Some Results of Resistance Distance and Kirchhoff Index Based on R-Graph

Qun Liu

Abstract—The resistance distance between any two vertices of a connected graph is defined as the effective resistance between them in the electrical network constructed from the graph by replacing each edge with a (unit) resistor. The Kirchhoff index \( K_f(G) \) is the sum of resistance distances between all pairs of vertices in \( G \). For a graph \( G \), let \( R(G) \) be the graph obtained from \( G \) by adding a new vertex corresponding to each edge of \( G \) and by joining each new vertex to the end vertices of the corresponding edge. Let \( G_1 \circ G_2, G_1 \oplus G_2 \) be the \( R \)-vertex corona and \( R \)-edge corona of \( G_1 \) and \( G_2 \). In this paper, formulate for the resistance distance and the Kirchhoff index in \( G_1 \circ G_2 \) and \( G_1 \oplus G_2 \) whenever \( G_1 \) and \( G_2 \) are arbitrary graphs are obtained. This improves and extends some earlier results.

Index Terms—Kirchhoff index, Resistance distance, \( R \)-vertex corona, \( R \)-edge corona, Generalized inverse

I. INTRODUCTION

In 1993, Klein and Randić [1] introduced a distance function named resistance distance on the basis of electrical network theory. The resistance distance \( r_{ij}(G) \) between any two vertices \( i \) and \( j \) in \( G \) is defined to be the effective resistance between them when unit resistors are placed on every edge of \( G \). The Kirchhoff index \( K_f(G) \) is the sum of resistance distances between all pairs of vertices of \( G \). The computation of two-point resistances in networks and the Kirchhoff index are classical problem in electric theory and graph theory. The resistance distance and the Kirchhoff index attracted extensive attention due to its wide applications in physics, chemistry and others. For more information on resistance distance and Kirchhoff index of graphs, the readers are referred to Refs. [2]-[12] and the references therein.

Let \( G = (V(G), E(G)) \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). Let \( d_i \) be the degree of vertex \( i \) in \( G \) and \( D_G = \text{diag}(d_1, d_2, \ldots, d_{|V(G)|}) \) the diagonal matrix with all vertex degrees of \( G \) as its diagonal entries. For a graph \( G \), let \( A_G \) and \( B_G \) denote the adjacency matrix and vertex-edge incidence matrix of \( G \), respectively. The matrix \( L_G = D_G - A_G \) is called the Laplacian matrix of \( G \), where \( D_G \) is the diagonal matrix of vertex degrees of \( G \). We use \( \mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0 \) to denote the spectrum of \( L_G \). If \( G \) is connected, then any principal submatrix of \( L_G \) is nonsingular.

In [13], new graph operations based on \( R(G) \) graphs: \( R \)-vertex corona and \( R \)-edge corona, are introduced, and their \( A \)-spectrum(resp. \( L \)-spectrum) are investigated. For a graph \( G \), let \( R(G) \) be the graph obtained from \( G \) by adding a new vertex corresponding to each edge of \( G \) and by joining each new vertex to the end vertices of the corresponding edge. Let \( I(G) \) be the set of newly added vertices, i.e. \( I(G) = V(R(G)) \setminus V(G) \). Let \( G_1 \) and \( G_2 \) be two vertex-disjoint graphs.

Definition 1. ([13]) The \( R \)-vertex corona of \( G_1 \) and \( G_2 \), denoted by \( G_1 \circ G_2 \), is the graph obtained from vertex disjoint \( R(G_1) \) and \( |V(G_1)| \) copies of \( G_2 \) by joining the ith vertex of \( V(G_1) \) to every vertex in the ith copy of \( G_2 \).

Definition 2. ([13]) The \( R \)-edge corona of \( G_1 \) and \( G_2 \), denoted by \( G_1 \oplus G_2 \), is the graph obtained from vertex disjoint \( R(G_1) \) and \( |V(G_1)| \) copies of \( G_2 \) by joining the ith vertex of \( V(G_1) \) to every vertex in the ith copy of \( G_2 \).

Note that if \( G_i \) has \( n_i \) vertices and \( m_k \) edges for \( i = 1, 2 \), then \( G_1 \circ G_2 \) has \( n_1 + n_1 + n_1 n_2 \) vertices and \( 3n_1 + n_1 n_2 + n_1 n_2 \) edges, \( G_1 \oplus G_2 \) has \( n_1 + n_1 + m_1 n_2 \) vertices and \( 3n_1 + n_1 n_2 + m_1 n_2 \) edges.

This paper is organized as follows. In Section 2, some auxiliary lemma are given. In Section 3, we obtain formulas for resistance distances of \( R \)-vertex corona and \( R \)-edge corona of two arbitrary graphs. In Section 4, we obtain formulas for Kirchhoff index of \( R \)-vertex corona and \( R \)-edge corona of two arbitrary graphs.

II. PRELIMINARIES

Let \( M \) be a square matrix \( M \). The \( \{1\} \)-inverse of \( M \) is a matrix \( X \) such that \( M X M = M \). If \( M \) is singular, then \( M \) has infinitely many \( \{1\} \)-inverse [14]. The group inverse of \( M \), denoted by \( M^\# \), is the unique matrix \( X \) such that \( M X M = M \) and \( MX = XM \). It is known [9, 11] that \( M^\# \) exists if and only if \( \text{rank}(M) = \text{rank}(M^2) \).

If \( M \) is real symmetric, then \( M^\# \) exists and \( M^\# \) is a symmetric \( \{1\} \)-inverse of \( M \). Actually, \( M^\# \) is equal to the Moore-Penrose inverse of \( M \) since \( M \) is symmetric [15].

We use \( M^{(1)} \) to denote any \( \{1\} \)-inverse of a matrix \( M \). Let \( (M)^{(1)}_{uv} \) denote the \((u,v)\)-entry of \( M \).

Lemma 1. ([16]) Let \( G \) be a connected graph. Then

\[
\begin{align*}
r_{uv}(G) &= (L_G^{(1)})_{uu} + (L_G^{(1)})_{uv} - (L_G^{(1)})_{uu} - (L_G^{(1)})_{uu} \\
&= (L_G^\#)_{uu} + (L_G^\#)_{uv} - 2(L_G^\#)_{uu}.
\end{align*}
\]

Let \( 1_n \) denotes the column vector of dimension \( n \) with all the entries equal one. We will often use \( 1_n \) to denote an all-ones column vector if the dimension can be read from the context.

Lemma 2. ([17]) For any graph, we have \( L_G^\# 1_n = 0 \).

Lemma 3. ([18]) Let

\[
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

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be a nonsingular matrix. If $A$ and $D$ are nonsingular, then
\[
M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} - A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} \\ (A - BD^{-1}C)^{-1} - A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} \end{pmatrix},
\]
where $S = D - CA^{-1}B$.

For a vertex $i$ of a graph $G$, let $T(i)$ denote the set of all neighbors of $i$ in $G$.

**Lemma 4.** ([17]) Let $G$ be a connected graph. For any $i, j \in V(G)$,
\[
r_{ij}(G) = d_i^{-1} \left( 1 + \sum_{k \in T(i)} r_{kj}(G) - d_i^{-1} \sum_{k, l \in T(i)} r_{kl}(G) \right).
\]

**Lemma 5.** ([16]) Let $G$ be a connected graph on $n$ vertices. Then
\[
Kf(G) = ntr(L_G^{(1)}) - 1^T L_G^{(1)} 1 = ntr(L_G^\#).
\]

**Lemma 6.** ([14]) Let $G$ be a connected graph of order $n$ with edge set $E$. Then
\[
\sum_{u < v, uv \in E} r_{uv}(G) = n - 1.
\]

**Lemma 7.** Let
\[
L = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}
\]
be the Laplacian matrix of a connected graph. If $D$ is nonsingular, then
\[
X = \begin{pmatrix} H^\# & -H^\#BD^{-1} \\ -D^{-1}B^TH^\#D^{-1} + D^{-1}B^TH^\#BD^{-1} \end{pmatrix}
\]
is a symmetric $(1)$-inverse of $L$, where $H = A - BD^{-1}B^T$.

**Proof** Let $H = A - BD^{-1}B^T$, and
\[
X = \begin{pmatrix} H^\# & -H^\#BD^{-1} \\ -D^{-1}B^TH^\#D^{-1} + D^{-1}B^TH^\#BD^{-1} \end{pmatrix} = \begin{pmatrix} H^\# & 0 \\ -D^{-1}B^TH^\# & I \\ 0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} D^{-1}B^TH^\#D^{-1} & 0 \\ 0 & D^{-1}B^TH^\#D^{-1} \end{pmatrix}.
\]
Since $D$ is nonsingular, then
\[
L = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} H & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}B^T & I \end{pmatrix}.
\]
By computation, we have
\[
LX = \begin{pmatrix} H & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}B^T & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & D^{-1}B^T \end{pmatrix} \begin{pmatrix} H^\# & 0 \\ 0 & D^{-1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} H^\# & 0 \\ 0 & D^{-1} \end{pmatrix} = \begin{pmatrix} H^\# & -H^\#BD^{-1} + BD^{-1} \end{pmatrix}.
\]
Hence $X$ is a symmetric $(1)$-inverse of $L$, where $H = A - BD^{-1}B^T$.

**Remarks:** The above result is similar to Lemma 2.8 in [16], this is another form of Lemma 2.8, but in the process of computing the resistance distance and Kirchhoff index of graph $G_1 \oplus G_2$ and $G_1 \odot G_2$, we use this formula to be more superior than Lemma 2.8 in [16].

**III. RESISTANCE DISTANCE IN R-VERTEX CORONA AND R-EDGE CORONA OF TWO GRAPHS**

We first give formulae for resistance distance between two arbitrary vertexes in $G_1 \odot G_2$.  

**Theorem 1.** Let $G_1$ be a graph with $n_1$ vertices and $m_1$ edges and $G_2$ be a graph with $n_2$ vertices and $m_2$ edges. Then the following holds:

(a) For any $i, j \in V(G_1)$, we have
\[
r_{ij}(G_1 \odot G_2) = \frac{2}{3} (L_{G_1}^\#)_{ii} + \frac{2}{3} (L_{G_2}^\#)_{jj} - \frac{4}{3} (L_{G_1}^\#)_{jj}.
\]

(b) For any $i, j \in V(G_2)$, we have
\[
r_{ij}(G_1 \odot G_2) = (I_{n_1} \otimes (L_{G_2} + I_{n_2})^{-1})_{jj} - 2(I_{n_1} \otimes (L_{G_2} + I_{n_2})^{-1})_{jj}.
\]

(c) For any $i \in V(G_1), j \in V(G_2)$, we have
\[
r_{ij}(G_1 \odot G_2) = \frac{2}{3} (L_{G_1}^\#)_{ii} + (I_{n_1} \otimes (L_{G_2} + I_{n_2})^{-1})_{jj} - \frac{2}{3} (L_{G_1}^\#)_{jj}.
\]

(d) For any $i \in I(G_1)$, $j \in V(G_1) \cup V(G_2)$, let $u_i v_i \in E(G_1)$ denote the edge corresponding to $i$, we have
\[
r_{ij}(G_1 \odot G_2) = \frac{1}{2} + \frac{1}{2} r_{u_i v_i} (G_1 \odot G_2) + \frac{1}{2} r_{v_i j} (G_1 \odot G_2) - \frac{1}{4} r_{u_i v_i} (G_1 \odot G_2).
\]

(e) For any $i, j \in I(G_1)$, let $u_i v_i, u_j v_j \in E(G_1)$ denote the edges corresponding to $i, j$, we have

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\[ r_{ij}(G_1 \otimes G_2) = 1 + \frac{1}{4}(r_{u_{ij}}(G_1 \otimes G_2) + r_{v_{ij}}(G_1 \otimes G_2) + r_{u_{ij}}(G_1 \otimes G_2) - r_{v_{ij}}(G_1 \otimes G_2)). \]

**Proof** Let \( R \) be the incidence matrix of \( G_1 \). Then with a proper labeling of vertices, the Laplacian matrix of \( G = G_1 \otimes G_2 \) can be written as:

\[
L(G) = \begin{pmatrix}
L_{G_1} + D_{G_1} + n_2I_{n_1} & -R & -I_{n_1} \otimes 1^T_{n_2} \\
-R^T & 2I_{m_1} & 0_{m_1 \times n_{12}} \\
-I_{n_1} \otimes 1_{n_2} & 0_{n_{12} \times n_{12}} & I_{n_1} \otimes (L_{G_2} + I_{n_2})
\end{pmatrix}
\]

where \( 0_{s \times t} \) denotes the \( s \times t \) matrix with all entries equal to zero.

Let \( A = L_{G_1} + D_{G_1} + n_2I_{n_1}, B = (-R - I_{n_1} \otimes 1^T_{n_2}) \),

\[
B^T = \begin{pmatrix} -I_{n_1} \otimes 1_{n_2} \\
2I_{m_1} \\
0_{n_{12} \times m_1}
\end{pmatrix}
\]

and

\[
D = \begin{pmatrix} 2I_{m_1} \\
0_{n_{12} \times m_1} \\
0_{n_{12} \times m_1}
\end{pmatrix} I_{n_1} \otimes (L_{G_2} + I_{n_2})
\]

We begin with the calculation about \( H \).

Let \( Q = I_{n_1} \otimes (L_{G_2} + I_{n_2})^{-1} \), then

\[
H = L_{G_1} + D_{G_1} + n_2I_{n_1} - (\begin{pmatrix} -R & -I_{n_1} \otimes 1^T_{n_2} \\
2I_{m_1} & 0_{m_1 \times n_{12}} \end{pmatrix} Q^{-1} \begin{pmatrix} -R^T \\
-I_{n_1} \otimes 1_{n_2} \end{pmatrix} - (\begin{pmatrix} -R \end{pmatrix} \begin{pmatrix} -I_{n_1} \otimes 1^T_{n_2} \end{pmatrix} - (\begin{pmatrix} -R \end{pmatrix} (\begin{pmatrix} -I_{n_1} \end{pmatrix} \otimes 1_{n_2}) - \frac{1}{2}RR^T - n_2I_{n_1}
\]

so, we have \( H^\# = \frac{2}{3}L^\#_{G_1} \).

Now we are ready to calculate \(-H^#BD^{-1}\) and \(-D^{-1}B^TH^#\).

Let \( K = I_{n_1} \otimes 1^T_{n_2} \), then

\[
-H^#BD^{-1} = -\begin{pmatrix} -R & -I_{n_1} \otimes 1^T_{n_2} \end{pmatrix} \begin{pmatrix} 2I_{m_1} & 0_{m_1 \times n_{12}} \end{pmatrix}^{-1} \begin{pmatrix} -R^T \\
-I_{n_1} \otimes 1_{n_2} \end{pmatrix}
\]

\[
= \left( \begin{pmatrix} -R & -I_{n_1} \otimes 1^T_{n_2} \\
2I_{m_1} & 0_{m_1 \times n_{12}} \end{pmatrix} H^\# \right)
\]

and

\[
-D^{-1}B^TH^# = -\left( \begin{pmatrix} 2I_{m_1} & 0_{m_1 \times n_{12}} \\
0_{m_1 \times n_{12}} & I_{n_1} \otimes (L_{G_2} + I_{n_2}) \end{pmatrix} \right)^{-1} \begin{pmatrix} -I_{n_1} \otimes 1_{n_2} \\
\frac{1}{2}R^T \end{pmatrix} H^# = \left( \begin{pmatrix} -1 \frac{1}{2} R^T H^# \\
-1 \frac{1}{2} R^T H^# \end{pmatrix} \right) \begin{pmatrix} -I_{n_1} \otimes 1_{n_2} \\
\frac{1}{2}R^T H^# \end{pmatrix} \begin{pmatrix} K \end{pmatrix} H^#.
\]

Next we are ready to compute the \( D^{-1}B^TH^#BD^{-1} \).
edges corresponding to $i, j$, by Lemma 4, we have

$$r_{ij}(G) = \frac{1}{2} + \frac{1}{2} r_{u_i j}(G_1 \odot G_2) + \frac{1}{2} r_{v_i j}(G_1 \odot G_2)$$

$$- \frac{1}{4} r_{u_i v_j}(G_1 \odot G_2)$$

$$= 1 + \frac{1}{4} \left( r_{u_i j}(G_1 \odot G_2) + r_{u_j v_i}(G_1 \odot G_2) + r_{v_i j}(G_1 \odot G_2) - r_{u_i v_j}(G_1 \odot G_2) \right),$$

as stated in (e).

The following result is proved in a way that is certainly similar in spirit to the proof of Theorem 1, but is a little more complicated in detail. Next, we will give the formulate for the resistance distance between two arbitrary vertexes in $G_1 \odot G_2$.

**Theorem 2.** Let $G_1$ be a graph on $n_1$ vertices and $m_1$ edges and $G_2$ be a graph on $n_2$ vertices and $m_2$ edges, let $G = G_1 \odot G_2$. Then the following holds:

(a) For any $i, j \in V(G_1)$, we have

$$r_{ij}(G) = \frac{2}{3} (L^\#_{G_1})_{ij} + \frac{2}{3} (L^\#_{G_2})_{ij} - \frac{4}{3} (L^\#_{G_1 \odot G_2})_{ij}.$$

(b) For any $i, j \in V(G_2)$, we have

$$r_{ij}(G) = (I_{m_1} \otimes (L_{G_2} + I_{n_2} - \frac{1}{n_2 + 2} j_{n_2 \times n_2})^{-1})_{ii}$$

$$+ (I_{m_1} \otimes (L_{G_2} + I_{n_2} - \frac{1}{n_2 + 2} j_{n_2 \times n_2})^{-1})_{jj}$$

$$- 2 (I_{m_1} \otimes (L_{G_2} + I_{n_2} - \frac{1}{n_2 + 2} j_{n_2 \times n_2})^{-1})_{ij}.$$

(c) For any $i \in V(G_1)$, $j \in V(G_2)$, we have

$$r_{ij}(G) = \frac{2}{3} (L^\#_{G_1})_{ii} + (I_{m_1} \otimes (L_{G_2} + I_{n_2} - \frac{1}{n_2 + 2} j_{n_2 \times n_2})^{-1})_{jj}$$

$$- (I_{m_1} \otimes (L_{G_2} + I_{n_2} - \frac{1}{n_2 + 2} j_{n_2 \times n_2})^{-1})_{ij}.$$

(d) For any $i \in I(G_1)$, $j \in V(G_1) \cup V(G_2)$, let $u_i v_j \in E(G_1)$ denote the edge corresponding to $i$, we have

$$r_{ij}(G) = \frac{1}{n_2 + 2} (1 + r_{u_i j}(G_1 \odot G_2) + r_{v_i j}(G_1 \odot G_2)$$

$$+ \sum_{k \in V(G_2)} r_{jk}(G_1 \odot G_2)) - \frac{1}{(n_2 + 2)^2}$$

$$\left( K f(G_2) + \sum_{k \in V(G_2)} r_{u_i k}(G_1 \odot G_2) + r_{u_v k}(G_1 \odot G_2) \right),$$

(e) For any $i, j \in I(G_1)$, let $u_i v_j, u_j v_i \in E(G_1)$ denote the edges corresponding to $i, j$, we have

$$r_{ij}(G) = \frac{1}{n_2 + 2} \left( 1 + r_{u_i j}(G_1 \odot G_2) + r_{v_i j}(G_1 \odot G_2)$$

$$+ \sum_{k \in V(G_2)} r_{ik}(G_1 \odot G_2) \right) - \frac{1}{(n_2 + 2)^2}$$

$$\left( K f(G_2) + \sum_{k \in V(G_2)} r_{u_i k}(G_1 \odot G_2) + r_{v_i k}(G_1 \odot G_2) \right).$$

Proof Let $R$ be the incidence matrix of $G_1$. Then with a proper labeling of vertices, the Laplacian matrix of $G_1 \odot G_2$ can be written as

$$L(G) =$$

$$\begin{pmatrix}
L_{G_1} + D_{G_1} & -R & 0_{n_1 \times m_2} \\
-R^T & (n_2 + 2) I_{m_1} & -I_{m_1} \otimes I_{n_2} \\
0_{m_1 \times m_2} & -I_{m_1} \otimes I_{n_2} & I_{m_1} \otimes (L_{G_2} + I_{n_2})
\end{pmatrix} ,$$

where $0_{s,t}$ denotes the $s \times t$ matrix with all entries equal to zero.

Let $A = L_{G_1} + D_{G_1}, B = (-R \ 0_{n_1 \times m_2})$, $B^T = (-R^T)$ and

$$D = \begin{pmatrix}
(n_2 + 2) I_{m_1} & -I_{m_1} \otimes I_{n_2} \\
-I_{m_1} \otimes I_{n_2} & I_{m_1} \otimes (L_{G_2} + I_{n_2})
\end{pmatrix} .$$

Note that $RR^T = D_{G_1} + AG_1$. Let $R_1 = [(n_2 + 2) I_{m_1} - (-I_{m_1} \otimes I_{n_2}) (I_{m_1} \otimes (L_{G_2} + I_{n_2}))^{-1} (-I_{m_1} \otimes I_{n_2})]^{-1} = \frac{1}{2} I_{m_1}$. By Lemma 3, we have

$$D^{-1} = \begin{pmatrix}
\frac{1}{2} I_{m_1} & \frac{1}{2} I_{m_1} \otimes I_{n_2} \\
\frac{1}{2} I_{m_1} \otimes I_{n_2} & F
\end{pmatrix} ,$$

where $F = I_{m_1} \otimes (L_{G_2} + I_{n_2} - \frac{1}{n_2 + 2} j_{n_2 \times n_2})^{-1}.$

We begin with the calculation about $H$.

Let $T = I_{m_1} \otimes (L_{G_2} + I_{n_2} - \frac{1}{n_2 + 2} j_{n_2 \times n_2})^{-1}, K = I_{m_1} \otimes I_{n_2}$, then

$$H = L_{G_1} + D_{G_1} = \begin{pmatrix}
-R & 0_{n_1 \times m_2} \\
\frac{1}{2} I_{m_1} \otimes I_{n_2} & T
\end{pmatrix}^{-1} \begin{pmatrix}
-R^T \\
0_{m_1 \times n_2}
\end{pmatrix}$$

$$= L_{G_1} + D_{G_1} + \begin{pmatrix}
\frac{1}{2} R & \frac{1}{2} R K \\
0_{m_1 \times n_2}
\end{pmatrix}$$

$$= L_{G_1} + D_{G_1} - \frac{1}{2} RR^T = \frac{1}{2} L_{G_1} .$$

So, we have $H^\# = \frac{3}{2} L_{G_1}$.

Now we are ready to calculate the $-H^#BD^{-1}$ and $-D^{-1}B^#H^#$.

$$-H^#BD^{-1} = -H^# \begin{pmatrix}
-R & 0_{n_1 \times m_2} \\
\frac{1}{2} I_{m_1} \otimes I_{n_2} & T
\end{pmatrix}^{-1} \begin{pmatrix}
-R \\
0_{n_1 \times m_2}
\end{pmatrix}$$

$$= \begin{pmatrix}
\frac{1}{2} H^# R & \frac{1}{2} H^# R K \\
\frac{1}{2} I_{m_1} \otimes I_{n_2} & \frac{1}{2} I_{m_1} \otimes I_{n_2} + \frac{1}{n_2 + 2} j_{n_2 \times n_2}
\end{pmatrix}^{-1}$$

and

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we have

\[-D^{-1}B^T H^# =
\]

\[-\left( \frac{1}{2} I_m \otimes 1_{n_2} \right) I_{m_1} \otimes \left( L_{G_2} + I_{n_2} - \frac{1}{n_2 + 2} j_{n_2 \times n_2} \right)^{-1} R \left( 0_{m_1 \times n_1} \otimes 1^T_{n_2} \right) (\frac{1}{2} R^T H^#) \left( 0_{m_1 \times n_1} \otimes 1^T_{n_2} \right)^{-1} \frac{1}{2} R^T H^# \left( 0_{m_1 \times n_1} \otimes 1^T_{n_2} \right) \right).\]

Next we are ready to compute the \( D^{-1}B^T H^# BD^{-1} \).

\[
D^{-1}B^T H^# BD^{-1} = \left( \frac{1}{2} R^T H^# \right) \left( -\frac{1}{2} R^T H^# \right) \left( -\frac{1}{2} R^T H^# \right)
\]

\[
-\left( \frac{1}{2} I_m \otimes 1_{n_2} \right) I_{m_1} \otimes \left( L_{G_2} + I_{n_2} - \frac{1}{n_2 + 2} j_{n_2 \times n_2} \right)^{-1} R \left( 0_{m_1 \times n_1} \otimes 1^T_{n_2} \right) (\frac{1}{2} R^T H^#) \left( 0_{m_1 \times n_1} \otimes 1^T_{n_2} \right)^{-1} \frac{1}{2} R^T H^# \left( 0_{m_1 \times n_1} \otimes 1^T_{n_2} \right) \right).
\]

Based on Lemma 3 and Lemma 7, the following matrix

\[
N = \left( \begin{array}{ccc}
\frac{1}{6} L^#_{G_1} & \frac{1}{6} L^#_{G_1} & \frac{1}{6} L^#_{G_1} \\
\frac{1}{6} R_{G_1} & \frac{1}{6} R_{G_1} & \frac{1}{6} R_{G_1} \\
\frac{1}{6} T_{G_1} & \frac{1}{6} T_{G_1} & \frac{1}{6} T_{G_1}
\end{array} \right)
\]

is a symmetric \( \{1\} \)-inverse of \( L(G_1 \otimes G_2) \), where \( K = I_{m_1} \otimes 1_{n_2}, T = I_{m_1} \otimes (L_{G_2} + I_{n_2} - \frac{1}{n_2 + 2} j_{n_2 \times n_2})^{-1}, Q = R^T L_{G_1} R \).

For any \( i, j \in V(G_1) \), by Lemma 1 and the Equation (2), we have

\[
r_{ij}(G_1 \otimes G_2) = \frac{2}{3}(L^#_{G_1})_{ii} + \frac{2}{3}(L^#_{G_1})_{jj} - \frac{4}{3}(L^#_{G_1})_{ij},
\]

denote the edge corresponding to \( i \), by Lemma 4, we have

\[
r_{ij}(G) = \frac{1}{n_2} \left( r_{ji}(G_1 \otimes G_2) + r_{ij}(G_1 \otimes G_2) + 1 \right)
\]

\[
+ \sum_{k \in V(G_2)} r_{ik}(G_1 \otimes G_2) \right) - \frac{1}{(n_2 + 2)^2} \left( Kf(G_2) + \sum_{k \in V(G_2)} r_{u,k}(G_1 \otimes G_2) + r_{v,k}(G_1 \otimes G_2) \right),
\]

as stated in (d).

For any \( i, j \in I(G_1) \), let \( u, v, u, v_j \in E(G_1) \) denote the edges corresponding to \( i, j \) respectively. By Lemma 4, we have

\[
r_{ij}(G) = \frac{1}{n_2 + 2} \left( 1 + r_{u,i}(G_1 \otimes G_2) + r_{v_j}(G_1 \otimes G_2) \right)
\]

\[
+ \sum_{k \in V(G_2)} r_{uk}(G_1 \otimes G_2) + \frac{1}{(n_2 + 2)^2} \left( Kf(G_2) + \sum_{k \in V(G_2)} r_{u,k}(G_1 \otimes G_2) + r_{v,k}(G_1 \otimes G_2) \right),
\]

as stated in (e).

IV. Kirchhoff Index in R-Vertex Corona and R-Edge Corona of Two Graphs

In this section, we will give the formulate for the Kirchhoff index in \( G_1 \otimes G_2 \) and \( G_1 \otimes G_2 \) whenever \( G_1 \) and \( G_2 \) are arbitrary graphs.

Theorem 3. Let \( G_1 \) be a graph with \( n_1 \) vertices and \( m_1 \) edges and \( G_2 \) be a graph with \( n_2 \) vertices and \( m_2 \) edges. Then

\[
Kf(G_1 \otimes G_2)
\]

\[
= \frac{2 + 2n_2}{3n_1} Kf(G_1) + \frac{3n_1 - n_1 + 1}{6} \pi \pi^T L^#_{G_1} R L^#_{G_1} R^T (R^T L^#_{G_1} R) 1^T (R^T L^#_{G_1} R) 1 - \frac{1}{2} \left( \sum_{i=1}^{n_1} \mu_i(G_2) + 1 \right)
\]

\[
+ \frac{3n_1 - n_1 + 1}{6} \pi \pi^T L^#_{G_1} R L^#_{G_1} R^T (R^T L^#_{G_1} R) 1 - \frac{1}{2} \left( \sum_{i=1}^{n_1} \mu_i(G_2) + 1 \right)
\]

\[
+ \frac{3n_1 - n_1 + 1}{6} \pi \pi^T L^#_{G_1} R L^#_{G_1} R^T (R^T L^#_{G_1} R) 1 - \frac{1}{2} \left( \sum_{i=1}^{n_1} \mu_i(G_2) + 1 \right)
\]

\[
= \frac{2 + 2n_2}{3n_1} Kf(G_1) + \frac{3n_1 - n_1 + 1}{6} \pi \pi^T L^#_{G_1} R L^#_{G_1} R^T (R^T L^#_{G_1} R) 1 - \frac{1}{2} \left( \sum_{i=1}^{n_1} \mu_i(G_2) + 1 \right)
\]

\[
+ \frac{3n_1 - n_1 + 1}{6} \pi \pi^T L^#_{G_1} R L^#_{G_1} R^T (R^T L^#_{G_1} R) 1 - \frac{1}{2} \left( \sum_{i=1}^{n_1} \mu_i(G_2) + 1 \right)
\]

where \( \pi = (d_1, d_2, \ldots, d_{n_1})^T \) and \( K = I_{n_1} \otimes 1^T_{n_2} \).

Proof Let \( L^#_{G_1 \otimes G_2} \) be the symmetric \( \{1\} \)-inverse of \( L_{G_1 \otimes G_2} \) Then
Let $G_1$ be an $r_1$-regular graph with $n_1$ vertices and $m_1$ edges and $G_2$ be a graph with $n_2$ vertices and $m_2$ edges. Then

$$Kf(G_1 \circ G_2) = (n_1 + m_1 + n_1n_2) \left( \frac{2 + 2n_2 + r_1}{3n_1} \right)$$

$$+ \frac{n_2}{6} \left[ 3m_1 - n_1 + 1 \right] - \frac{n_1 - 1}{6}.$$ 

Proof Since $R_1 = r_1$ and $L^\# \pi_1 = 0$, then $1^T R^T L^\#_G R_1 = 1^T (R^T L^\#_G K) 1 = 0$ and $\pi^T L^\#_G \pi = r_1^2 1^T L^\#_G 1 = 0$. Then the required result is obtained by plugging $1^T R^T L^\#_G, 1^T (R^T L^\#_G K) 1$ and $\pi^T L^\#_G \pi$ into Theorem 3.

Theorem 4. Let $G_1$ be a graph with $n_1$ vertices and $m_1$ edges, $G_2$ be a graph with $n_2$ vertices and $m_2$ edges. Then

$$Kf(G_1 \circ G_2) = (n_1 + m_2 + m_1n_2) \left( \frac{2 + r_1}{3n_1} Kf(G_1) \right)$$

$$+ \frac{1}{6} tr(D_G \circ L^\#_G) + \frac{1}{3} tr(K^T R^T L^\#_G, RK)$$

$$+ m_1 \sum_{i=1}^{n_2} \frac{1}{\mu_i(G_2)} + \frac{3m_1 - n_1 + 1}{6}$$

$$- \frac{n_1 - 1}{6} + \frac{n_1}{6} \frac{3m_1 - n_1 + 1}{2},$$

where $K = I_{m_1} \otimes I_{n_2}$ and $\pi = (d_1, d_2, \ldots, d_{n_2})^T$.

Proof The proof is similar to that of Theorem 3 and hence we omit details.

Corollary 2. Let $G_1$ be an $r_1$-regular graph with $n_1$ vertices and $m_1$ edges and $G_2$ be a graph with $n_2$ vertices and $m_2$ edges. Then

$$Kf(G_1 \circ G_2) = (n_1 + m_2 + m_1n_2) \left( \frac{2 + r_1}{3n_1} Kf(G_1) \right)$$

$$+ \frac{1}{6} tr(K^T R^T L^\#_G, RK) + m_1$$

$$+ \frac{1}{3} \sum_{i=1}^{n_2} \frac{1}{\mu_i(G_2)} + \frac{3m_1 - n_1 + 1}{3} - \frac{m_1 + 4m_1n_2}{2}.$$
where $K = I_{m_1} \otimes I_{m_2}^T$.

**Remark:** From Theorem 3.2 and Theorem 4.2, in [13], one can immediately get the Kirchhoff index of a graph by computing the Laplacian eigenvalues of the graph. However, the expression given in Theorem 3.2 and Theorem 4.2 in [13] are somewhat complicated. In the above, by the result of Theorem 3 and Theorem 4 in a different way we obtain a much simpler expression.

**V. CONCLUSION**

We use the Laplacian generalized inverse approach to obtain the formulate for the resistance distance and the Kirchhoff index in $G_1 \odot G_2$ and $G_1 \ominus G_2$ whenever $G_1$ and $G_2$ are arbitrary graphs. This approach is more simpler than the computation of the results in [13] and improves and extends some earlier results.

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**REFERENCES**