Neutrix Limit on Divergent Series

Huizeng Qin, Youmin Lu

***Abstract*—In this paper, we give a simple and straight forward definition for the neutrix limit of divergent series and use it to study the neutrix limits of the divergent series ralated to the Rieman-Zeta function $\zeta(\alpha)$ and some series related to the polylogarith function $\mathrm{Li}_n(z)$. Then, we apply the reults to some specific examles. Our reults on Riemann-Zeta function and its derivatives are consistent with the tranditional ones.**

Key Words: Neutrix limit, divergent series, Riemann-zeta function, polylogarithm function.

I. Introduction

The concepts of neutrix and neutrix limit due to van der Corput [1] have been studied by many mathematicians and used widely in many areas of mathematics, physics and statistics. For example, neutrix calculus or neutrix limit has been used to study the polygamma function [2], the q-analogue of the incomplete gamma function [3], the delta function [4, 5], the q-gamma function [6], quantum field theory [7], the beta function [8], the gamma function [9] and the incomplete beta function [10, 11]. In this section, we first use three examples to explain the rationale for introducing our definition of neutrix limit and then give a simple and more straightforward definition for the neutrix limit of divergent series.

As well-known, the harmonic series $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ is divergent, but the limit $\lim\limits_{m\to\infty}(\sum\limits_{n=1}^{m}\frac{1}{n}-\ln m)=\gamma$ gives us the famous Euler-Mascheroni constant. More generally, the Riemann Zeta function $\zeta(\alpha)$ is defined as

$$\zeta(\alpha)=\sum_{l=1}^{\infty}\frac{1}{l^{\alpha}}, \tag{1.1}$$

or

$$\zeta(\alpha)=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}\frac{t^{\alpha-1}dt}{e^t-1}. \tag{1.2}$$

for $\mathrm{Re}\,\alpha>1$ By the neutrix calculus or the reflection functional equation,

$$\zeta(1-\alpha)=\frac{2\Gamma(\alpha)\zeta(\alpha)}{(2\pi)^{\alpha}}\cos\frac{\alpha\pi}{2}. \tag{1.3}$$

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Huizeng Qin is with the Institute of Applied Mathematics, Shandong University of Technology Zibo, Shandong, P. R. China.

Youmin Lu is with the Department of Mathematics and Computer Science, Bloomsburg University, Bloomsburg, PA 17815 USA; Email: ylu@bloomu.edu, Phone: (570)389-4625; Fax: (570)389-3599.

According to the traditional definition of series,

$$\lim_{n\longrightarrow\infty}S_n(\alpha)=\zeta(\alpha)\text{ for }\mathrm{Re}\alpha>1 \tag{1.4}$$

where $S_n(\alpha)=\sum\limits_{l=1}^{n}\frac{1}{l^{\alpha}}$ is the partial sum of the series (1.1). Although (1.4) is not defined when $\mathrm{Re}\,\alpha\le 1$, $\zeta(\alpha)$ has a unique analytic continuation to the entire complex plane, excluding the point $\alpha=1$.We need to find out how $\zeta(\alpha)$ is related to this series as $\mathrm{Re}(\alpha)<1$. Similarly, the series $\sum\limits_{l=0}^{\infty}z^l$ has the partial sum

$$S_n(z)=\frac{1}{1-z}-\frac{z^{n+1}}{1-z}. \tag{1.5}$$

which is divergent as $|z|>1$. By the Abel-Plana formula [12], we have

$$\begin{aligned}S_n(z)=&\int_0^n z^x dx+\frac{1+z^n}{2}\\&+i\int_0^{\infty}\frac{z^{iy}-z^{-iy}-z^{n+iy}+z^{n-iy}}{e^{2\pi y}-1}dy\\=&\frac{1}{2}-\frac{1}{\ln z}+i\int_0^{\infty}\frac{z^{iy}-z^{-iy}}{e^{2\pi y}-1}dy\\&+z^n\left(\frac{1}{\ln z}+\frac{1}{2}-i\int_0^{\infty}\frac{z^{iy}-z^{-iy}}{e^{2\pi y}-1}dy\right).\end{aligned} \tag{1.6}$$

Following (1.5) and (1.6), we can get

$$\begin{aligned}&\frac{1}{1-z}-\frac{1}{2}+\frac{1}{\ln z}-\\&i\int_0^{\infty}\frac{z^{iy}-z^{-iy}}{e^{2\pi y}-1}dy\\=&z^n\left(\frac{1}{1-z}-\frac{1}{2}+\frac{1}{\ln z}-i\int_0^{\infty}\frac{z^{iy}-z^{-iy}}{e^{2\pi y}-1}dy\right).\end{aligned} \tag{1.7}$$

Thus,

$$\begin{aligned}&i\int_0^{\infty}\frac{e^{iy\ln z}-e^{-iy\ln z}}{e^{2\pi y}-1}dy\\=&i\int_0^{\infty}\frac{z^{iy}-z^{-iy}}{e^{2\pi y}-1}dy=\frac{1}{1-z}-\frac{1}{2}+\frac{1}{\ln z}.\end{aligned} \tag{1.8}$$

In fact, if the term $\frac{z^{n+1}}{1-z}$ in equation (1.5) and $z^n\left(\frac{1}{\ln z}+\frac{1}{2}-i\int_0^{\infty}\frac{z^{iy}-z^{-iy}}{e^{2\pi y}-1}dy\right)$ in equation (1.6) were deleted, we would have the following result:

$$N.\sum_{l=0}^{\infty}z^l=\frac{1}{1-z}=\frac{1}{2}-\frac{1}{\ln z}+i\int_0^{\infty}\frac{z^{iy}-z^{-iy}}{e^{2\pi y}-1}dy, \tag{1.9}$$

where $N.\sum\limits_{l=0}^{\infty}z^l$ is some "finite part" of $\sum\limits_{l=0}^{\infty}z^l$ called the neutrix limit of the series. From (1.9), we can also obtain

(1.8). By using the idea from the examples above, we define the neutrix limit of a divergent series as following.

Definition 1: For a series $\sum\limits_{l=1}^{\infty} a_l$, if its partial sum S_n can be written as

$$S_n = A(n, \ln n)n^{\lambda} + B(n, \ln n) \ln^{\mu} n + C(n), \quad (1.10)$$

where λ and μ are complex numbers satisfying $0 < \mathrm{Re}\,\lambda < 1$ and $0 < \mathrm{Re}\,\mu < 1$, $A(x, y)$ and $B(x, y)$ are polynomial functions of x and y, and $\lim\limits_{n\longrightarrow\infty} C(n) = C$ for a constant C, then its neutrix limit $N.\sum\limits_{l=1}^{\infty} a_l$ is defined to be C and denoted by

$$N.\sum_{l=1}^{\infty} a_l = \underset{n\longrightarrow\infty}{N-\lim} S_n = C. \quad (1.11)$$

Definition 2: For a series $\sum\limits_{l=0}^{\infty} a_l z^l$, if its partial sum $S_n(z)$ can be written as

$$S_n(z) = \left(A(n, \ln n, z)n^{\lambda} + B(n, \ln n, z) \ln^{\mu} n\right) z^n \quad (1.12)$$
$$+ C(n, z),$$

where λ and μ are complex numbers satisfying $0 < \mathrm{Re}\,\lambda < 1$ and $0 < \mathrm{Re}\,\mu < 1$, $A(x, y, z)$ and $B(x, y, z)$ are polynomial functions of x and y and analytic functions on z, $\lim\limits_{n\longrightarrow\infty} C(n, z) = C(z)$ for an analytic function $C(z)$, then its neutrix limit is defined to be $C(z)$ and denoted by

$$N.\sum_{l=1}^{\infty} a_l z^l = \underset{n\longrightarrow\infty}{N-\lim} S_n(z) = C(z). \quad (1.13)$$

In this paper, we first study the neutrix limit of the series which define the Riemann-zeta function $\zeta(\alpha)$ and the polylogarithm function $Li_{\alpha}(z)$. By using the neutrix limit method, we can extend both Riemann zeta function $\zeta(\alpha)$ and polylogarithm function $Li_{\alpha}(z)$ to the tntire complex plane and obtain results that are consistent with the traditional results. In the last section, we apply the results to several examples.

II. Neutrix limit, Riemann Zeta Function and Its Derivatives

In this section, we study the neutrix limit of $\sum\limits_{l=1}^{\infty} l^{\alpha}$ for $\mathrm{Re}\,\alpha \geqslant -1$ and show that the results obtained by using our definition of neutrix limit are consistent with the traditional ones.

Theorem 3: For any complex number α satisfying $\mathrm{Re}\,\alpha > -1$ and $\mathrm{Re}\,\alpha \neq 0$,

$$N.\sum_{j=1}^{\infty} j^{\alpha} = -\frac{2\Gamma(\alpha+1)\zeta(\alpha+1)}{(2\pi)^{\alpha+1}} \sin\frac{\alpha\pi}{2} \quad (2.1)$$

and furthermore

$$N.\sum_{j=1}^{\infty} \frac{1}{j} = \gamma, \quad (2.2)$$

where γ is the Euler-Mascheroni constant.

Proof: Case I. $\mathrm{Re}\,\alpha > 0$.

Using the Abel-Plana formula [12] with $f(t) = t^{\alpha}$, we have

$$\begin{aligned}\sum_{j=0}^{n} j^{\alpha} &= \int_0^n x^{\alpha} dx + \frac{n^{\alpha}}{2} \quad (2.3)\\ &+ i\int_0^{\infty} \frac{(iy)^{\alpha} - (n+iy)^{\alpha} - (-iy)^{\alpha} + (n-iy)^{\alpha}}{e^{2\pi y}-1} dy\\ &= \frac{n^{\alpha+1}}{\alpha+1} + \frac{n^{\alpha}}{2} + i\int_0^{\infty} \frac{(iy)^{\alpha} - (-iy)^{\alpha}}{e^{2\pi y}-1} dy\\ &- i\int_0^{\infty} \frac{(n+iy)^{\alpha} - (n-iy)^{\alpha}}{e^{2\pi y}-1} dy.\end{aligned}$$

Since

$$\int_0^{\infty} \frac{y^{\alpha}}{e^{2\pi y}-1} dy = \frac{\Gamma(\alpha+1)\zeta(\alpha+1)}{(2\pi)^{\alpha+1}},$$

one has

$$\begin{aligned}& i\int_0^{\infty} \frac{(iy)^{\alpha} - (-iy)^{\alpha}}{e^{2\pi y}-1} dy \quad (2.4)\\ =& i\left(i^{\alpha} - (-i)^{\alpha}\right)\int_0^{\infty} \frac{y^{\alpha}}{e^{2\pi y}-1} dy\\ =& -\frac{2\Gamma(\alpha+1)\zeta(\alpha+1)}{(2\pi)^{\alpha+1}} \sin\frac{\alpha\pi}{2}.\end{aligned}$$

Let m be any integer satisfying $2m-1 > \mathrm{Re}\,\alpha$. Then,

$$\begin{aligned}& i\int_0^{\infty} \frac{(n+iy)^{\alpha} - (n-iy)^{\alpha}}{e^{2\pi y}-1} dy \quad (2.5)\\ =& \int_0^{\infty} (\sum_{l=1}^{m} \frac{(-1)^l (\alpha)_{2l-1} n^{\alpha-2l+1} y^{2l-1}}{(2l-1)!}\\ &- \frac{(-1)^m (\alpha)_{2m} (n+i\theta_n y)^{\alpha-2m} y^{2m}}{(2m)!}) \frac{dy}{e^{2\pi y}-1}\\ =& \sum_{l=1}^{m} \frac{(-1)^l (\alpha)_{2l-1} \zeta(2l) n^{\alpha-2l+1}}{(2\pi)^{2l}} + O(\frac{1}{n})\end{aligned}$$

for $\mathrm{Re}\,\alpha \neq 1, 3, 5, \cdots$, where $(x)_n$ is the Pochhammer Symbol and $\theta_n \in (0, 1)$. Substituting (2.4) and (2.5) into (2.3), we get

$$\begin{aligned}& \underset{n\longrightarrow\infty}{N-\lim} \sum_{j=0}^{n} j^{\alpha}\\ =& \underset{n\longrightarrow\infty}{N-\lim} (\frac{n^{\alpha+1}}{\alpha+1} + \frac{n^{\alpha}}{2} - \frac{2\Gamma(\alpha+1)\zeta(\alpha+1)}{(2\pi)^{\alpha+1}} \sin\frac{\alpha\pi}{2}\\ &+ 2\sum_{l=1}^{m} \frac{(-1)^{l-1} (\alpha)_{2l-1} \zeta(2l) n^{\alpha-2l+1}}{(2\pi)^{2l}} + O(\frac{1}{n})\\ =& -\frac{2\Gamma(\alpha+1)\zeta(\alpha+1)}{(2\pi)^{\alpha+1}} \sin\frac{\alpha\pi}{2}.\end{aligned}$$

Therefore, (2.1) holds. When $\mathrm{Re}\,\alpha = 1, 3, 5, \cdots$ or $\mathrm{Im}\,\alpha \neq 0$ and $\mathrm{Re}\,\alpha = 0$, (2.1) is also established by analytic continuation.

Case II. $-1 < \mathrm{Re}\,\alpha < 0$.

In this case, (2.3) can be written as

$$\begin{aligned}
&\int_1^n x^\alpha dx + \frac{1+n^\alpha}{2}\\
&+ i\int_0^\infty \frac{(1+iy)^\alpha - (n+iy)^\alpha}{e^{2\pi y}-1}dy\\
&- i\int_0^\infty \frac{(1-iy)^\alpha - (n-iy)^\alpha}{e^{2\pi y}-1}dy\\
=&\frac{1}{2} - \frac{1}{\alpha+1} + i\int_0^\infty \frac{(1+iy)^\alpha - (1-iy)^\alpha}{e^{2\pi y}-1}dy\\
&+ \frac{n^{\alpha+1}}{\alpha+1} + \frac{n^\alpha}{2} - i\int_0^\infty \frac{(n+iy)^\alpha - (n-iy)^\alpha}{e^{2\pi y}-1}dy
\end{aligned}$$

and hence,

$$\begin{aligned}
&N-\lim_{n\longrightarrow\infty}\sum_{j=1}^n j^\alpha\\
=&\frac{1}{2} - \frac{1}{\alpha+1} + i\int_0^\infty \frac{(1+iy)^\alpha - (1-iy)^\alpha}{e^{2\pi y}-1}dy\\
=&\frac{1}{2} - \frac{1}{\alpha+1} + 2\int_0^\infty \frac{\sin(-\alpha\arctan y)}{(1+y^2)^{-\frac{\alpha}{2}}(e^{2\pi y}-1)}dy\\
=&\zeta(-\alpha),
\end{aligned}$$

where Jensen's formula [12] is used. Thus, (2.1) also holds for complex numbers α satisfying $-1 < \operatorname{Re}\alpha < 0$.

For $\alpha = -1$, (2.3) can be written as

$$\begin{aligned}
&N-\lim_{n\longrightarrow\infty}\sum_{j=1}^n \frac{1}{j}\\
=&N-\lim_{n\longrightarrow\infty}(\int_1^n \frac{1}{x}dx + \frac{1}{2} + \frac{1}{2n}\\
&+ i\int_0^\infty \frac{1}{e^{2\pi y}-1}\left(\frac{1}{1+iy} - \frac{1}{n+iy}\right)dy\\
&- i\int_0^\infty \frac{1}{e^{2\pi y}-1}\left(\frac{1}{1-iy} - \frac{1}{n-iy}\right)dy)\\
=&\frac{1}{2} + 2\int_0^\infty \frac{y}{(e^{2\pi y}-1)(1+y^2)}dy = \gamma.
\end{aligned}$$

Therefore, (2.2) holds. ■

Theorem 4: 1) For any complex number α satisfying $\operatorname{Re}\alpha > -1$ and any integer p,

$$\begin{aligned}
&N.\sum_{j=1}^\infty j^\alpha \ln^p j \qquad (2.6)\\
=&-\frac{p!}{(2\pi)^\alpha}\sum_{i=0}^p \frac{\zeta^{(p-i)}(\alpha+1)}{(p-i)!}\sum_{j=0}^i \frac{\Gamma^{(i-j)}(\alpha+1)}{(i-j)!}A_j
\end{aligned}$$

where $A_j = \sum_{l=0}^j \frac{(-1)^l \pi^{j-l-1}\ln^l(2\pi)}{(j-l)!l!2^{j-l}}\sin\frac{(\alpha+j-l)\pi}{2}$, $\Gamma^{(n)}(z)$ is defined by the following recurrence formula

$$\Gamma^{(n)}(z) = \sum_{k=0}^{n-1}\left(\begin{array}{c} n-1\\ k\end{array}\right)\psi^{(n-1-k)}(z)\Gamma^{(k)}(z), \qquad (2.7)$$

$\psi(x)$ is the Digamma function defined by

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z) = -\gamma - \frac{1}{z} + \sum_{l=1}^\infty\left(\frac{1}{l} - \frac{1}{l+z}\right), \quad (2.8)$$

$$\psi^{(k)}(z) = k!(-1)^{k+1}\zeta(k+1,z), k = 1,2,\cdots, \qquad (2.9)$$

and $\zeta(k,z)$ is the Hurwitz zeta function defined by

$$\zeta(k,z) = \sum_{l=0}^\infty \frac{1}{(l+z)^k}.$$

2) For any natural number p,

$$\begin{aligned}
&N.\sum_{j=1}^\infty j^{-1}\ln^p j \qquad (2.10)\\
=&\sum_{j=0}^{[\frac{p}{2}]}\frac{(-1)^{p-j}C_p^{2j}}{2^{p-2j-1}}\int_0^\infty yh(y,2j)dy\\
&-\sum_{j=0}^{[\frac{p-1}{2}]}\frac{(-1)^{p-j}C_p^{2j+1}}{2^{p-2j-2}}\int_0^\infty h(y,2j+1)dy
\end{aligned}$$

where $h(y,n) = \frac{(\tan^{-1}y)^n\ln^{p-n}(1+y^2)}{(e^{2\pi y}-1)(1+y^2)}$.

Proof: 1) By using the Abel-Plana formula with $f(t) = t^\alpha\ln^p t$ for any integer $p > 0$,

$$\begin{aligned}
&\sum_{j=1}^n j^\alpha\ln^p l \qquad (2.11)\\
=&\int_1^n x^\alpha\ln^p x dx + i\int_0^\infty \frac{(1+iy)^\alpha\ln^p(1+iy)}{e^{2\pi y}-1}dy\\
&- i\int_0^\infty \frac{(1-iy)^\alpha\ln^p(1-iy)}{e^{2\pi y}-1}dy + \frac{n^\alpha\ln^p n}{2}\\
&- i\int_0^\infty \frac{(n+iy)^\alpha\ln^p(n+iy)}{e^{2\pi y}-1}dy,\\
&+ i\int_0^\infty \frac{(n-iy)^\alpha\ln^p(n-iy)}{e^{2\pi y}-1}dy,
\end{aligned}$$

Using the Taylor expansion, we have

$$\begin{aligned}
&i\int_0^\infty \frac{(1+iy)^\alpha\ln^p(1+iy)}{e^{2\pi y}-1}dy \qquad (2.12)\\
&- i\int_0^\infty \frac{(1-iy)^\alpha\ln^p(1-iy)}{e^{2\pi y}-1}dy\\
=&\frac{d^p}{d\alpha^p}i\int_0^\infty \frac{(n+iy)^\alpha - (n-iy)^\alpha}{e^{2\pi y}-1}dy\\
=&2\int_0^\infty (\sum_{l=1}^m \frac{(-1)^l y^{2l-1}}{(2l-1)!}\frac{d^p}{d\alpha^p}\left[g(l,1)n^{\alpha-2l+1}\right]\\
&+ \frac{(-1)^m y^{2m}}{(2m)!}\frac{d^p}{d\alpha^p}\left[g(m,\frac{1}{2})(n+i\theta_n y)^{\alpha-2m}\right])\\
&\times\frac{dy}{e^{2\pi y}-1}\\
=&2\sum_{l=1}^m \frac{(-1)^l\zeta(2l)}{(2\pi)^{2l}}\frac{d^p}{d\alpha^p}\left[g(l,1)n^{\alpha-2l+1}\right] + O(\frac{1}{n}),
\end{aligned}$$

where $\alpha-2m+1<0$ for any integer $m\geqslant 0$ and $g(n,k)=(\alpha-2(n-k))_{2n-2k+1}$. By definition (1.13) and

$$\begin{aligned}&\int_1^n x^\alpha \ln^p x dx \qquad (2.13)\\ =&\frac{1}{\alpha+1}\int_1^n \ln^p x dx^{\alpha+1}\\ =&\frac{n^{\alpha+1}\ln^p n}{\alpha+1}-\frac{p}{\alpha+1}\int_1^n x^\alpha \ln^{p-1} x dx\\ =&\cdots\\ =&p!\sum_{k=0}^{p}\frac{(-1)^k n^{\alpha+1}\ln^{p-k} n}{(p-k)!(\alpha+1)^{k+1}}-\frac{(-1)^p p!}{(\alpha+1)^{p+1}},\end{aligned}$$

we have

$$\begin{aligned}&N.\sum_{j=1}^{\infty} j^\alpha \ln^p l \qquad (2.14)\\ =&N-\lim_{n\longrightarrow\infty}\sum_{j=1}^{n} j^\alpha \ln^p l\\ =&-\frac{(-1)^p p!}{(\alpha+1)^{p+1}}\\ &+i\int_0^\infty \frac{(1+iy)^\alpha \ln^p(1+iy)}{e^{2\pi y}-1}dy\\ &-i\int_0^\infty \frac{(1-iy)^\alpha \ln^p(1-iy)}{e^{2\pi y}-1}dy\\ =&-\frac{(-1)^p p!}{(\alpha+1)^{p+1}}-2\frac{d^p}{d\alpha^p}\int_0^\infty \frac{\mathrm{Im}((1+iy)^\alpha}{e^{2\pi y}-1}dy\\ =&-\frac{d^p}{d\alpha^p}\left(\frac{2\Gamma(\alpha+1)\zeta(\alpha+1)}{(2\pi)^{\alpha+1}}\sin\frac{\alpha\pi}{2}\right).\end{aligned}$$

It follows the general Leibniz differentiation rule that (2.6) holds.

2) For $\alpha=-1$, we can use the Abel-Plana formula with $f(t)=\frac{\ln^p t}{t}$ to get

$$\begin{aligned}&N.\sum_{j=1}^{\infty}\frac{\ln^p l}{j}\\ =&N-\lim_{n\longrightarrow\infty}(i\int_0^\infty\left(\frac{\ln^p(1+iy)}{1+iy}-\frac{\ln^p(1-iy)}{1-iy}\right)\\ &\times\frac{dy}{e^{2\pi y}-1}+\frac{\ln^{p+1} n}{p+1}+\frac{\ln^p n}{2n}\\ &+i\int_0^\infty\left(-\frac{\ln^p(n+iy)}{n+iy}+\frac{\ln^p(n-iy)}{n-iy}\right)\\ &\times\frac{1}{e^{2\pi y}-1}dy)\\ =&i\int_0^\infty\frac{(1-iy)\ln^p(1+iy)}{(e^{2\pi y}-1)(1+y^2)}dy\\ &-i\int_0^\infty\frac{(1+iy)\ln^p(1-iy)}{(e^{2\pi y}-1)(1+y^2)}dy\\ =&\sum_{j=0}^{[\frac{p}{2}]}(\frac{(-1)^{p-j}C_p^{2j}}{2^{p-2j-1}}\\ &\times\int_0^\infty\frac{y\left(\tan^{-1}y\right)^{2j}\ln^{p-2j}\left(1+y^2\right)}{(e^{2\pi y}-1)(1+y^2)}dy)\\ &-\sum_{j=0}^{[\frac{p-1}{2}]}(\frac{(-1)^{p-j}C_p^{2j+1}}{2^{p-2j-2}}\\ &\times\int_0^\infty\frac{\left(\tan^{-1}y\right)^{2j+1}\ln^{p-2j-1}\left(1+y^2\right)}{(e^{2\pi y}-1)(1+y^2)}dy).\end{aligned}$$

Therefore (2.10) holds. ■

Remark 5: By (2.1) we have

$$\frac{d^p}{d\alpha^p}N.\sum_{j=1}^{\infty}j^\alpha=\frac{d^p}{d\alpha^p}\left(-\frac{2\Gamma(\alpha+1)\zeta(\alpha+1)}{(2\pi)^{\alpha+1}}\sin\frac{\alpha\pi}{2}\right). \qquad (2.15)$$

Comparing (2.6) with (2.15), we have showed that the derivative operator and the neutrix limit operator are exchangeable for the power series $\sum_{j=1}^{\infty}j^\alpha$:

$$\frac{d^p}{d\alpha^p}N.\sum_{j=1}^{\infty}j^\alpha=N.\sum_{j=1}^{\infty}j^\alpha\ln^p l. \qquad (2.16)$$

Remark 6: Olver [12] gives the following result

$$\zeta(1-\alpha)=\frac{\Gamma(\alpha)\zeta(\alpha)}{2^{\alpha-1}\pi^\alpha}\cos\frac{\alpha\pi}{2}. \qquad (2.17)$$

It follows that

$$\zeta(-\alpha)=-\frac{2\Gamma(\alpha+1)\zeta(\alpha+1)}{(2\pi)^{\alpha+1}}\sin\frac{\alpha\pi}{2}. \qquad (2.18)$$

By Theorem 3, we see that

$$\zeta(-\alpha)=N.\sum_{j=1}^{\infty}j^\alpha \qquad (2.19)$$

for $\mathrm{Re}\,\alpha>-1$. Differentiating (2.19) with respect to α and using (2.16), we see that

$$(-1)^p\zeta^{(p)}(-\alpha)=\frac{d^p}{dx^p}N.\sum_{j=1}^{\infty}j^\alpha=N.\sum_{j=1}^{\infty}j^\alpha\ln^p j \qquad (2.20)$$

for $\mathrm{Re}\,\alpha>-1$. Particularly,

$$(-1)^p\zeta^{(p)}(1)=N.\sum_{j=1}^{\infty}\frac{\ln^p j}{j}=\gamma_p \qquad (2.21)$$

where γ_p is the Stieltjes constant. By (2.21), we have the following Taylor expansion

$$\zeta(\alpha)=\tau(\alpha)+\sum_{j=0}^{\infty}\frac{(-1)^j\gamma_j}{j!}(\alpha-1)^j \tag{2.22}$$

where $\gamma_0=\gamma$ and $\tau(\alpha)=\begin{cases}0, \text{if } \alpha=1\\ \frac{1}{\alpha-1}, \text{if } \alpha\neq 1\end{cases}$.

III. Neutrix Limit and the Polylogarithm Function

The polylogarithm function $Li_n(z)$, also known as the Jonquiere's function, is the function defined by

$$Li_n(z)=\sum_{j=1}^{\infty}\frac{z^j}{j^n} \tag{3.1}$$

over the open unit disk in the complex plane. When $|z|>1$, $Li_n(z)$ are divergent series. we consider their neutrix limit in this section.

Theorem 7: For any complex numbers z and α satisfying $\alpha\neq -1$,

$$N.\sum_{j=1}^{\infty}j^{\alpha}z^j=\frac{\Gamma(\alpha+1)}{(-\ln z)^{\alpha+1}}-\frac{\Gamma(\alpha+1)}{(2\pi)^{\alpha+1}} \times(e^{\frac{(\alpha-1)\pi i}{2}}\zeta(\alpha+1,1-\frac{i\ln z}{2\pi}) +e^{\frac{(1-\alpha)\pi i}{2}}\zeta(\alpha+1,1+\frac{i\ln z}{2\pi})) \tag{3.2}$$

or

$$\begin{aligned}N.\sum_{j=1}^{\infty}j^{\alpha}z^j=&\frac{i\Gamma(\alpha+1)\cos\frac{\alpha\pi}{2}}{(2\pi)^{\alpha+1}}\\ &\times\left(\zeta(\alpha+1,1-\frac{i\ln z}{2\pi})-\zeta(\alpha+1,1+\frac{i\ln z}{2\pi})\right)\\ &+\frac{\Gamma(\alpha+1)}{(-\ln z)^{\alpha+1}}-\frac{\Gamma(\alpha+1)\sin\frac{\alpha\pi}{2}}{(2\pi)^{\alpha+1}}\\ &\times\left(\zeta(\alpha+1,1-\frac{i\ln z}{2\pi})+\zeta(\alpha+1,1+\frac{i\ln z}{2\pi})\right).\end{aligned} \tag{3.3}$$

Proof: Using the Abel-Plana formula with $f(t)=t^{\alpha}z^t$, we have

$$\begin{aligned}&\sum_{j=1}^{n}j^{\alpha}z^j\\ =&\int_1^n t^{\alpha}z^t dt+\frac{n^{\alpha}z^n+z}{2}\\ &+i\int_0^{\infty}(\frac{(1+iy)^{\alpha}z^{1+iy}-(1-iy)^{\alpha}z^{1-iy}}{e^{2\pi y}-1}\\ &-\frac{(n+iy)^{\alpha}z^{n+iy}-(n-iy)^{\alpha}z^{n-iy}}{e^{2\pi y}-1})dy\end{aligned} \tag{3.4}$$

for any complex number $\alpha\neq -1$, and

$$\begin{aligned}z=\sum_{j=0}^{1}j^{\alpha}z^j=&\int_0^1 t^{\alpha}z^t dt+\frac{z}{2}\\ &+i\int_0^{\infty}(\frac{(iy)^{\alpha}z^{iy}-(-iy)^{\alpha}z^{-iy}}{e^{2\pi y}-1}\\ &-\frac{(1+iy)^{\alpha}z^{1+iy}-(1-iy)^{\alpha}z^{n-iy}}{e^{2\pi y}-1})dy\end{aligned} \tag{3.5}$$

for any complex number α satisfying $\mathrm{Re}\,\alpha>0$. Similar to the proof of Theorem 3, we have

$$\begin{aligned}&\sum_{j=1}^{n}j^{\alpha}z^j\\ =&\frac{n^{\alpha}z^n}{2}+i\int_0^{\infty}\frac{-(n+iy)^{\alpha}z^{n+iy}+(n-iy)^{\alpha}z^{n-iy}}{e^{2\pi y}-1}dy\\ &+\frac{1}{(-\ln z)^{\alpha+1}}\int_0^{-n\ln z}(t^{\alpha}e^{-t}dt\\ &+i\int_0^{\infty}\frac{(iy)^{\alpha}z^{iy}-(-iy)^{\alpha}z^{-iy}}{e^{2\pi y}-1})dy.\end{aligned} \tag{3.6}$$

Using the Taylor expansion, one can also get

$$\begin{aligned}&\int_0^{\infty}\frac{(n+iy)^{\alpha}z^{iy}-(n-iy)^{\alpha}z^{-iy}}{e^{2\pi y}-1}dy\\ =&2\int_0^{\infty}(\sum_{l=0}^{m-1}\frac{(-1)^{l-1}(\alpha)_{2l}n^{\alpha-2l}y^{2l}\sin{(y\ln z)}}{(2l-1)!}\\ &+2\sum_{l=1}^{m}\frac{(-1)^{l}(\alpha)_{2l-1}n^{\alpha-2l+1}y^{2l-1}\cos{(y\ln z)}}{(2l-1)!}\\ &+\frac{(-1)^m(\alpha)_{2m}(n+i\theta_1 y)^{\alpha-2m}y^{2m}}{(2m)!}\\ &-\frac{(-1)^m(\alpha)_{2m}(n-i\theta_2 y)^{\alpha-2m}y^{2m}}{(2m)!})\frac{dy}{e^{2\pi y}-1}\\ =&2\sum_{l=0}^{m-1}\frac{(-1)^{l-1}(\alpha)_{2l}n^{\alpha-2l}y^{2l}}{(2l-1)!}\int_0^{\infty}\frac{y^{2l}\sin{(y\ln z)}\,dy}{e^{2\pi y}-1}\\ &+2\sum_{l=1}^{m}(\frac{(-1)^{l}(\alpha)_{2l-1}n^{\alpha-2l+1}}{(2l-1)!}\\ &\times\int_0^{\infty}\frac{y^{2l-1}\cos{(y\ln z)}\,dy}{e^{2\pi y}-1})+O(\frac{1}{n}).\end{aligned} \tag{3.7}$$

With careful calculation, we have

$$\begin{aligned}&i\int_0^{\infty}\frac{(iy)^{\alpha}z^{iy}-(-iy)^{\alpha}z^{-iy}}{e^{2\pi y}-1}dy\\ =&2\cos\frac{\alpha\pi}{2}\int_0^{\infty}\frac{y^{\alpha}\sin{(y\ln z)}}{e^{2\pi y}-1}dy\\ &+2\sin\frac{\alpha\pi}{2}\int_0^{\infty}\frac{y^{\alpha}\cos{(y\ln z)}}{e^{2\pi y}-1}dy,\end{aligned} \tag{3.8}$$

$$
\begin{aligned}
&2\int_0^\infty \frac{y^\alpha \sin(y\ln z)}{e^{2\pi y}-1}dy \qquad (3.9)\\
=&i\int_0^\infty \frac{y^\alpha\left(e^{iy\ln z}-e^{-iy\ln z}\right)}{e^{2\pi y}-1}dy\\
=&i\sum_{j=1}^\infty\int_0^\infty\left(e^{-(2\pi j-i\ln z)y}-e^{-(2\pi j+i\ln z)y}\right)y^\alpha dy\\
=&\frac{i\Gamma(\alpha+1)}{(2\pi)^{\alpha+1}}(\zeta(\alpha+1,1-\frac{i\ln z}{2\pi})\\
&-\zeta(\alpha+1,1+\frac{i\ln z}{2\pi})),
\end{aligned}
$$

and

$$
\begin{aligned}
&2\int_0^\infty \frac{y^\alpha \cos(y\ln z)}{e^{2\pi y}-1}dy \qquad (3.10)\\
=&\int_0^\infty \frac{y^\alpha\left(e^{iy\ln z}+e^{-iy\ln z}\right)}{e^{2\pi y}-1}dy\\
=&\sum_{j=1}^\infty\int_0^\infty y^\alpha\left(e^{-(2\pi-i\ln z)jy}+e^{-(2\pi+i\ln z)jy}\right)dy\\
=&\frac{\Gamma(\alpha+1)}{(2\pi)^{\alpha+1}}(\zeta(\alpha+1,1-\frac{i\ln z}{2\pi})\\
&+\zeta(\alpha+1,1+\frac{i\ln z}{2\pi}).
\end{aligned}
$$

By (3.6)-(3.10) and definition (1.12), we conclude that (3.2) holds. For $\mathrm{Re}\,\alpha>-1$, (3.2) can also be established by using analytic continuation. ∎

If we let $z=e^{2\pi qi}$ in (3.2), the following corollary can be established.

Corollary 8: For any complex number α satisfying $\mathrm{Re}\,\alpha>0$ or $\mathrm{Im}\,\alpha\neq 0$ and $\mathrm{Re}\,\alpha=0$,

$$
\begin{aligned}
&N.\sum_{n=1}^\infty n^\alpha e^{2\pi qni} \qquad (3.11)\\
=&\frac{\Gamma(\alpha+1)}{(2\pi)^{\alpha+1}}(\zeta(\alpha+1,q)e^{\frac{\pi(1+\alpha)i}{2}}\\
&+\zeta(\alpha+1,1-q)e^{-\frac{\pi(1+\alpha)i}{2}}
\end{aligned}
$$

and furthermore,

$$
\begin{aligned}
&N.\sum_{n=1}^\infty n^\alpha\cos(2\pi qn)\\
=&-\tfrac{(\zeta(\alpha+1,q)+\zeta(\alpha+1,1-q))\Gamma(\alpha+1)}{(2\pi)^{\alpha+1}}\sin\tfrac{\alpha\pi}{2},\\
&N.\sum_{n=1}^\infty n^\alpha\sin(2\pi qn) \qquad (3.12)\\
=&\tfrac{(\zeta(\alpha+1,q)-\zeta(\alpha+1,1-q))\Gamma(\alpha+1)}{(2\pi)^{\alpha+1}}\cos\tfrac{\alpha\pi}{2}.
\end{aligned}
$$

Remark 9: By (3.2), we may obtain the following expressions for the polylogarithm function $Li_n(z)$ and its derivatives.

$$
\begin{aligned}
&Li_{-\alpha}(z) \qquad (3.13)\\
=&\frac{\Gamma(\alpha+1)}{(-\ln z)^{\alpha+1}}\\
&-\frac{\Gamma(\alpha+1)}{(2\pi)^{\alpha+1}}(e^{\frac{(\alpha-1)\pi i}{2}}\zeta(\alpha+1,1-\frac{i\ln z}{2\pi})\\
&+e^{\frac{(1-\alpha)\pi i}{2}}\zeta(\alpha+1,1+\frac{i\ln z}{2\pi})),
\end{aligned}
$$

$$
\begin{aligned}
&Li_{-\alpha}(-1) \qquad (3.14)\\
=&-\frac{2\left(1-2^{-\alpha-1}\right)\Gamma(\alpha+1)\zeta(\alpha+1)}{\pi^{\alpha+1}}\\
&\times\cos\frac{(\alpha-1)\pi}{2},
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{d}{d\alpha}Li_{-\alpha}(-1) \qquad (3.15)\\
=&-\frac{2^{-\alpha}\Gamma(\alpha+1)\zeta(\alpha+1)\ln 2}{\pi^{\alpha+1}}\cos\frac{(\alpha-1)\pi}{2}\\
&+\frac{2\left(1-2^{-\alpha-1}\right)(\ln\pi-\psi(\alpha+1))}{\pi^{\alpha+1}}\\
&\times\frac{\Gamma(\alpha+1)\zeta(\alpha+1)}{\pi^{\alpha+1}}\cos\frac{(\alpha-1)\pi}{2}\\
&-\frac{2\left(1-2^{-\alpha-1}\right)\Gamma(\alpha+1)\zeta'(\alpha+1)}{\pi^{\alpha+1}}\\
&\times\cos\frac{(\alpha-1)\pi}{2}\\
&+\frac{\left(1-2^{-\alpha-1}\right)\Gamma(\alpha+1)\zeta(\alpha+1)}{\pi^\alpha}\sin\frac{(\alpha-1)\pi}{2}.
\end{aligned}
$$

In particular, the following results are true.

$$
\begin{aligned}
Li_0(-1)&=-\frac{1}{2},\\
Li_1(-1)&=-\ln 2,\\
Li_{-2n}(-1)&=0,
\end{aligned}
$$

$$
\begin{aligned}
&Li_{1-2n}(-1)\\
=&\frac{2(-1)^n\left(2^{2n}-1\right)(2n-1)!}{(2\pi)^{2n}}\zeta(2n),
\end{aligned}
$$

$$
\frac{d}{d\alpha}Li_\alpha(-1)|_{\alpha=0}=-\frac{1}{2}\ln\frac{\pi}{2},
$$

$$
\begin{aligned}
&\frac{d}{d\alpha}Li_\alpha(-1)|_{\alpha=-2n}\\
=&\frac{(-1)^n\left(1-2^{-2n-1}\right)(2n)!)\zeta(2n+1)}{\pi^{2n}},
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d}{d\alpha}Li_\alpha(-1)|_{\alpha=1-2n}\\
=&\frac{(-1)^{n-1}(2n-1)!}{\pi^{2n}}(2^{1-2n}\zeta(2n)\ln 2\\
&+2\left(1-2^{-2l}\right)\zeta'(2n)\\
&-2\left(1-2^{-2n}\right)(\ln\pi+\gamma-H_{2n-1})\zeta(2n)).
\end{aligned}
$$

IV. Examples of Applications

By (1.9), we can get

$$Li_{-n}(z) = N.\sum_{j=1}^{\infty} j^n z^j = Q_n(z), \tag{4.1}$$

where $Q_k(z) = z\frac{d}{dz}Q_{k-1}(z)$ and $Q_0(z) = \frac{z}{1-z}$. This result is consistent with the results established by Adamchik [15].

Let

$$I_m^k = \int_0^1 x^k \left\{\frac{1}{x}\right\}^m dx \tag{4.2}$$

for integers $k \geqslant 0$ and $m \geqslant 1$. We have the following results.

Example 10: For integers k and m,

$$\begin{aligned} I_m^k =& \frac{(k-m)!m!}{(k+1)!}(\zeta(k+1-m) \\ & - \sum_{u=0}^{m} C_{k+u-m}^u (\zeta(k+1+u-m)-1)) \end{aligned} \tag{4.3}$$

when $k \geqslant m \geq 1$ and

$$\begin{aligned} I_m^k =& C_m^{k+1} \sum_{l=1}^{\infty} \frac{(-1)^l \zeta(l+1)}{m+l-k} \\ & - \frac{m!}{(k+1)!} \sum_{i=1}^{k} \frac{i!\,(\zeta(i+1)-1)}{(m-k+i)!} + \frac{1}{m-k} C_m^{k+1} \end{aligned} \tag{4.4}$$

or

$$\begin{aligned} I_m^k =& C_m^{k+1} (\sum_{l=1}^{\infty} \frac{(-1)^l (\zeta(l+1)-1)}{m-k+l} \\ & - (-1)^{m-k}\left(\ln 2 + \sum_{l=1}^{m-k} \frac{(-1)^l}{l}\right)) \\ & - \frac{m!}{(k+1)!} \sum_{i=1}^{k} \frac{i!\,(\zeta(i+1)-1)}{(m+i-k)!} \end{aligned} \tag{4.5}$$

when $0 \leq k < m$.

Proof: When $k \geq m \geq 1$,

$$\begin{aligned} I_m^k =& \int_1^{\infty} t^{-k-2} \{t\}^m \, dt \\ =& \sum_{n=1}^{\infty} \int_0^1 \frac{t^m}{(n+t)^{k+2}} dt \\ =& \sum_{n=1}^{\infty} \sum_{i=0}^{m} (-1)^i C_m^i n^i \int_0^1 \frac{1}{(n+t)^{k+2+i-m}} dt \\ =& \sum_{i=0}^{m} \frac{(-1)^i C_m^i}{k+i+1-m} \sum_{n=1}^{\infty} (\frac{1}{n^{k+1-m}} \\ & - \frac{n^i}{(n+1)^{k+i+1-m}}) \\ =& \sum_{i=0}^{m} \frac{(-1)^i C_m^i}{k+i+1-m} (N.\sum_{n=1}^{\infty} \frac{1}{n^{k+1-m}} \\ & - \sum_{u=0}^{i} (-1)^u C_i^u N.\sum_{n=1}^{\infty} \frac{1}{(n+1)^{k+u+1-m}}) \\ =& \sum_{i=0}^{m} \frac{(-1)^i C_m^i}{k+i+1-m} \zeta(k+1-m) \\ & - \sum_{i=0}^{m} (\frac{(-1)^i C_m^i}{k+i+1-m} \\ & \times \sum_{u=0}^{i} (-1)^u C_i^u (\zeta(k+1+u-m)-1)) \\ =& \zeta(k+1-m) \sum_{i=0}^{m} C_m^i \frac{(-1)^i}{k+i+1-m} \\ & - \sum_{u=0}^{m} (C_m^u (\zeta(k+1+u-m)-1) \times \\ & \sum_{i=0}^{m-u} C_{m-u}^i \frac{(-1)^i}{k+i+u+1-m}). \end{aligned} \tag{4.6}$$

Using

$$\sum_{i=0}^{n} C_n^i \frac{(-1)^i}{x+i} = \frac{n!}{x_{n+1}},$$

in (4.6), we see that (4.3) holds.

When $0 \le k < m$, one can use integration by parts to get

$$\begin{aligned} I_m^k =& \sum_{n=1}^{\infty}\Big(\frac{(m-k)_{k+1}}{(k+1)!}\int_0^1 \frac{x^{m-k-1}}{n+x}dx \\ &- \sum_{i=0}^{k}\frac{(m+1-i)_i}{(k+1-i)_{i+1}(n+1)^{k+1-i}}\Big) \\ =& \sum_{n=1}^{\infty}\Big(C_m^{k+1}\sum_{l=0}^{\infty}\frac{(-1)^l}{(m+l-k)}\frac{1}{n^{l+1}} \\ &- \frac{m!}{(k+1)!}\sum_{i=0}^{k}\frac{(k-i)!}{(m-i)!(n+1)^{k+1-i}}\Big) \\ =& C_m^{k+1}\sum_{l=1}^{\infty}\frac{(-1)^l\zeta(l+1)}{m+l-k} \\ &- \frac{m!}{(k+1)!}\sum_{i=1}^{k}\frac{i!\,(\zeta(i+1)-1)}{(m-k+i)!} + C_m^{k+1}\frac{1}{(m-k)}. \end{aligned}$$

Thus, (4.4) holds. ■

Remark 11: Let $k = 0$ in (4.5). Then,

$$I_m^0 = m\sum_{l=1}^{\infty}\frac{(-1)^l\zeta(l+1)}{m+l} + 1. \tag{4.7}$$

Qin and Lu [16] give the result

$$\begin{aligned} I_m^0 =& \ln(2\pi) - \frac{m\gamma}{2} - \frac{1}{m-1} \\ &- 2\sum_{l=1}^{\left[\frac{m-1}{2}\right]}\frac{(-1)^l(m)_{2l}\zeta'(2l)}{(2\pi)^{2l}} \\ &- \sum_{l=1}^{\left[\frac{m-2}{2}\right]}\frac{(-1)^l(m)_{2l+1}\zeta(2l+1)}{2\,(2\pi)^{2l}} \end{aligned} \tag{4.8}$$

for $m > 1$ and $I_1^0 = 1 - \gamma$. Combining (4.7) and (4.8), we can get

$$\begin{aligned} &\sum_{l=1}^{\infty}\frac{(-1)^l\zeta(l+1)}{m+l} \\ =& \frac{\ln(2\pi)}{m} - \frac{\gamma}{2} - \frac{1}{m-1} \\ &- 2\sum_{l=1}^{\left[\frac{m-1}{2}\right]}\frac{(-1)^l(m-1)_{2l-1}\zeta'(2l)}{(2\pi)^{2l}} \\ &- \sum_{l=1}^{\left[\frac{m-2}{2}\right]}\frac{(-1)^l(m-1)_{2l}\zeta(2l+1)}{2\,(2\pi)^{2l}} \end{aligned} \tag{4.9}$$

for $m > 1$ and $\sum_{l=1}^{\infty}\frac{(-1)^l\zeta(l+1)}{1+l} = -\gamma$. Thus (4.4) can also be written into a closed form

$$\begin{aligned} I_m^k =& C_m^{k+1}\Big(\frac{\ln(2\pi)+1}{m-k} - \frac{1}{m-k-1} \\ &- \frac{\gamma}{2} - \frac{1}{m-k}\sum_{i=1}^{k}\frac{(\zeta(i+1)-1)}{C_{m-k+i}^{m-k}} \\ &- 2\sum_{l=1}^{\left[\frac{m-k-1}{2}\right]}\frac{(-1)^l(m-k-1)_{2l-1}\zeta'(2l)}{(2\pi)^{2l}} \\ &- \sum_{l=1}^{\left[\frac{m-k-2}{2}\right]}\frac{(-1)^l(m-k-1)_{2l}\zeta(2l+1)}{2\,(2\pi)^{2l}}\Big) \end{aligned} \tag{4.10}$$

for $k < m-1$, and

$$I_m^{m-1} = H_m - \gamma - \sum_{i=1}^{m-1}\frac{\zeta(i+1)}{1+i} \tag{4.11}$$

where $H_m = \sum_{i=1}^{m}\frac{1}{i}$.

We now consider the series I_k defined by

$$\sum_{n_1,\dots,n_k=1}^{\infty}(-1)^n\left(H_n - \ln n - \gamma\right)|_{n=n_1+n_2+\dots+n_k}. \tag{4.12}$$

This series is a condition for the convergence of some other series, in which the order of summation usually can not be changed. By using neutrix limit, we have the following results.

Example 12: For any integer $k > 0$,

$$\begin{aligned} I_k =& \frac{(-1)^k\left(H_{k-1} - \ln 2\right)}{2^k} \\ &+ \frac{1}{(k-1)!}\sum_{l=0}^{k-1}s(k,l+1)Li_{1-l}(-1) \\ &+ \frac{1}{(k-1)!}\sum_{l=0}^{k-1}\Big(s(k,l+1)\times \\ &\left(\frac{d}{d\alpha}Li_\alpha(-1)|_{\alpha=-l} - \gamma Li_{-l}(-1)\right)\Big) \end{aligned} \tag{4.13}$$

and therefore, we have a closed from

$$\begin{aligned} I_k =& \frac{(-1)^k\left(H_{k-1} - \ln 2\right)}{2^k} \\ &+ \frac{(\gamma - \ln(2\pi))\,s(k,1) - s(k,2)}{2(k-1)!} \\ &+ \frac{1}{(k-1)!}\sum_{l=1}^{\left[\frac{k}{2}\right]}\frac{(-1)^{l-1}s(k,2l)(2l-1)!}{\pi^{2l}} \\ &\times(2^{1-2l}\zeta(2l)\ln 2 + 2(1-2^{-2l})\zeta'(2l) \\ &- (2-2^{1-2l})(\ln\pi - H_{2l-1})\zeta(2l)) \\ &+ \frac{1}{(k-1)!}\sum_{l=1}^{\left[\frac{k-1}{2}\right]}\Big(\frac{(-1)^l s(k,2l+1)((2-2^{1-2l})}{\pi^{2l}} \\ &\times\frac{(2l-1)!\zeta(2l)}{\pi^{2l}} + \frac{(1-2^{-2l-1})(2l)!\zeta(2l+1)}{\pi^{2l}}\Big) \end{aligned} \tag{4.14}$$

where $s(i,k)$ is the Stirling number of the first kind.

Proof: Exchanging the summation order of the series, we have

$$\begin{aligned} I_k =& N.\sum_{n=k}^{\infty}((-1)^n\left(H_n-\ln n-\gamma\right) \\ &\times \sum_{n_1 n_2+\cdots+n_k=n} 1) \\ =& N.\sum_{n=k}^{\infty}(-1)^n C_{n-1}^{k-1}\left(H_n-\ln n-\gamma\right) \\ =& \frac{(-1)^k}{(k-1)!}\frac{d^k}{dz^k}N.\sum_{n=1}^{\infty}\frac{z^n H_{n-1}}{n}|_{z=-1} \\ &+\frac{1}{(k-1)!}N.\sum_{n=1}^{\infty}\frac{(-1)^n (n)_k}{n}\left(\frac{1}{n}-\ln n-\gamma\right) \end{aligned} \tag{4.15}$$

for integer $k>0$. Putting

$$(n)_k=\sum_{j=0}^{k}s(k,j)n^j=\sum_{j=0}^{k-1}s(k,j+1)n^{j+1} \tag{4.16}$$

and $N.\sum_{n=1}^{\infty}\frac{z^n H_{n-1}}{n}=\frac{1}{2}\ln^2(1-z)$ into (4.15), we have

$$\begin{aligned} I_k =& \frac{(-1)^k}{2(k-1)!}\frac{d^k}{dz^k}\ln^2(1-z)|_{z=-1} \\ &+\frac{1}{(k-1)!}\sum_{l=0}^{k-1}(s(k,l+1) \\ &\times N.\sum_{n=1}^{\infty}(-1)^n\left(n^{l-1}-n^l\ln n-\gamma n^l\right)). \end{aligned} \tag{4.17}$$

It follows that

$$\begin{aligned} N.\sum_{n=1}^{\infty}(-1)^n n^l =& Li_{-l}(-1), \\ N.\sum_{n=1}^{\infty}(-1)^n n^l\ln n =& \frac{d}{d\alpha}Li_{-\alpha}(-1)|_{\alpha=l}, \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} &\frac{1}{2}\frac{d^k}{dz^k}\ln^2(1-z) \\ =&-\frac{d^{k-1}}{dz^{k-1}}\frac{\ln(1-z)}{1-z} \\ =&-\frac{d^{k-2}}{dz^{k-2}}\left(\frac{\ln(1-z)}{(1-z)^2}-\frac{1}{(1-z)^2}\right) \\ =&-\frac{d^{k-3}}{dz^{k-3}}\left(\frac{2\ln(1-z)}{(1-z)^3}-\frac{1}{(1-z)^3}\right) \\ &+\frac{k!}{(1-z)^k} \\ =&\frac{(k-1)!H_{k-1}}{(1-z)^k}-\frac{(k-1)!\ln(1-z)}{(1-z)^k}. \end{aligned} \tag{4.19}$$

By (4.17), (4.18) and (4.19), we obtain (4.13). By using (3.4)-(3.12) and (4.13), we can also conclude that (4.14) is true. ■

Now, we consider how to use neutrix limit to compute the values of the double series

$$S(p,q)=\sum_{n,m=1}^{\infty}\frac{(-1)^{n+m}\ln^p(n+m)}{(n+m)^q}. \tag{4.20}$$

Example 13: For integers $p\geq 1$ and $q\geqslant 1$,

$$\begin{aligned} S(p,q) =& (-1)^p\frac{d^p}{d\alpha^p}Li_\alpha(-1)|_{\alpha=q-1} \\ &-(-1)^p\frac{d^p}{d\alpha^p}Li_\alpha(-1)|_{\alpha=q}. \end{aligned} \tag{4.21}$$

This result can be obtained by exchanging the order of summation,

$$\begin{aligned} S(p,q) =& \sum_{k=2}^{\infty}\frac{(-1)^k\ln^p k}{k^q}\sum_{n+m=k}1 \\ =& N.\sum_{k=2}^{\infty}\frac{(-1)^k(k-1)\ln^p k}{k^q} \\ =& N.\sum_{k=2}^{\infty}\frac{(-1)^k\ln^p k}{k^{q-1}}-N.\sum_{k=2}^{\infty}\frac{(-1)^k}{k^q}\ln^p k \\ =& (-1)^p\frac{d^p}{d\alpha^p}Li_\alpha(-1)|_{\alpha=q-1} \\ &-(-1)^p\frac{d^p}{d\alpha^p}Li_\alpha(-1)|_{\alpha=q}. \end{aligned} \tag{4.22}$$

Since $\frac{d^p}{d\alpha^p}Li_\alpha(-1)|_{\alpha=q}$ is in closed form for integers $p>0$ and $q\geqslant 0$, $S(p,q)$ must also be closed. For example, by using (4.21) we can obtain closed forms of $S(3,1)$ and $S(3,4)$:

$$\begin{aligned} S(3,1) =& -\gamma^3+\frac{3\gamma^2\ln 2}{2}-\frac{\pi^2\ln 2}{8}-\frac{\ln^3 2}{2} \\ &-\gamma\ln^3 2+\frac{\ln^4 2}{4}-\frac{3\gamma^2\ln\pi}{2} \\ &+\frac{\pi^2\ln\pi}{8}-\frac{3\ln^2 2\ln\pi}{2} \\ &-\frac{3\ln 2\ln^2\pi}{2}+\frac{\ln^3\pi}{2}-3\gamma\gamma_1 \\ &+3\gamma_1\ln 2-3\gamma_1\ln^2 2 \\ &-3\gamma_1\ln\pi-\frac{3\gamma_2}{2}-3\gamma_2\ln 2+\zeta(3), \\ S(3,4) =& -\frac{\pi^4}{720}\ln^3 2 \\ &+\frac{\zeta(3)}{4}\ln^3 2+\frac{3\zeta'(3)}{4}\ln^2 2 \\ &+\frac{3\zeta'(4)}{8}\ln^2 2-\frac{3\zeta''(4)}{8}\ln 2 \\ &+\frac{3\zeta'''(3)}{4}-\frac{7\zeta'''(3)}{8}. \end{aligned}$$

V. Conclusion

A series may diverge, but its ”finite part” may be important after its ”infinite part” is neglected. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but it gives the famous Euler-Mascheroni constant $\gamma = \lim_{m\to\infty} (\sum_{n=1}^{m} \frac{1}{n} - \ln m)$ after its ”infinite part” $\ln m$ is neglected. We follow the traditional idea of neutrix calculus to give a simple and straight forward definition of neutrix limit for divergent series which defines precisely the ”infinite part” and ”finite part” of a divergent series. Combining this definition with the Abel-Plana formula, we have studied the series that define the Riemann-zeta function $\zeta(\alpha)$ and the polylogarithm function $Li_n(z)$ respectively. Our examples show that this definition is very easy to use and the results obtained are consistent with the traditional ones.

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