.5×10^{-3}
Optimal C	33.57	34.25	35.46	35.83	36.797
Optimal T	6.85×10^4	7.78×10^4	8.23×10^4	8.15×10^4	7.29×10^4
Optimal M	4.48×10^{-4}	4.62×10^{-4}	4.77×10^{-4}	4.97×10^{-4}	5.36×10^{-4}

V. CONCLUSION

In this article, we consider two dependent failure processes: degradation failure caused by continuous smooth degradation with the gamma process and traumatic failure caused by the random shock process. The dependent of the two processes lies in that with the degradation level increasing the probability of traumatic failure caused by the random shock is increasing. Optimal maintenance settings were determined based on minimizing the average long run cost rate. In addition, we discussed how the different parameters effect on the optimal M and T, and it indicated that the proposed reliability and maintenance models are robust and valid.

TABLE III
THE SENSITIVITY ANALYSIS OF THE PARAMETER λ

λ	1×10^{-5}	2×10^{-5}	3×10^{-5}	4×10^{-5}	5×10^{-5}
Optimal C	27.54	32.38	38.01	41.53	43.87
Optimal T	9.37×10^4	9.03×10^4	7.83×10^4	7.26×10^4	6.45×10^4
Optimal M	5.12×10^{-4}	4.87×10^{-4}	4.70×10^{-4}	5.12×10^{-4}	4.63×10^{-4}

For future research directions, we can consider the more complicated case, such as random degradation failure threshold. Furthermore, in this paper, we had not considered the random shock process affected the degradation process, so the international maintenance model is also an interesting direction.

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