General Disposition Strategies of Series Configuration Queueing Systems

Yu-Li Tsai*, Member IAENG, Daichi Yanagisawa, and Katsuhiro Nishinari

Abstract—**In this paper, we suggest general disposition strategies for open queueing networks consisting of the arbitrary number of service stations with different service rate for each service station. Poisson arrivals and exponential service times are assumed to apply Markov chain analysis. We apply matrix-geometric method to evaluate steady-state probabilities of the quasi-birth-death process. We define important performance measures including mean number in the system, mean waiting time in the system and blocking probability of the service stations in front of the terminal service station. The exact formulae of stability conditions for the system with both the same and different service rates are derived. Disposition strategies for increasing operational efficiency of the system are proposed for the queueing system with the arbitrary number of service stations.**

Keywords—**Disposition Strategy, Matrix-geometric method, Performance Analysis, Simulation**

I. Introduction

Series configuration queueing systems with no intermediate waiting line between service stations are very popular in modern automated production system. This kind of system with four service stations is depicted in **Fig. 1**. Designing efficient and high performance of this kind of system is very important, since the automobile industry is a major composition of modern economy. We can further save huge production costs by organizing optimized each production line working simultaneously around the world at a time. It is obvious that we can apply simulation results of this queueing system to real industrial applications, such as automobile assembly line, supply chain management or other similar systems. Furthermore, companies in automobile industry can benefit their business through the insights of our studies. Generally, the assumption that the disposition strategies are all the same for the system with the arbitrary number of service stations is reasonable by intuitive considerations. However, the results of the simulation show that the disposition strategies for the series configuration queueing system with different service rate of service stations depending on the number of service stations of the system.

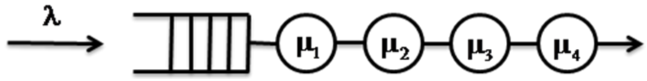

Fig 1. Series configuration queueing system with four service stations.

Traditional exact iteration relationships of the steady-state probabilities are almost impossible to be derived in the case of complex queueing networks (e.g. series configurations systems and systems with breakdowns). In addition, successful derivation of stability conditions is the first prerequisite to further investigate the mean-value analysis of system performance. With the development of modern computational facilities, numerical analysis becomes significant to investigate operational performance of complex queueing systems. Matrix-geometric method provides a powerful framework to deal with queueing systems with complex Markovian structures. We can design algorithms to evaluate steady-state probabilities and concurrently check normalization conditions through this method. Exact stability conditions can also be derived in a systematic way. Advanced analysis of significant performance measures based on evaluating correct steady-state probabilities and stability conditions of the system in accordance with each different quasi-birth and death process. In this study, we derived the exact formulae of stability conditions of the series configurations system with four service stations. Computational simulations reveal several meaningful considerations of this kind of popular queueing system. General dispositions for improving operational efficiency of the system are proposed through numerical results. We expect that our results can provide insights for real industrial applications.

Open queueing networks with no intermediate waiting queue between service stations and blocking phenomena was first investigated by Hunt [1]. He treated other various cases of the series queueing system including queue with infinite capacity between service stations, finite capacity queue between service stations and the unpaced belt-production line. Neuts [2] developed mathematical analysis and related applications of matrix-geometric method. Zhou et al. [3] studied a two-stage tandem queueing network with Markovian arrival process inputs and buffer sharing. An exact analysis of the queueing network was investigated by Markov process. They proposed a matrix filtration technique to derive the probability distribution of queue length arrivals. Optimal design of unpaced assembly lines was studied by Hillier [4]. Allocation of work to the service stations and the allocation of buffer storage space between the service stations are two major topics in their research. They derived exact solutions for small lines with a fixed kind of processing

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time distribution. An approximation methods for fluid flow production lines with multi-server workstations and finite buffers was proposed by Bierbooms et al. [5]. The method is based on decomposition of the production line into single-buffer subsystems. Bierbooms et al. [6] further developed an approximation method to determine the throughput and mean sojourn time of single server tandem queues with general service times and finite buffers by decomposition method. Numerical cases presented that this approach performs accurate estimates for the throughput and mean sojourn time than existing methods. Ke and Tsai [7] first considered the disposition strategy for a self-blocking open queueing system consisting of two service stations with different service rate. Sakuma and Inoie [8] studied and assembly-like queueing system with generally distributed time-constraints. They applied Whitt's approximation to obtain the stationary distribution. Shin and Moon [9] proposed an approximation method for throughput in tandem queues with multiple independent reliable servers at each stage and finite buffers between service stations. Hudson et al. [10] gave complete reviews for the topics about unbalanced unpaced serial production lines. Several unanswered questions about the performance of assembly line are described in this work. Sani and Daman [11] studied a queueing system consisting of two service stations with an exponential server and a general service under a controlled queue discipline. They applied supplementary variable method to derive steady-state distribution of the number of customers in the system. Ramasamy et al. [12] discussed the steady-state analysis of heterogeneous services of a queueing system, called Geo/G/2 queueing system. Embedded method and supplementary variable technique are applied to investigate the system performances.

Tsai et al. [13] compared the disposition strategies for the open queueing networks with two and three service stations. They discovered that the mean waiting time in the system can be reduced significantly by applying appropriate disposition strategy for setting higher service rate for specific service stations. Tsai et al. [14] further developed performance analysis of series configuration queueing system consisting of four service stations.

II. Problem Formulation and Notations

The queueing system consists of independent service stations in series configuration and operates simultaneously. Every customer follows Poisson arrival process with mean arrival rate λ. The time to serve a customer in each service station is exponentially distributed with mean service time $\frac{1}{\mu}$. Each customer should enter all of the service stations from the first service station to the terminal station in order. A complete service is defined as a customer enters to each service station in order and finishes the works in each station. There are no intermediate waiting queues between each service station. The distinctive phenomenon so called blocking after service happens in the case that a customer completes the service in a service station, but another customer in the next station has not finished the service yet. The customer who completed the service is blocked by the customer who is still receiving the service located next station. In this system, the blocking phenomenon happens in the station-1, the station-2 and the station-3. A queue with infinite capacity is allowed in front of the first service station. In addition, only a customer can enter each service station at a time and the service rate is independent of the number of customers. The service of the system obeys the first come first serve (FCFS) discipline.

We use P_{n_1,n_2,n_3,n_4,n_5} to denote the steady-state probability P_{n_1,n_2,n_3,n_4,n_5} of n_1 customer in the station-4 and n_2 customer in the station-3 and n_3 customer in the station-2 and n_4 customer in the station-1 and n_5 customer in the queue.

III. Modeling Framework

Let $\mathbf{P}=[\mathbf{P_0},\mathbf{P_1},\mathbf{P_2},...]$ denote the steady-state probability vector corresponding to the transition matrix Q. The steady-state equation of the quasi-birth-death process is $\mathbf{P}\mathrm{Q}=\mathbf{0}$, with the normalization constraint $\mathbf{P1}=1$. We can obtain the following set of matrix equations with a finite dimension:

$$\mathbf{P_0}\mathrm{B}_{0,0}+\mathbf{P_1}\mathrm{B}_{1,0}=\mathbf{0}, \tag{1}$$

$$\mathbf{P_0}\mathrm{B}_{0,1}+\mathbf{P_1}\mathrm{A}_1+\mathbf{P}_2\mathrm{A}_2=\mathbf{0}, \tag{2}$$

$$\mathbf{P}_i\mathrm{A}_0+\mathbf{P}_{i+1}\mathrm{A}_1+\mathbf{P}_{i+2}\mathrm{A}_2=\mathbf{0}, \qquad i\geq 1. \tag{3}$$

The following recurrence relation can be constructed with a rate matrix R

$$\mathbf{P}_i=\mathbf{P}_{i-1}\mathrm{R}=\mathbf{P}_1\mathrm{R}^{i-1}, \qquad i\geq 1. \tag{4}$$

The unknown rate matrix R can be obtained by substituting (4) into (3), we obtain the following characteristic equation of the recurrence relation

$$\mathrm{A}_0+\mathrm{R}\mathrm{A}_1+\mathrm{R}^2\mathrm{A}_2=0. \tag{5}$$

The simplified matrix equations of (1) and (2) can be represented as

$$\mathbf{P_0}\mathrm{B}_{0,0}+\mathbf{P_1}\mathrm{B}_{1,0}=\mathbf{0}, \tag{6}$$

$$\mathbf{P_0}\mathrm{B}_{0,1}+\mathbf{P_1}(\mathrm{A}_1+\mathrm{R}\mathrm{A}_2)=\mathbf{0}. \tag{7}$$

According to Bloch et al. [15], the normalization condition equation that only involves $\mathbf{P_0}$ and $\mathbf{P_1}$ is given by

$$\mathbf{P_0}\mathbf{1}+\mathbf{P_1}(\mathrm{I}-\mathrm{R})^{-1}\mathbf{1}=1, \tag{8}$$

where I is the identity matrix with same size as the rate matrix R.

The rate matrix R in (5) is solved by iterative method. Collecting (6), (7) and (8) together, the steady-state probability vector of $\mathbf{P_0}$ and $\mathbf{P}_1$ can be obtained by solving following matrix equation

$$(\mathbf{P_0},\mathbf{P_1})\begin{pmatrix}\mathrm{B}_{0,0} & \mathrm{B}_{0,1}^{*} & \mathbf{1}\\ \mathrm{B}_{1,0} & (\mathrm{A}_1+\mathrm{R}\mathrm{A}_2)^{*} & (\mathrm{I}-\mathrm{R})^{-1}\mathbf{1}\end{pmatrix}=(\mathbf{0},1). \tag{9}$$

where $(.)^{*}$ indicates that the last column of the included matrix is removed to avoid linear dependency.

- Stability Conditions

The stability condition is given by Neuts [2] for the ergodicity of steady-state probabilities:

$$\mathbf{P}_\mathrm{A}\mathrm{A}_0\mathbf{1}<\mathbf{P}_\mathrm{A}\mathrm{A}_2\mathbf{1}, \tag{10}$$

where $\mathbf{P}_\mathrm{A}$ is the steady-state probability vector corresponding to the generator matrix A.

Theorem 1. The stability conditions for the series configuration system consisting of four service stations.
The following inequalities are necessary conditions for the system to be stable.
(1) For $\mu_1 \neq \mu_2 \neq \mu_3 \neq \mu_4$

$$\lambda < \frac{N}{D}, \tag{11}$$

where the exact results of the N and the D are shown in the supplementary document.

(2) Special case: $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu$

$$\lambda < \frac{4024}{7817}\mu. \tag{12}$$

The maximum utilization of the system consisting of four service stations is approximate 0.514 indicated by (12).

IV. PERFORMANCE METRICS AND DISPOSITION STRATEGY

Performance measures including mean number in the system, mean number in the queue, mean waiting time in the system, mean waiting time in the queue and blocking probability of the service stations in front of the terminal station for the series configuration system consisting of four service stations are defined. In addition, we propose general disposition strategies for the series configuration queueing system consisting of the arbitrary number of service stations based on the numerical results in this section.

- Performance measures

Performance measures for the system consisting of four service stations are defined by
(1) Mean number of customers in the system

$$\begin{aligned}
L = {} & (P_{0,0,0,1,0} + P_{0,0,1,0,0} + P_{0,1,0,0,0} + P_{1,0,0,0,0} + P_{1,b,0,0,0} + P_{0,0,1,b,0} + P_{0,1,b,0,0} + P_{0,1,b,b,0} + P_{1,b,b,0,0}) \\
& + 2\,(P_{0,0,0,1,1} + P_{0,0,1,1,0} + P_{0,1,0,1,0} + P_{1,0,0,1,0} + P_{1,0,1,0,0} + P_{1,1,0,0,0} + P_{0,1,1,0,0} \\
& + P_{1,b,0,1,0} + P_{0,0,1,b,1} + P_{0,1,1,b,0} + P_{1,1,b,0,0} + P_{0,1,b,1,0} + P_{1,0,1,b,0} + P_{1,b,1,0,0}) \\
& + 3\,(P_{0,0,0,1,2} + P_{0,0,1,1,1} + P_{0,1,0,1,1} + P_{1,0,0,1,1} + P_{1,0,1,1,0} + P_{1,1,0,1,0} + P_{0,1,1,1,0} + P_{1,1,1,0,0}) \\
& + \sum_{n=4}^{\infty} (P_{0,0,0,1,n-1} + P_{0,0,1,1,n-2} + P_{0,1,0,1,n-2} + P_{1,0,0,1,n-2} + P_{1,0,1,1,n-3} + P_{1,1,0,1,n-3} + P_{0,1,1,1,n-3} + P_{1,1,1,1,n-4}) \cdot n \\
& + \sum_{n=3}^{\infty} (P_{1,b,0,1,n-2} + P_{0,0,1,b,n-1} + P_{0,1,1,b,n-2} + P_{1,1,b,1,n-3} + P_{0,1,b,1,n-2} + P_{1,0,1,b,n-2} + P_{1,b,1,1,n-3} + P_{1,1,1,b,n-3}) \cdot n \\
& + \sum_{n=2}^{\infty} (P_{0,1,b,b,n-1} + P_{1,b,b,1,n-2} + P_{1,1,b,b,n-2} + P_{1,b,1,b,n-2}) \cdot n \\
& + \sum_{n=1}^{\infty} (P_{1,b,b,b,n-1}) \cdot n.
\end{aligned} \tag{13}$$

(2) Mean number of customers in the queue

$$\begin{aligned}
L_q = {} & (P_{0,0,0,1,1} + P_{0,0,1,b,1} + P_{0,0,1,1,1} + P_{0,1,0,1,1} + P_{1,0,0,1,1} + P_{1,0,1,1,1} + P_{1,1,0,1,1} \\
& + P_{1,b,0,1,1} + P_{0,1,1,1,1} + P_{0,1,1,b,1} + P_{0,1,b,b,1} + P_{0,1,b,1,1} + P_{1,0,1,b,1}) \\
& + 2\,(P_{0,0,0,1,2} + P_{0,0,1,1,2} + P_{0,1,0,1,2} + P_{1,0,0,1,2} + P_{0,0,1,b,2}) \\
& + 3\,(P_{0,0,0,1,3}) + \sum_{n=4}^{\infty} (P_{0,0,0,1,n}) \cdot n + \sum_{n=3}^{\infty} (P_{0,0,1,1,n} + P_{0,1,0,1,n} + P_{1,0,0,1,n} + P_{0,0,1,b,n}) \cdot n \\
& + \sum_{n=2}^{\infty} (P_{1,0,1,1,n} + P_{1,1,0,1,n} + P_{1,b,0,1,n} + P_{0,1,1,1,n} + P_{0,1,1,b,n} + P_{0,1,b,b,n} + P_{0,1,b,1,n} + P_{1,0,1,b,n}) \cdot n \\
& + \sum_{n=1}^{\infty} (P_{1,1,1,1,n} + P_{1,1,b,1,n} + P_{1,b,b,1,n} + P_{1,b,1,1,n} + P_{1,1,1,b,n} + P_{1,b,b,b,n} + P_{1,1,b,b,n} + P_{1,b,1,b,n}) \cdot n.
\end{aligned} \tag{14}$$

(3) Mean waiting time in the system (Little's Law)

$$W = \frac{L}{\lambda}. \tag{15}$$

(4) Mean waiting time in the queue (Little's Law)

$$W_q = \frac{L_q}{\lambda}. \tag{16}$$

(5) Blocking probability of the customer in the station-1

$$\begin{aligned}
P_{b,1} = \sum_{n=0}^{\infty} & P_{0,0,1,b,n} + P_{0,1,1,b,n} + P_{0,1,b,b,n} + P_{1,0,1,b,n} \\
& + P_{1,1,1,b,n} + P_{1,b,b,b,n} + P_{1,1,b,b,n} + P_{1,b,1,b,n}.
\end{aligned} \tag{17}$$

(6) Blocking probability of the customer in the station-2

$$\begin{aligned}
P_{b,2} = \sum_{n=0}^{\infty} & P_{1,1,b,1,n} + P_{0,1,b,1,n} + P_{0,1,b,b,n} \\
& + P_{1,b,b,1,n} + P_{1,b,b,b,n} + P_{1,1,b,b,n}.
\end{aligned} \tag{18}$$

(7) Blocking probability of the customer in the station-3

$$\begin{aligned}
P_{b,3} = \sum_{n=0}^{\infty} & P_{1,b,0,1,n} + P_{1,b,1,1,n} + P_{1,b,b,1,n} \\
& + P_{1,b,b,b,n} + P_{1,b,1,b,n}.
\end{aligned} \tag{19}$$

Proposition 3.1. Disposition strategies for the series configuration queueing system consisting of the arbitrary number of service stations with different service rates are different.
We propose different disposition strategies for the system based on our previous research Tsai et al. [13] and this work in order to increase the operational efficiency.
(1) Series configuration queueing system with **the odd number** of service stations
It is better to arrange lower service rate for the first service station compared with other service stations in the system in order to obtain the best operational efficiency for the system with the odd number of service stations.

(2) Series configuration queueing system with **the even number** of service stations
We suggest setting higher service rates for the service stations in front of the terminal station as possibly as we can. In this way, the mean waiting time in the system would be the shortest compared with other disposition strategies.

V. NUMERICAL RESULTS

In this section, we illustrate numerical experiments for the queueing system consisting of four stations. Performance metrics of the system with equivalent service rates (i.e. $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu$) and different service rates are presented. We will suggest better disposition strategies to increase operational efficiency for the system according to the results of simulations.

- **Same service rates for each service station**

First, we study the increasing trends of mean number in the system and blocking probabilities as a function of mean arrival rate λ. **Fig. 2.** presents the mean number in the system. It is observed that the upper bound of the stability condition of the mean number in the system approaches to $\frac{4024}{7817}$ (≈ 0.514), which proves the correctness of the exact results we derived in the section 3.2. Blocking probability of the station-1, the station-2 and the station-3 as a function of mean arrival rate of the system consisting of four service stations is shown in **Fig. 3**. Furthermore, it is investigated that the blocking probability of the station-1 is higher than that of the station-2 and of the station-3 in this case.

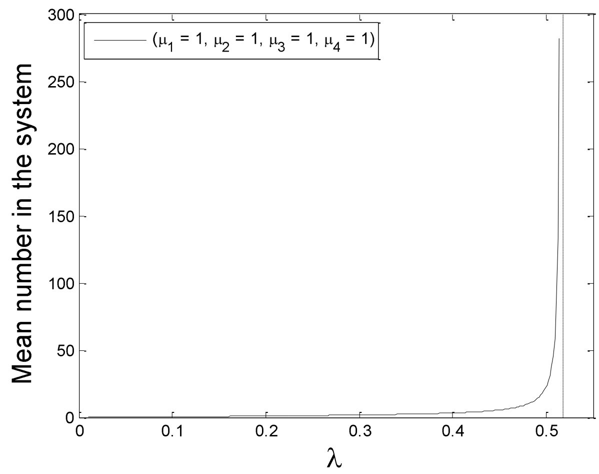

Fig 2. Mean number in the system.

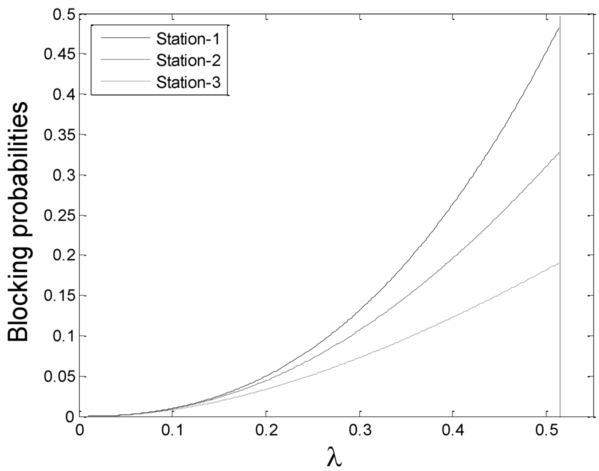

Fig 3. Blocking probability with $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$.

- **Controlling the service rates of the three service stations**

In the case of different service rates, we study the conditions that we can concurrently control the service rates of three service stations and the service rate of only one service station for the system consisting of four service stations.

First, we investigate the cases that we are able to control three service rates of the service stations in this system. We set $\mu_1 = 2, \mu_2 = 2, \mu_3 = 2, \mu_4 = 1$ and $\mu_1 = 2, \mu_2 = 2, \mu_3 = 1, \mu_4 = 2$ and $\mu_1 = 2, \mu_2 = 1, \mu_3 = 2, \mu_4 = 2$ and $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2, \mu_4 = 2$, then vary the mean arrival rate λ from 0.01 to 0.7. It is suggested to set higher service rates for the station-1, the station-2 and the station-3 in order to obtain the best operational efficiency for the system, as shown in **Fig 4**. This best disposition strategy for the system consisting of four service stations is accordant with the result of the system comprising two service stations indicated by Tsai et al. [13].

Since the mean waiting time in the system of the case $\mu_1 = 2, \mu_2 = 1, \mu_3 = 2, \mu_4 = 2$ is always higher than that of the case $\mu_1 = 2, \mu_2 = 2, \mu_3 = 1, \mu_4 = 2$ for all mean arrival rates. We just compare the cases between $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2, \mu_4 = 2$ and $\mu_1 = 2, \mu_2 = 2, \mu_3 = 1, \mu_4 = 2$. It is investigated that the mean waiting time of the system of the case $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2, \mu_4 = 2$ is higher than that of the case $\mu_1 = 2, \mu_2 = 2, \mu_3 = 1, \mu_4 = 2$ when mean arrival rate is lower than 0.65. This result shows that when the mean arrival rate is lower than 0.65, the case $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2, \mu_4 = 2$ causes longer time for the customers waiting in the queue as show in **Fig 4**. The mean waiting time in the queue of the case $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2, \mu_4 = 2$ becomes shorter than that of the case $\mu_1 = 2, \mu_2 = 2, \mu_3 = 1, \mu_4 = 2$ when the mean arrival rate is greater than 0.65.

We suggest the case $\mu_1 = 2, \mu_2 = 2, \mu_3 = 2, \mu_4 = 1$ as the best disposition strategy, when we are able to control service rates of three service stations for the system.

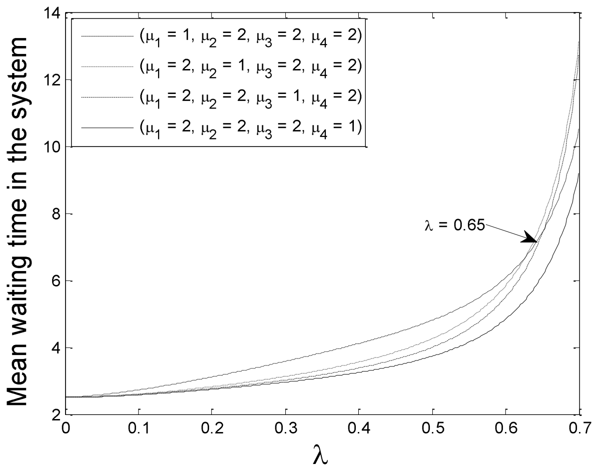

Fig 4. Mean waiting time in the system.
(Controlling the service rates of the three service stations)

- **Controlling the service rates of only one service station**

Next, we study the cases of controlling service rate of one service station, we set $\mu_1 = 2, \mu_2 = 1, \mu_3 = 1, \mu_4 = 1$ and $\mu_1 = 1, \mu_2 = 2, \mu_3 = 1, \mu_4 = 1$ and $\mu_1 = 1, \mu_2 = 1, \mu_3 = 2, \mu_4 = 1$ and $\mu_1 = 1, \mu_2 = 1, \mu_3 = 1, \mu_4 = 2$ then vary the mean arrival rate λ from 0.01 to 0.5. It is investigated that the mean waiting time is the greatest in the case of $\mu_1 = 1, \mu_2 = 1, \mu_3 = 1, \mu_4 = 2$ compared with other three cases as shown in **Fig 5**. This disposition strategy makes the customers in the queue difficult to enter the service stations, since the mean waiting time in the queue is higher than other three cases.

Similar to the case studies of controlling three service stations in previous section, we note that the mean waiting time in the system in the case of $\mu_1 = 1, \mu_2 = 2, \mu_3 = 1, \mu_4 = 1$ is always lower than that of the case $\mu_1 = 1, \mu_2 = 1, \mu_3 = 2, \mu_4 = 1$. We compare the case $\mu_1 = 2, \mu_2 = 1, \mu_3 = 1, \mu_4 = 1$ with the case

$\mu_1 = 1, \mu_2 = 2, \mu_3 = 1, \mu_4 = 1$ for discussing the disposition strategies. We consider that the mean waiting time in the system of the case $\mu_1 = 2, \mu_2 = 1, \mu_3 = 1, \mu_4 = 1$ is lower than that of the case $\mu_1 = 1, \mu_2 = 2, \mu_3 = 1, \mu_4 = 1$ when the mean arrival rate is lower than 0.42. It is noted the mean waiting time in the queue is almost the same for both cases, so setting higher service rate for the station-1 is better to make customer to enter the service stations when mean arrival rate is lower than 0.42, as shown in **Fig 5**. When the mean arrival is greater than 0.42, it is observed that the mean waiting time in the system in the case of $\mu_1 = 2, \mu_2 = 1, \mu_3 = 1, \mu_4 = 1$ is larger than that of the case $\mu_1 = 1, \mu_2 = 2, \mu_3 = 1, \mu_4 = 1$. While the increasing of the mean arrival rate, the setting of lower service rates in the station-1 and the station-2 and the station-3 makes customers take longer waiting time in the queue.

We suggest that setting $\mu_1 = 2, \mu_2 = 1, \mu_3 = 1, \mu_3 = 1$ as the best disposition strategy when the mean arrival rate is lower than 0.42. On the other hand, for the case that we can control only one of the service rates for the system consisting of four service stations, we observe that case of $\mu_1 = 1, \mu_2 = 2, \mu_3 = 1, \mu_4 = 1$ is a relatively better disposition strategy compared with the case $\mu_1 = 2, \mu_2 = 1, \mu_3 = 1, \mu_3 = 1$ when the mean arrival rate becomes larger than 0.42.

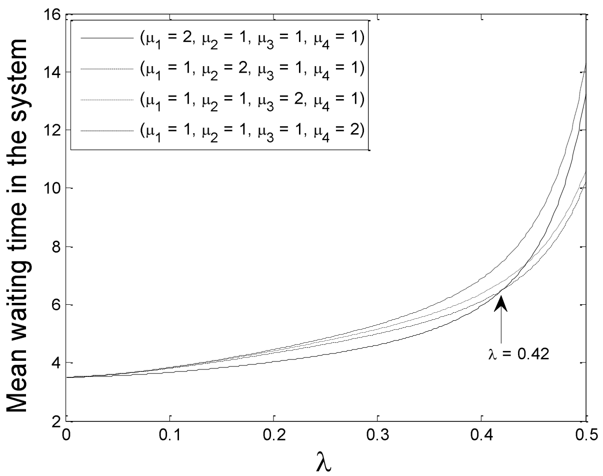

Fig 5. Mean waiting time in the system.
(Controlling the service rates of only one service station)

- **Controlling the service rates of two service stations**

We continue to study the cases for controlling two service stations. We set $\mu_1 = 1, \mu_2 = 1, \mu_3 = 2, \mu_4 = 2$ and $\mu_1 = 2, \mu_2 = 1, \mu_3 = 1, \mu_4 = 2$ and $\mu_1 = 2, \mu_2 = 2, \mu_3 = 1, \mu_4 = 1$ and $\mu_1 = 1, \mu_2 = 2, \mu_3 = 1, \mu_4 = 2$ and $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2, \mu_4 = 1$ and $\mu_1 = 2, \mu_2 = 1, \mu_3 = 2, \mu_4 = 1$ then vary the mean arrival rate λ from 0.01 to 0.6. The six patterns of mean waiting time in the system by controlling service rates of two service stations are shown in **Fig 6.** and **Fig 7.**, respectively. It is noted that the cases $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2, \mu_4 = 1$ and $\mu_1 = 2, \mu_2 = 1, \mu_3 = 2, \mu_4 = 1$ and $\mu_1 = 2, \mu_2 = 2, \mu_3 = 1, \mu_4 = 1$ are relatively better disposition strategies in all of six cases, so we plot these cases together again to determine the best disposition strategy.

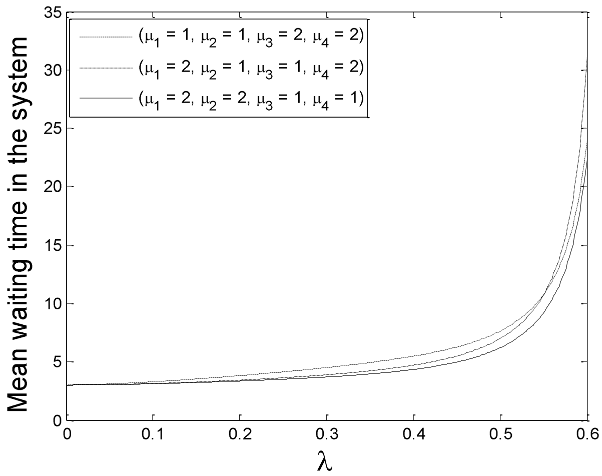

Fig 6. Mean waiting time in the system.
(Controlling the service rates of two service stations)

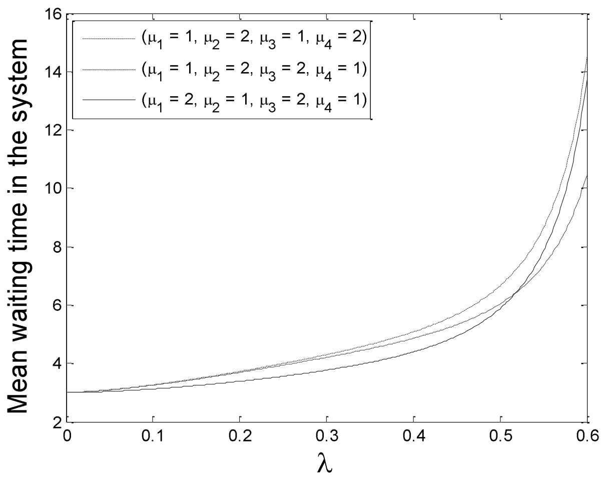

Fig 7. Mean waiting time in the system.
(Controlling the service rates of two service stations)

We observed that the mean waiting time in the system in the cases of $\mu_1 = 2, \mu_2 = 1, \mu_3 = 2, \mu_4 = 1$ and $\mu_1 = 2, \mu_2 = 2, \mu_3 = 1, \mu_4 = 1$ are larger than that of the case $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2, \mu_4 = 1$ when the mean arrival rate is lower than 0.52. On the other hand, the case $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2, \mu_4 = 1$ performs better than the other cases while the mean arrival rate is greater than 0.52 as show in **Fig 8.** It is clear that the mean waiting time in the queue of the cases of $\mu_1 = 2, \mu_2 = 1, \mu_3 = 2, \mu_4 = 1$ and $\mu_1 = 2, \mu_2 = 2, \mu_3 = 1, \mu_4 = 1$ grow very quickly compared with the case $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2, \mu_4 = 1$ when mean arrival rate is greater than 0.52 as shown in **Fig 9.** Because the difference of the mean waiting time in the system of these cases are not significant, while the mean arrival rate is lower than 0.52. We suggest that, for average performances, the case $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2, \mu_4 = 1$ is the best disposition strategy to increase the efficiency of the system when we are able to control service rates of two service stations for the system.

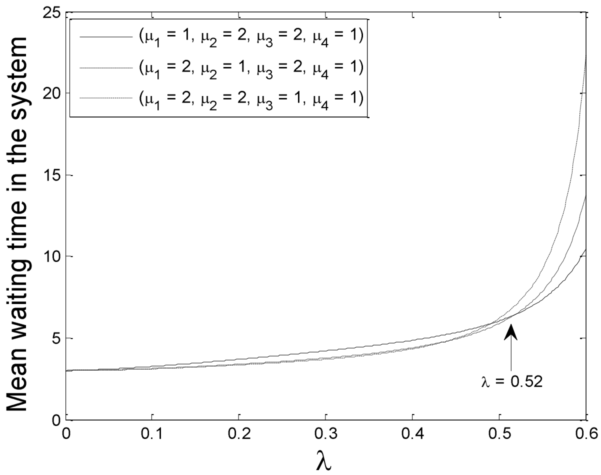

Fig 8. Mean waiting time in the system.
(Controlling the service rates of two service stations)

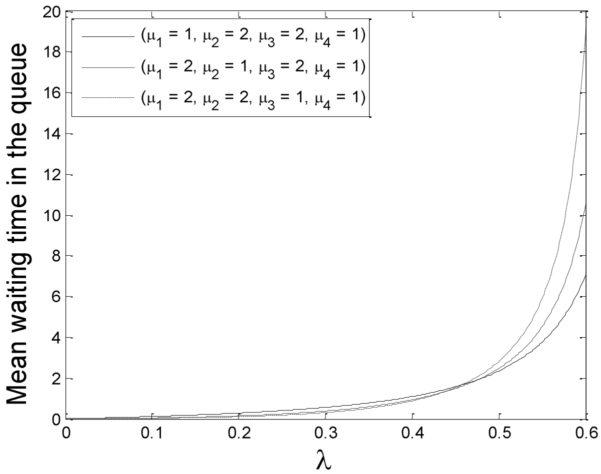

Fig 9. Mean waiting time in the queue.
(Controlling the service rates of two service stations)

- **Mean waiting time in the system with fixed total service rate of service stations**

We finally investigate the disposition strategies of the series configuration queueing system constrained by the fixed total service rate of service stations. For instance, when we can control the service rate of three service stations, the fixed total service rate of service stations is 7 (i.e. summing service rate of each service station, $2+2+2+1=7$ in the case of controlling three service stations). The results of simulation show that disposing equivalent service rate for each service station is the best disposition strategy for the series configurations queueing system as shown in **Fig 10.** and **Fig 11.** and **Fig 12.**, respectively. This disposition strategy is especially suitable for automation production system with computer integrated manufacturing, because keeping equivalent float service rate for each service station is more difficult than focusing on setting integer service rate for specific service stations in practical applications.

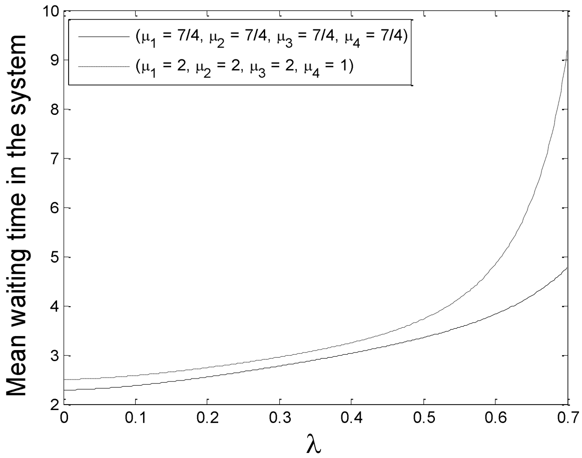

Fig 10. Mean waiting time in the system.
(Controlling the service rates of three service stations with total fixed service rate of 7)

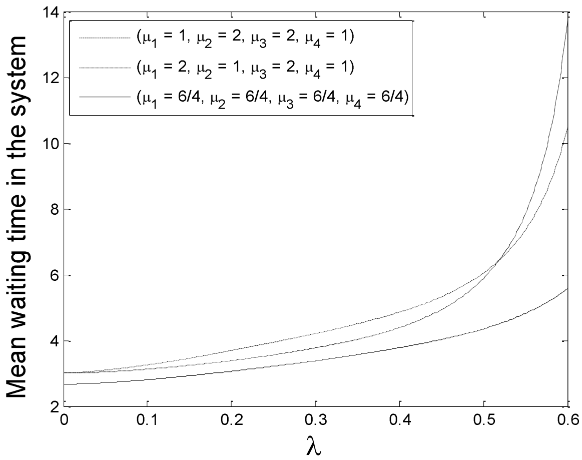

Fig 11. Mean waiting time in the system.
(Controlling the service rates of two service stations with total fixed service rate of 6)

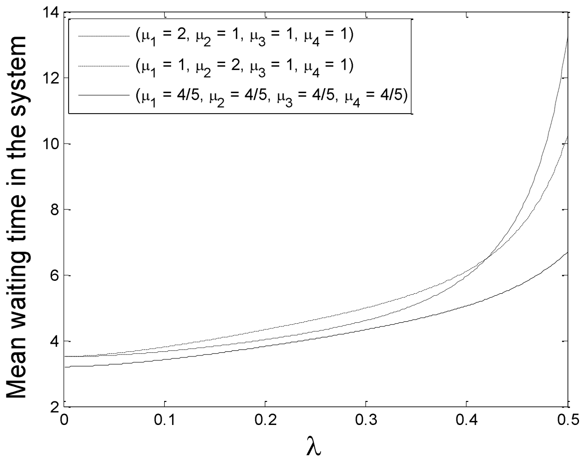

Fig 12. Mean waiting time in the system.
(Controlling the service rates of only one service station with total fixed service rate of 5)

VI. Conclusion

We propose general disposition strategies for the series configuration queueing system based on a series of research, Tsai et al. [13] and in this work. We discover that the disposition strategies for improving the operational efficiency of the series configuration queueing system depend on the number of service stations of the system. Exact formulae of stability condition with same and different service rates of each service station are derived, so that we can confirm the correctness of the estimated steady-state probabilities. The results can be applied in automobile production lines and related similar queueing systems.

Numerical results verify the consistency of stability condition of the system with equivalent service rate of each service station. Moreover, the maximum utilization of the system decrease as the number of the service stations increase due to enlarging blocking probability of the station-1 of the series configuration queueing system. The numerical result of the blocking probability of the station-1 in the condition of very high arrival rate of the system reflects that the derived theoretical result of maximum utilization approaches to 0.514 for the system with four service stations.

Future research will focus on the transient analysis of the series configuration system in order to study the dynamic behavior of the system. Statistical analysis with real data in the automobile production line will also be conducted to further validate our theoretical results and inferences.

Acknowledgment

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