Interactive Decision Making for Multiobjective Fuzzy Random Programming Problems with Simple Recourse through a Fractile Model

Hitoshi Yano and Rongrong Zhang

Abstract—**In this paper, we focus on multiobjective fuzzy random programming problems with simple recourse through a fractile optimization model, in which fuzzy random variables coefficients are involved in equality constraints, and random variables coefficients are involved in the objective functions. In the proposed method, equality constraints with fuzzy random variables are defined on the basis of a possibility measure and a two-stage programming method. To deal with the objective functions involving random variable coefficients, a fractile optimization model is applied. For a given permissible possibility level and/or permissible probability levels specified by the decision maker, several kinds of Pareto optimality concepts are introduced. Interactive decision making methods are proposed to obtain a satisfactory solution from among a Pareto optimal solution set. The proposed method is applied to a farm planning problem in the Philippines, in which it is assumed that the amount of water resource in dry season is represented as a fuzzy random variable.**

Index Terms—**multiobjective programming, simple recourse programming, a possibility measure, fuzzy random variables, a satisfactory solution.**

I. Introduction

During the past six decades, various types of stochastic programming approaches have been proposed to deal with mathematical programming problems with random variable coefficients. Such approaches can be classified into two groups, one is two-stage programming methods [2], [4], [7], [18], [20] and the other is chance constraints methods [3], [15], [16]. In two-stage programming problems, the first-stage is to minimize the penalty cost for the violation of the equality constraints under the assumption that the decision variables are fixed, and the second-stage is to minimize the original objective function and the corresponding penalty cost. For chance constraint programming problems, a probability maximization model and a fractile optimization model were proposed. In a probability maximization model, the probability that the objective function is smaller than a certain target value is maximized. A fractile optimization model can be regarded as a complementary to the corresponding probability maximization model, in which a target variable is optimized under the condition that the decision maker specifies the probability level that the objective function is smaller than the target variable.

Two-stage programming methods have been applied to various types of water resource allocation problems with random inflow in future [17], [19]. However, if probability density functions of random variables are unknown or the problem is a large scale one with random variables, it may be extremely hard to solve the corresponding two-stage programming problem. From such a point of view, inexact two-stage programming methods have been proposed [6], [12].

As an extension for multiobjective programming problems, Sakawa et al. [14] proposed an interactive fuzzy decision making method for multiobjective stochastic programming problems with simple recourse. However, in the real world decision making situations, it seems to be natural to consider that the uncertainty is expressed by not only fuzziness but also randomness simultaneously. From such a point of view, interactive decision making methods for multiobjective fuzzy random programming problems have been proposed [9], [10], in which chance constraint methods and a possibility measure are applied to deal with fuzzy random variable coefficients [11].

In this paper, we focus on multiobjective fuzzy random programming problems with simple recourse through a fractile optimization model, where the coefficients of equality constraints are defined by LR-type fuzzy random variables [9], [10], and the coefficients of the objective functions are defined by random variables. We propose interactive decision making methods to obtain a satisfactory solution from among a Pareto optimal solution set. In section II, we focus on multiobjective programming problems, in which the coefficients of equality constraints are fuzzy random variables. After the decision maker specifies a permissible possibility level for fuzzy random variables, multiobjective fuzzy random simple recourse programming problems are formulated, and the corresponding Pareto optimal solution concept is defined. To obtain a satisfactory solution from among a Pareto optimal solution set, an interactive algorithm (Algorithm 1) is developed [22]. In section III, we further focus on multiobjective programming problems, in which the coefficients of equality constraints are fuzzy random variables while the coefficients of the objective functions are random variables. After the decision maker specifies not only a permissible possibility level for fuzzy random variables but also permissible probability levels for the objective functions, multiobjective fuzzy random simple recourse programming problems through a fractile optimization model are formulated, and the corresponding Pareto optimal solution concept is defined. To obtain a satisfactory solution from among a Pareto optimal solution set, an interactive algorithm (Algorithm 2) is developed. From a view point that permissible probability levels and the corresponding objective functions

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conflict with each other, the another interactive algorithm (Algorithm 4) is also developed, in which the decision maker is requested to specify not permissible probability levels but the membership functions of permissible probability levels. In Algorithm 3, permissible probability levels are automatically set as appropriate values through the fuzzy decision [1], [13], [24]. In section IV, to show the efficiency of the proposed method, we apply Algorithm 2 to a farm planning problem in the Philippines [23], in which it is assumed that the amount of water resource in dry season is represented as a fuzzy random variable and the profit coefficients of seven crops are represented as random variables.

II. Multiobjective Fuzzy Random Programming Problems with Simple Recourse

In this section, we focus on multiobjective programming problems involving fuzzy random variable coefficients in the right-hand sides of the equality constraints.

[MOP1]

$$\min_{\boldsymbol{x}\in X}\ (\boldsymbol{c}_1\boldsymbol{x},\cdots,\boldsymbol{c}_k\boldsymbol{x}) \tag{1}$$

subject to

$$A\boldsymbol{x}=\tilde{\bar{\boldsymbol{d}}} \tag{2}$$

where $\boldsymbol{c}_\ell=(c_{\ell 1},\cdots,c_{\ell n}),\ell=1,\cdots,k$ are n dimensional coefficient row vectors of objective function, $\boldsymbol{x}=(x_1,\cdots,x_n)^T\geq\boldsymbol{0}$ is an n dimensional decision variable column vector, X is a linear constraint set with respect to $\boldsymbol{x}$. A is an $(m\times n)$ dimensional coefficient matrix, $\tilde{\bar{\boldsymbol{d}}}=(\tilde{\bar{d}}_1,\cdots,\tilde{\bar{d}}_m)^T$ is an m dimensional coefficient column vector whose elements are fuzzy random variables [11] (The symbols "–" and "~" mean randomness and fuzziness respectively).

In order to deal with fuzzy random variables efficiently, Katagiri et al. [9], [10] defined a special type of a fuzzy random variable based on the concept of LR fuzzy numbers [5], which is called an LR-type fuzzy random variable. Under the occurrence of each elementary event ω, $\tilde{\bar{d}}_i(\omega)$ is a realization of an LR-type fuzzy random variable $\tilde{\bar{d}}_i$, which is an LR fuzzy number [5] whose membership function is defined as follows.

$$\mu_{\tilde{\bar{d}}_i(\omega)}(s)=\begin{cases} L\left(\frac{\bar{b}_i(\omega)-s}{\alpha_i}\right), & s\leq\bar{b}_i(\omega)\\ R\left(\frac{s-\bar{b}_i(\omega)}{\beta_i}\right), & s>\bar{b}_i(\omega)\end{cases} \tag{3}$$

where the function $L(t)\stackrel{\text{def}}{=}\max\{0,l(t)\}$ is a real-valued continuous function from $[0,\infty)$ to $[0,1]$, and $l(t)$ is a strictly decreasing continuous function satisfying $l(0)=1$. Also, $R(t)\stackrel{\text{def}}{=}\max\{0,r(t)\}$ satisfies the same conditions. $\alpha_{ij}(>0)$ and $\beta_{ij}(>0)$ are called left and right spreads [5]. The mean value $\bar{b}_i$ is a random variable, whose probability density function and cumulative distribution function are defined as $h_i(\cdot)$ and $H_i(\cdot)$ respectively. It is assumed that random variables $\bar{b}_i,i=1,\cdots,m$ are independent with each other.

Since it is difficult to deal with MOP1 directly, we introduce a permissible possibility level $\gamma(0<\gamma\leq 1)$ based on a concept of a possibility measure [5] for the equality constraints (2),

$$\mathrm{Pos}(\boldsymbol{a}_i\boldsymbol{x}=\tilde{\bar{\boldsymbol{d}}}_i(\omega))\geq\gamma,i=1,\cdots,m, \tag{4}$$

where $\boldsymbol{a}_i=(a_{i1},\cdots,a_{in}),i=1,\cdots,m$ are n-dimensional row vectors of A. From the property of LR fuzzy numbers, the i-th inequality condition (4) can be transformed into the following two inequalities.

$$\bar{b}_i(\omega)-L^{-1}(\gamma)\alpha_i\leq\boldsymbol{a}_i\boldsymbol{x}\leq\bar{b}_i(\omega)+R^{-1}(\gamma)\beta_i \tag{5}$$

For the above two inequalities (5), we introduce two vectors

$$\boldsymbol{y}^+=(y_1^+,\cdots,y_m^+)^T\geq\boldsymbol{0},\boldsymbol{y}^-=(y_1^-,\cdots,y_m^-)^T\geq\boldsymbol{0},$$

where (y_i^+,y_i^-) represent the shortage and the excess for the interval (5), and the following relations hold [21].

(1) For the case $\bar{b}_i(\omega)-L^{-1}(\gamma)\alpha_i>\boldsymbol{a}_i\boldsymbol{x}$, it holds that $y_i^+=\bar{b}_i(\omega)-L^{-1}(\gamma)\alpha_i-\boldsymbol{a}_i\boldsymbol{x}>0,y_i^-=0$.

(2) For the case $\bar{b}_i(\omega)+R^{-1}(\gamma)\beta_i<\boldsymbol{a}_i\boldsymbol{x}$, it holds that $y_i^+=0,y_i^-=\boldsymbol{a}_i\boldsymbol{x}-(\bar{b}_i(\omega)+R^{-1}(\gamma)\beta_i)>0$.

(3) For the case $\bar{b}_i(\omega)-L^{-1}(\gamma)\alpha_i\leq\boldsymbol{a}_i\boldsymbol{x}\leq\bar{b}_i(\omega)+R^{-1}(\gamma)\beta_i$, it holds that $y_i^+=0,y_i^-=0$.

Yano [21] has already formulated fuzzy random simple recourse programming problems using $(\boldsymbol{y}^+,\boldsymbol{y}^-)$. In this paper, as a extension of [21], we formulated a multiobjective fuzzy random simple recourse programming problem (MOP2) as follows.

[MOP2]

$$\left.\begin{array}{l} \min\limits_{\boldsymbol{x}\in X}\ \boldsymbol{c}_1\boldsymbol{x}+E\left[\min\limits_{\boldsymbol{y}^+,\boldsymbol{y}^-}\ (\boldsymbol{q}_1^+\boldsymbol{y}^++\boldsymbol{q}_1^-\boldsymbol{y}^-)\right]\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \min\limits_{\boldsymbol{x}\in X}\ \boldsymbol{c}_k\boldsymbol{x}+E\left[\min\limits_{\boldsymbol{y}^+,\boldsymbol{y}^-}\ (\boldsymbol{q}_k^+\boldsymbol{y}^++\boldsymbol{q}_k^-\boldsymbol{y}^-)\right]\end{array}\right\} \tag{6}$$

subject to

$$\begin{array}{l}\boldsymbol{a}_i\boldsymbol{x}+y_i^+\geq\bar{b}_i(\omega)-L^{-1}(\gamma)\alpha_i,i=1,\cdots,m\\ \boldsymbol{a}_i\boldsymbol{x}-y_i^-\leq\bar{b}_i(\omega)+R^{-1}(\gamma)\beta_i,i=1,\cdots,m\\ \boldsymbol{x}\in X,\boldsymbol{y}^+\geq\boldsymbol{0},\boldsymbol{y}^-\geq\boldsymbol{0}\end{array}$$

where

$$\boldsymbol{q}_\ell^+ = (q_{\ell 1}^+,\cdots,q_{\ell m}^+)\geq\boldsymbol{0},\ell=1,\cdots,k \tag{7}$$

$$\boldsymbol{q}_\ell^- = (q_{\ell 1}^-,\cdots,q_{\ell m}^-)\geq\boldsymbol{0},\ell=1,\cdots,k \tag{8}$$

are m dimensional weighting row vectors for $\boldsymbol{y}^+$ and $\boldsymbol{y}^-$ respectively. For the ℓ-th objective function of (6), the second term can be transformed into follows [21].

$$\begin{aligned} & E\left[\min_{\boldsymbol{y}^+,\boldsymbol{y}^-}\ (\boldsymbol{q}_\ell^+\boldsymbol{y}^++\boldsymbol{q}_\ell^-\boldsymbol{y}^-)\right]\\ = & \sum_{i=1}^m q_{\ell i}^+\Big(E[\bar{b}_i]-\boldsymbol{a}_i\boldsymbol{x}-L^{-1}(\gamma)\alpha_i\Big)\\ & +\sum_{i=1}^m q_{\ell i}^+\Big\{(\boldsymbol{a}_i\boldsymbol{x}+L^{-1}(\gamma)\alpha_i)H_i(\boldsymbol{a}_i\boldsymbol{x}+L^{-1}(\gamma)\alpha_i)\\ & -\int_{-\infty}^{\boldsymbol{a}_i\boldsymbol{x}+L^{-1}(\gamma)\alpha_i} b_ih_i(b_i)db_i\Big\}\\ & +\sum_{i=1}^m q_{\ell i}^-\Big\{(\boldsymbol{a}_i\boldsymbol{x}-R^{-1}(\gamma)\beta_i)H_i(\boldsymbol{a}_i\boldsymbol{x}-R^{-1}(\gamma)\beta_i)\\ & -\int_{-\infty}^{\boldsymbol{a}_i\boldsymbol{x}-R^{-1}(\gamma)\beta_i} b_ih_i(b_i)db_i\Big\}\\ \stackrel{\text{def}}{=} & d_\ell(\boldsymbol{x},\gamma)\end{aligned} \tag{9}$$

In the following, we define the objective functions as:

$$z_\ell(\boldsymbol{x},\gamma) \stackrel{\text{def}}{=} \boldsymbol{c}_\ell \boldsymbol{x} + d_\ell(\boldsymbol{x},\gamma), \ell = 1,\cdots,k. \quad (10)$$

Then, a multiobjective programming problem (1) can be reduced to a multiobjective fuzzy random simple recourse programming problem, in which a permissible possibility level γ is a parameter specified by the decision maker.

[MOP3(γ)]

$$\min_{\boldsymbol{x}\in X} \ (z_1(\boldsymbol{x},\gamma),\cdots,z_k(\boldsymbol{x},\gamma)) \quad (11)$$

Now, we can define a Pareto optimal solution concept for (11).

Definition 1

$\boldsymbol{x}^* \in X$ is said to be a γ-Pareto optimal solution to MOP3(γ), if and only if there does not exist another $\boldsymbol{x} \in X$ such that $z_\ell(\boldsymbol{x},\gamma) \le z_\ell(\boldsymbol{x}^*,\gamma), \ell = 1,\cdots,k$ with strict inequality holding for at least one ℓ.

For generating a candidate of a satisfactory solution which is also a γ-Pareto optimal solution, the decision maker is asked to specify a permissible possibility level γ and the reference objective values $\hat{z}_\ell, \ell = 1,\cdots,k$ [13]. Once γ and $\hat{z}_\ell, \ell = 1,\cdots,k$ are specified, the corresponding γ-Pareto optimal solution, which is in a sense close to his/her requirement or better than that if the reference objective values are attainable, is obtained by solving the following minmax problem [13].

[MINMAX1($\hat{\boldsymbol{z}}$,γ)]

$$\min_{\boldsymbol{x}\in X,\lambda\in \mathrm{R}^1} \lambda \quad (12)$$

$$\text{s.t.} \quad z_\ell(\boldsymbol{x},\gamma) - \hat{z}_\ell \le \lambda, \ell = 1,\cdots,k \quad (13)$$

The relationships between the optimal solution $(\boldsymbol{x}^*,\lambda^*)$ of MINMAX1($\hat{\boldsymbol{z}}$,γ) and γ-Pareto optimal solutions to MOP3(γ) can be characterized by the following theorem.

Theorem 1

(1) If $\boldsymbol{x}^* \in X, \lambda^* \in \mathrm{R}^1$ is a unique optimal solution of MINMAX1($\hat{\boldsymbol{z}}$,γ), then $\boldsymbol{x}^* \in X$ is a γ-Pareto optimal solution to MOP3(γ).

(2) If $\boldsymbol{x}^* \in X$ is a γ-Pareto optimal solution to MOP3(γ), then $\boldsymbol{x}^* \in X$ $\lambda^* \stackrel{\text{def}}{=} z_\ell(\boldsymbol{x}^*,\gamma) - \hat{z}_\ell, \ell = 1,\cdots,k$ is an optimal solution of MINMAX1($\hat{\boldsymbol{z}}$,γ) for some reference objective values $\hat{\boldsymbol{z}} = (\hat{z}_1,\cdots,\hat{z}_k)$.

(Proof)

(1) Assume that $\boldsymbol{x}^* \in X$ is not a γ-Pareto optimal solution to MOP3(γ). Then, there exists $\boldsymbol{x} \in X$ such that $z_\ell(\boldsymbol{x},\gamma) \le z_\ell(\boldsymbol{x}^*,\gamma), \ell = 1,\cdots,k$ with strict inequality holding for at least one ℓ. This means that $z_\ell(\boldsymbol{x},\gamma) - \hat{z}_\ell \le z_\ell(\boldsymbol{x}^*,\gamma) - \hat{z}_\ell \le \lambda^*, \ell = 1,\cdots,k$, which contradicts the fact that $\boldsymbol{x}^* \in X$ is a unique optimal solution to MINMAX1($\hat{\boldsymbol{z}}$,γ).

(2) Assume that $\boldsymbol{x}^* \in X, \lambda^* \in \mathrm{R}^1$ is not an optimal solution to MINMAX1($\hat{\boldsymbol{z}},\gamma$) for any reference objective values $\hat{\boldsymbol{z}} = (\hat{z}_1,\cdots,\hat{z}_k)$, which satisfy the equalities $\lambda^* = z_\ell(\boldsymbol{x}^*,\gamma) - \hat{z}_\ell, \ell = 1,\cdots,k$. Then, there exists some $\boldsymbol{x} \in X, \lambda < \lambda^*$ such that $z_\ell(\boldsymbol{x},\gamma) - \hat{z}_\ell \le \lambda, \ell = 1,\cdots,k$. This means that $z_\ell(\boldsymbol{x},\gamma) < z_\ell(\boldsymbol{x}^*,\gamma), \ell = 1,\cdots,k$, which contradicts the fact that $\boldsymbol{x}^* \in X$ is a γ-Pareto optimal solution to MOP3(γ).

Unfortunately, it is not guaranteed that $(\boldsymbol{x}^*,\lambda^*)$ is a γ-Pareto optimal solution to MOP3(γ), if $(\boldsymbol{x}^*,\lambda^*)$ is not unique. In order to guarantee γ-Pareto optimality, we solve a γ-Pareto optimality test problem for $(\boldsymbol{x}^*,\lambda^*)$.

Theorem 2

Let $\boldsymbol{x}^* \in X$, $\lambda^* \in \mathrm{R}^1$ be an optimal solution to MINMAX1($\hat{\boldsymbol{z}}$,γ), in which $\lambda^* = z_\ell(\boldsymbol{x}^*,\gamma) - \hat{z}_\ell, \ell = 1,\cdots,k$. Corresponding to the optimal solution $\boldsymbol{x}^* \in X$, solve the following γ-Pareto optimality test problem.

$$\max_{\boldsymbol{x}\in X,\boldsymbol{\epsilon}=(\epsilon_1,\cdots,\epsilon_k)\ge \boldsymbol{0}} \sum_{\ell=1}^{k} \epsilon_\ell \quad (14)$$

subject to

$$z_\ell(\boldsymbol{x},\gamma) - \hat{z}_\ell + \epsilon_\ell \le \lambda^*, \ell = 1,\cdots,k$$

Let $\check{\boldsymbol{x}} \in X, \check{\epsilon}_\ell \ge 0, \ell = 1,\cdots,k$ be an optimal solution to (14). If $\sum_{\ell=1}^{k} \check{\epsilon}_\ell = 0$, then $\boldsymbol{x}^* \in X$ is a γ-Pareto optimal solution to (11).

On the other hand, the partial differentiation of $z_\ell(\boldsymbol{x},\gamma), \ell = 1,\cdots,k$ for $x_s, s = 1,\cdots,n$ and $x_t, t = 1,\cdots,n$ can be calculated as follows.

$$\begin{aligned} &\frac{\partial z_\ell(\boldsymbol{x},\gamma)}{\partial x_s \partial x_t} \\ = &\sum_{i=1}^{m} q_{\ell i}^{+} a_{is} a_{it} h_i(\boldsymbol{a}_i\boldsymbol{x} + L^{-1}(\gamma)\alpha_i) \\ &+\sum_{i=1}^{m} q_{\ell i}^{-} a_{is} a_{it} h_i(\boldsymbol{a}_i\boldsymbol{x} - R^{-1}(\gamma)\beta_i) \end{aligned} \quad (15)$$

The Hessian matrix for $z_\ell(\boldsymbol{x},\gamma)$ can be written as:

$$\begin{aligned} &\nabla^2 z_\ell(\boldsymbol{x},\gamma) \\ = &\sum_{i=1}^{m} q_{\ell i}^{+} h_i(\boldsymbol{a}_i\boldsymbol{x} + L^{-1}(\gamma)\alpha_i) \cdot A_i \\ &+\sum_{i=1}^{m} q_{\ell i}^{-} h_i(\boldsymbol{a}_i\boldsymbol{x} - R^{-1}(\gamma)\beta_i) \cdot A_i, \end{aligned} \quad (16)$$

where $A_i, i = 1,\cdots,m$ are $(n \times n)$-dimensional matrices defined as follows.

$$A_i \stackrel{\text{def}}{=} \begin{pmatrix} a_{i1}^2 & \cdots & a_{i1}a_{in} \\ \vdots & \ddots & \vdots \\ a_{in}a_{i1} & \cdots & a_{in}^2 \end{pmatrix}, i = 1,\cdots,m \quad (17)$$

Because of the property of the Hessian matrix for $z_\ell(\boldsymbol{x},\gamma), \ell = 1,\cdots,k$, the following theorem holds.

Theorem 3

MINMAX1($\hat{\boldsymbol{z}},\gamma$) is a convex programming problem.

(Proof)

From the definition (17), it holds that $A_i = \boldsymbol{a}_i^T \cdot \boldsymbol{a}_i$. Therefore, the following relation holds for any n-dimensional column vector $\boldsymbol{y} \in \mathrm{R}^1$.

$$\begin{aligned} \boldsymbol{y}^T A_i \boldsymbol{y} &= \boldsymbol{y}^T \cdot (\boldsymbol{a}_i^T \cdot \boldsymbol{a}_i) \cdot \boldsymbol{y} \\ &= (\boldsymbol{y}^T \cdot \boldsymbol{a}_i^T) \cdot (\boldsymbol{a}_i \cdot \boldsymbol{y}) \\ &= (\boldsymbol{a}_i \cdot \boldsymbol{y})^T \cdot (\boldsymbol{a}_i \cdot \boldsymbol{y}) \ge 0 \end{aligned}$$

This means that matrices $A_i, i = 1,\cdots,m$ are positive semidefinite. Because of the assumptions that probability density functions $h_i(\cdot) \ge 0, i = 1,\cdots,m$, and $q_{\ell i}^{+} \ge 0, q_{\ell i}^{-} \ge 0, \ell = 1,\cdots,k, i = 1,\cdots,m$, the following relation holds for each of the Hessian matrices $\nabla^2 z_\ell(\boldsymbol{x},\gamma), \ell = 1,\cdots,k$.

$$\begin{aligned} &\boldsymbol{y}^T \nabla^2 z_\ell(\boldsymbol{x},\gamma) \boldsymbol{y} \\ = &\sum_{i=1}^{m} q_{\ell i}^{+} h_i(\boldsymbol{a}_i\boldsymbol{x} + L^{-1}(\gamma)\alpha_i) \cdot \boldsymbol{y}^T A_i \boldsymbol{y} \\ &+\sum_{i=1}^{m} q_{\ell i}^{-} h_i(\boldsymbol{a}_i\boldsymbol{x} - R^{-1}(\gamma)\beta_i) \cdot \boldsymbol{y}^T A_i \boldsymbol{y} \ge 0 \end{aligned}$$

This means that MINMAX1$(\hat{\boldsymbol{z}},\gamma)$ is a convex programming problem.

The relationship between a permissible possibility level γ and the optimal objective function value $z_\ell(\boldsymbol{x}^*,\gamma)$ of MINMAX1$(\hat{\boldsymbol{z}},\gamma)$ can be characterized by the following theorem.

Theorem 4

For the optimal solution $\boldsymbol{x}^* \in X$ of MINMAX1$(\hat{\boldsymbol{z}},\gamma)$, the following relation holds.

$$\begin{aligned}
&\frac{\partial z_\ell(\boldsymbol{x}^*,\gamma)}{\partial\gamma}\\
= \quad & -\sum_{i=1}^{m} q_{\ell i}^{+}\frac{\partial L^{-1}(\gamma)}{\partial\gamma}\alpha_i\\
&+\sum_{i=1}^{m} q_{\ell i}^{+}\frac{\partial L^{-1}(\gamma)}{\partial\gamma}\alpha_i H_i(\boldsymbol{a}_i\boldsymbol{x}^*+L^{-1}(\gamma)\alpha_i)\\
&-\sum_{i=1}^{m} q_{\ell i}^{-}\frac{\partial R^{-1}(\gamma)}{\partial\gamma}\beta_i H_i(\boldsymbol{a}_i\boldsymbol{x}^*-R^{-1}(\gamma)\beta_i)
\end{aligned} \tag{18}$$

Now, following the above discussions, we can present an interactive algorithm to derive a satisfactory solution from among a γ-Pareto optimal solution set to (11).

[Algorithm 1]

Step 1: Set a permissible possibility level $\gamma = 1$.

Step 2: The decision maker sets the initial reference objective values $\hat{z}_\ell$ for $z_\ell(\boldsymbol{x},\gamma), \ell = 1,\cdots,k$.

Step 3: Solve MINMAX1$(\hat{\boldsymbol{z}},\gamma)$ and obtain the corresponding optimal solution $(\boldsymbol{x}^*,\lambda^*)$. For the optimal solution $\boldsymbol{x}^*$, a γ-Pareto optimality test problem is solved.

Step 4: If the decision maker is satisfied with the current value of the γ-Pareto optimal solution $z_\ell(\boldsymbol{x}^*,\gamma), \ell = 1,\cdots,k$, then stop. Otherwise, the decision maker updates his/her reference objective values $\hat{z}_\ell, \ell = 1,\cdots,k$, and/or a permissible possibility level γ, and return to Step 3.

III. Multiobjective Fuzzy Random Programming Problems with Simple Recourse Through a Fractile Model

In this section, we further consider multiobjective programming problems, in which the coefficients of equality constraints are fuzzy random variables while the coefficients of the objective functions are random variables.

[MOP4]

$$\min_{\boldsymbol{x}\in X}\ (\bar{\boldsymbol{c}}_1\boldsymbol{x},\cdots,\bar{\boldsymbol{c}}_k\boldsymbol{x}) \tag{19}$$

subject to

$$A\boldsymbol{x} = \tilde{\bar{\boldsymbol{d}}} \tag{20}$$

where $\bar{\boldsymbol{c}}_\ell = (\bar{c}_{\ell 1},\cdots,\bar{c}_{\ell n}), \ell = 1,\cdots,k$ is an n dimensional random variable coefficient row vectors of the objective function $\bar{\boldsymbol{c}}_\ell\boldsymbol{x}$. $\boldsymbol{x} = (x_1,\cdots,x_n)^T \geq \boldsymbol{0}$ is an n dimensional decision variable column vector, X is a linear constraint set with respect to $\boldsymbol{x}$. A is an $(m\times n)$ dimensional coefficient matrix, $\tilde{\bar{\boldsymbol{d}}} = (\tilde{\bar{d}}_1,\cdots,\tilde{\bar{d}}_m)^T$ is an m dimensional coefficient column vector whose element is an LR-type fuzzy random variable $\tilde{\bar{d}}_i$ [9], [10] defined by (3).

In the following, let us assume that the each element $\bar{c}_{\ell j}$ is a Gaussian random variable :

$$\bar{c}_{\ell j} \sim \mathrm{N}(E[\bar{c}_{\ell j}],\sigma_{\ell jj}),$$

and the positive definite variance covariance matrices $V_\ell, \ell = 1,\cdots,k$ between Gaussian random variables $\bar{c}_{\ell j}, j = 1,\cdots,n$ are given as:

$$V_\ell = \begin{pmatrix} \sigma_{\ell 11} & \sigma_{\ell 12} & \cdots & \sigma_{\ell 1n}\\ \sigma_{\ell 21} & \sigma_{\ell 22} & \cdots & \sigma_{\ell 2n}\\ \cdots & \cdots & \cdots & \cdots\\ \sigma_{\ell n1} & \sigma_{\ell n2} & \cdots & \sigma_{\ell nn}\end{pmatrix}, i = 1,\cdots,k. \tag{21}$$

We denote the vectors of the expectation for the random variable row vector $\bar{\boldsymbol{c}}_\ell$ as $\boldsymbol{E}[\bar{\boldsymbol{c}}_\ell] = (E[\bar{c}_{\ell 1}],\cdots,E[\bar{c}_{\ell n}]), \ell = 1,\cdots,k$. Then, using the variance covariance matrix V_ℓ, the objective function $\bar{\boldsymbol{c}}_\ell\boldsymbol{x}$ becomes a Gaussian random variable.

$$\bar{\boldsymbol{c}}_\ell\boldsymbol{x} \sim \mathrm{N}(\boldsymbol{E}[\bar{\boldsymbol{c}}_\ell]\boldsymbol{x},\boldsymbol{x}^T V_\ell\boldsymbol{x}), \ell = 1,\cdots,k \tag{22}$$

According to the discussion in the previous section, for a permissible possibility level, MOP4 can be reduced to the following multiobjective stochastic programming problem.

[MOP5(γ)]

$$\min_{\boldsymbol{x}\in X}\ (\bar{\boldsymbol{c}}_1\boldsymbol{x}+d_1(\boldsymbol{x},\gamma),\cdots,\bar{\boldsymbol{c}}_k\boldsymbol{x}+d_k(\boldsymbol{x},\gamma)) \tag{23}$$

where $d_\ell(\boldsymbol{x},\gamma), \ell = 1,\cdots,k$ are penalty costs defined by (9). If the decision maker specifies permissible probability levels $\hat{p}_\ell, \ell = 1,\cdots,k$ for $\bar{\boldsymbol{c}}_\ell\boldsymbol{x}$, the multiobjective stochastic problem (23) can be transformed into the following multiobjective programming problem through a fractile optimization model [15], [16].

[MOP6$(\gamma,\hat{\boldsymbol{p}})$]

$$\min_{\boldsymbol{x}\in X}(f_1(\boldsymbol{x},\gamma,\hat{p}_1),\cdots,f_k(\boldsymbol{x},\gamma,\hat{p}_k)) \tag{24}$$

where $f_\ell(\boldsymbol{x},\gamma,\hat{p}_\ell)$ is defined as follows.

$$\begin{aligned} f_\ell(\boldsymbol{x},\gamma,\hat{p}_\ell) \quad &\stackrel{\text{def}}{=} \quad \boldsymbol{E}[\bar{\boldsymbol{c}}_\ell]\boldsymbol{x}+\Phi^{-1}(\hat{p}_\ell)\cdot\sqrt{\boldsymbol{x}^T V_\ell\boldsymbol{x}}\\ &\quad\quad +d_\ell(\boldsymbol{x},\gamma)\end{aligned} \tag{25}$$

In MOP6$(\gamma,\hat{\boldsymbol{p}})$, it is assumed that $\Phi^{-1}(\cdot)$ is an inverse function of a cumulative distribution function for N(0,1) and $0.5 < \hat{p}_\ell < 1, \ell = 1,\cdots,k$. It should be noted here that the problem (24) can be regarded as a generalized version of (11), since $f_\ell(\boldsymbol{x},\gamma,0.5)$ is equivalent to $z_\ell(\boldsymbol{x},\gamma)$ if $\boldsymbol{E}[\bar{\boldsymbol{c}}_\ell]$ is replaced by $\boldsymbol{c}_\ell$, and $f_\ell(\boldsymbol{x},\gamma,\hat{p}_\ell), \ell = 1,\cdots,k$ are convex functions with respect to $\boldsymbol{x}\in X$ because of Theorem 3.

Similar to Definition 1, we can define a Pareto optimal solution concept to (24).

Definition 2

$\boldsymbol{x}^* \in X$ is said to be a $(\gamma,\ \hat{\boldsymbol{p}})$-Pareto optimal solution to MOP6$(\gamma,\hat{\boldsymbol{p}})$, if and only if there does not exist another $\boldsymbol{x}\in X$ such that $f_\ell(\boldsymbol{x},\gamma,\hat{p}_\ell) \leq f_\ell(\boldsymbol{x}^*,\gamma,\hat{p}_\ell), \ell = 1,\cdots,k$ with strict inequality holding for at least one ℓ.

For the reference objective values $\hat{f}_\ell, \ell = 1,\cdots,k$ specified by the decision maker, the corresponding $(\gamma,\ \hat{\boldsymbol{p}})$-Pareto optimal solution is obtained by solving the following minimax problem [13].

[MINMAX2$(\hat{\boldsymbol{f}},\ \gamma,\ \hat{\boldsymbol{p}})$]

$$\min_{\boldsymbol{x}\in X,\lambda\in\mathrm{R}^1}\ \lambda \tag{26}$$

$$\text{s.t.} \qquad f_\ell(\boldsymbol{x},\gamma,\hat{\boldsymbol{p}})-\hat{f}_\ell \leq \lambda, \ell = 1,\cdots,k \tag{27}$$

Similar to Theorem 3, MINMAX2$(\hat{\boldsymbol{f}},\ \gamma,\ \hat{\boldsymbol{p}})$ becomes a convex programming problem.

The relationships between the optimal solution $(\boldsymbol{x}^*,\lambda^*)$ of MINMAX2$(\hat{\boldsymbol{f}},\ \gamma,\ \hat{\boldsymbol{p}})$ and $(\gamma,\ \hat{\boldsymbol{p}})$-Pareto optimal solutions to MOP6$(\gamma,\hat{\boldsymbol{p}})$ can be characterized by the following theorem.

Theorem 5
(1) If $\boldsymbol{x}^* \in X, \lambda^* \in \mathrm{R}^1$ is a unique optimal solution of MINMAX2($\hat{\boldsymbol{f}}, \gamma, \hat{\boldsymbol{p}}$), then $\boldsymbol{x}^* \in X$ is a ($\gamma, \hat{\boldsymbol{p}}$)-Pareto optimal solution to MOP6($\gamma, \hat{\boldsymbol{p}}$).
(2) If $\boldsymbol{x}^* \in X$ is a (γ, $\hat{\boldsymbol{p}}$)-Pareto optimal solution to MOP6($\gamma, \hat{\boldsymbol{p}}$), then $\boldsymbol{x}^* \in X$ $\lambda^* \stackrel{\mathrm{def}}{=} f_\ell(\boldsymbol{x}^*, \gamma, \hat{\boldsymbol{p}}), \ell = 1, \cdots, k$ is an optimal solution of MINMAX2($\hat{\boldsymbol{f}}$, γ, $\hat{\boldsymbol{p}}$) for some reference objective values $\hat{\boldsymbol{f}} = (\hat{f}_1, \cdots, \hat{f}_k)$.

We can present the interactive algorithm to obtain a satisfactory solution from among a (γ, $\hat{\boldsymbol{p}}$)-Pareto optimal solution set.

[Algorithm 2]
Step 1: The decision maker sets a permissible possibility level γ and permissible probability levels $\hat{p}_\ell, (0.5 < \hat{p}_\ell < 1), \ell = 1, \cdots, k$.
Step 2: The decision maker sets the initial reference objective values $\hat{f}_\ell, \ell = 1, \cdots, k$ for $f_\ell(\boldsymbol{x}, \gamma, \hat{\boldsymbol{p}}), \ell = 1, \cdots, k$.
Step 3: Solve MINMAX2($\hat{\boldsymbol{f}}$, γ, $\hat{\boldsymbol{p}}$) and obtain the corresponding optimal solution $(\boldsymbol{x}^*, \lambda^*)$. For the optimal solution $\boldsymbol{x}^*$, a (γ, $\hat{\boldsymbol{p}}$)-Pareto optimality test problem is solved.
Step 4: If the decision maker is satisfied with the current value $f_\ell(\boldsymbol{x}^*, \gamma, \hat{\boldsymbol{p}}), \ell = 1, \cdots, k$ then stop. Otherwise, the decision maker updates his/her reference objective values $\hat{f}_\ell, \ell = 1, \cdots, k$, a permissible possibility level γ, and/or permissible probability levels $\hat{p}_\ell, \ell = 1, \cdots, k$, and return to Step 3.

In order to deal with MOP6($\gamma, \hat{\boldsymbol{p}}$), the decision maker must specify permissible probability levels $\hat{\boldsymbol{p}}$ in advance. However, in general, the decision maker seems to prefer not only the less value of the objective function $f_\ell(\boldsymbol{x}, \gamma, \hat{p}_\ell)$ but also the larger value of the permissible probability level $\hat{p}_\ell$. From such a point of view, we consider the following multiobjective programming problem which can be regarded as a natural extension of MOP6($\gamma, \hat{\boldsymbol{p}}$).

[MOP7(γ)]

$$\min_{\boldsymbol{x} \in X, \hat{p}_\ell \in (0,1), \ell=1,\cdots,k} \quad (f_1(\boldsymbol{x}, \gamma, \hat{p}_1), \cdots, f_k(\boldsymbol{x}, \gamma, \hat{p}_k), -\hat{p}_1, \cdots, -\hat{p}_k)$$

where permissible probability levels $\hat{p}_\ell, \ell = 1, \cdots, k$ are not fixed values but decision variables.

Considering the imprecise nature of the decision maker's judgment, we assume that the decision maker has a fuzzy goal for each objective function in MOP7(γ). Such a fuzzy goal can be quantified by eliciting the corresponding membership function. Let us denote a membership function of an objective function $f_\ell(\boldsymbol{x}, \gamma, \hat{p}_\ell)$ as $\mu_{f_\ell}(f_\ell(\boldsymbol{x}, \gamma, \hat{p}_\ell))$, and a membership function of a permissible probability level $\hat{p}_\ell$ as $\mu_{\hat{p}_\ell}(\hat{p}_\ell)$ respectively. Then, MOP7(γ) can be transformed into the following problem.

[MOP8(γ)]

$$\max_{\boldsymbol{x} \in X, \hat{p}_\ell \in (0,1), \ell=1,\cdots,k} \quad (\mu_{f_1}(f_1(\boldsymbol{x}, \gamma, \hat{p}_1)), \cdots, \mu_{f_k}(f_k(\boldsymbol{x}, \gamma, \hat{p}_k)), \mu_{\hat{p}_1}(\hat{p}_1), \cdots, \mu_{\hat{p}_k}(\hat{p}_k)) \tag{28}$$

In the following, we make the following assumptions with respect to the membership functions $\mu_{f_\ell}(f_\ell(\boldsymbol{x}, \gamma, \hat{p}_\ell))$, $\mu_{\hat{p}_\ell}(\hat{p}_\ell), \ell = 1, \cdots, k$.

Assumption 1
$\mu_{\hat{p}_\ell}(\hat{p}_\ell), \ell = 1, \cdots, k$ are strictly increasing and continuous with respect to $\hat{p}_\ell \in P_\ell \stackrel{\mathrm{def}}{=} [\hat{p}_{\ell\min}, \hat{p}_{\ell\max}] \subset (0.5, 1)$, where $\mu_{\hat{p}_\ell}(\hat{p}_\ell) = 0$ if $0 \le \hat{p}_\ell \le \hat{p}_{\ell\min}$, and $\mu_{\hat{p}_\ell}(\hat{p}_\ell) = 1$ if $\hat{p}_{\ell\max} \le \hat{p}_\ell \le 1$.

Assumption 2
$\mu_{f_\ell}(f_\ell(\boldsymbol{x}, \gamma, \hat{p}_\ell)), \ell = 1, \cdots, k$ are strictly decreasing and continuous with respect to $f_\ell(\boldsymbol{x}, \gamma, \hat{p}_\ell) \in [f_{\ell\min}, f_{\ell\max}]$, where $\mu_{f_\ell}(f_\ell(\boldsymbol{x}, \gamma, \hat{p}_\ell)) = 0$ if $f_\ell(\boldsymbol{x}, \gamma, \hat{p}_\ell) \ge f_{\ell\max}$, and $\mu_{f_\ell}(f_\ell(\boldsymbol{x}, \gamma, \hat{p}_\ell)) = 1$ if $f_\ell(\boldsymbol{x}, \gamma, \hat{p}_\ell) \le f_{\ell\min}$.

In order to determine these membership functions appropriately, let us assume that the decision maker sets $\hat{p}_{\ell\min}$, $\hat{p}_{\ell\max}$, $f_{\ell\min}$ and $f_{\ell\max}$ as follows.

At first, the decision maker specifies $\hat{p}_{\ell\min}(> 0.5)$ and $\hat{p}_{\ell\max}$ in his/her subjective manner, where $\hat{p}_{\ell\min}$ is an acceptable minimum value and $\hat{p}_{\ell\max}$ is a sufficiently satisfactory minimum value, and sets the intervals :

$$P_\ell \stackrel{\mathrm{def}}{=} [\hat{p}_{\ell\min}, \hat{p}_{\ell\max}], \ell = 1, \cdots, k.$$

Corresponding to the interval P_ℓ, let us denote the interval of $\mu_{f_\ell}(f_\ell(\boldsymbol{x}, \gamma, \hat{p}_\ell))$ as $[f_{\ell\min}, f_{\ell\max}]$, where $f_{\ell\min}$ is a sufficiently satisfactory maximum value and $f_{\ell\max}$ is an acceptable maximum value. $f_{\ell\min}$ can be obtained by solving the following problem.

$$f_{\ell\min} \stackrel{\mathrm{def}}{=} \min_{\boldsymbol{x} \in X} f_\ell(\boldsymbol{x}, \gamma, \hat{p}_{\ell\min}) \tag{29}$$

In order to obtain $f_{\ell\max}$, we first solve the following k programming problems, $\min_{\boldsymbol{x} \in X} f_\ell(\boldsymbol{x}, \gamma, \hat{p}_{\ell\max}), \ell = 1, \cdots, k$. Let $\boldsymbol{x}_\ell, \ell = 1, \cdots, k$ be the corresponding optimal solution. Using the optimal solutions $\boldsymbol{x}_\ell, \ell = 1, \cdots, k$, $f_{\ell\max}$ can be obtained as follows.

$$f_{\ell\max} \stackrel{\mathrm{def}}{=} \max_{\ell=1,\cdots,k, \ell \ne i} f_\ell(\boldsymbol{x}_\ell, \gamma, \hat{p}_{\ell\max}) \tag{30}$$

It should be noted here that, from (25), $\mu_{f_\ell}(f_\ell(\boldsymbol{x}, \gamma, \hat{p}_\ell))$ and $\mu_{\hat{p}_\ell}(\hat{p}_\ell)$ perfectly conflict each other with respect to $\hat{p}_\ell$. Here, let us assume that the decision maker adopts the fuzzy decision [1], [13], [24] in order to integrate both the membership functions $\mu_{f_\ell}(f_\ell(\boldsymbol{x}, \gamma, \hat{p}_\ell))$ and $\mu_{\hat{p}_\ell}(\hat{p}_\ell)$. Then, the integrated membership function $\mu_{D_{f_\ell}}(\boldsymbol{x}, \gamma, \hat{p}_\ell)$ can be defined as follows.

$$\mu_{D_{f_\ell}}(\boldsymbol{x}, \gamma, \hat{p}_\ell) \stackrel{\mathrm{def}}{=} \min\{\mu_{\hat{p}_\ell}(\hat{p}_\ell), \mu_{f_\ell}(f_\ell(\boldsymbol{x}, \gamma, \hat{p}_\ell))\} \tag{31}$$

Using the integrated membership functions $\mu_{D_{f_\ell}}(\boldsymbol{x}, \gamma, \hat{p}_\ell)$, $\ell = 1, \cdots, k$, MOP8(γ) can be transformed into the following form.

[MOP9(γ)]

$$\max_{\boldsymbol{x} \in X, \hat{p}_\ell \in P_\ell, \ell=1,\cdots,k} \left(\mu_{D_{f_1}}(\boldsymbol{x}, \gamma, \hat{p}_1), \cdots, \mu_{D_{f_k}}(\boldsymbol{x}, \gamma, \hat{p}_k)\right)$$

In order to deal with MOP9(γ), we introduce an M-γ-Pareto optimal solution concept.

Definition 3
$\boldsymbol{x}^* \in X, \hat{p}_\ell^* \in P_\ell, \ell = 1, \cdots, k$ is said to be an M-γ-Pareto optimal solution to MOP9(γ), if and only if there does not exist another $\boldsymbol{x} \in X, \hat{p}_\ell \in P_\ell, \ell = 1, \cdots, k$ such that $\mu_{D_{f_\ell}}(\boldsymbol{x}, \gamma, \hat{p}_\ell) \ge \mu_{D_{f_\ell}}(\boldsymbol{x}^*, \gamma, \hat{p}_\ell^*)$ $\ell = 1, \cdots, k$, with strict inequality holding for at least one ℓ.

For generating a candidate of a satisfactory solution which is also M-γ-Pareto optimal, the decision maker is asked to specify the reference membership values [14]. Once the reference membership values $\hat{\boldsymbol{\mu}} = (\hat{\mu}_1, \cdots, \hat{\mu}_k)$ are specified, the corresponding M-γ-Pareto optimal solution is obtained by solving the following minmax problem.

[MINMAX3($\hat{\boldsymbol{\mu}}, \gamma$)]

$$\min_{\boldsymbol{x} \in X, \hat{p}_\ell \in P_\ell, \ell=1,\cdots,k, \lambda \in \Lambda} \lambda \tag{32}$$

subject to

$$\hat{\mu}_\ell - \mu_{f_\ell}(f_\ell(\boldsymbol{x}, \gamma, \hat{p}_\ell)) \leq \lambda, \ell = 1, \cdots, k \tag{33}$$

$$\hat{\mu}_\ell - \mu_{\hat{p}_\ell}(\hat{p}_\ell) \leq \lambda, \ell = 1, \cdots, k \tag{34}$$

where

$$\Lambda \stackrel{\text{def}}{=} [\max_{i=1,\cdots,k} \hat{\mu}_i - 1, \min_{i=1,\cdots,k} \hat{\mu}_i].$$

Because of Assumption 2, the constraints (33) can be transformed as follows.

$$\hat{p}_\ell \leq \Phi\left(\frac{\mu_{f_\ell}^{-1}(\hat{\mu}_\ell - \lambda) - \boldsymbol{E}[\bar{\boldsymbol{c}}_\ell]\boldsymbol{x} - d_\ell(\boldsymbol{x}, \gamma)}{\sqrt{\boldsymbol{x}^T V_\ell \boldsymbol{x}}}\right) \tag{35}$$

From the constraints (34) and Assumption 1, it holds that $\hat{p}_\ell \geq \mu_{\hat{p}_\ell}^{-1}(\hat{\mu}_\ell - \lambda)$. Therefore, the constraint (35) can be reduced to the following inequality where a permissible probability level $\hat{p}_\ell$ is disappeared.

$$\begin{aligned} & \mu_{f_\ell}^{-1}(\hat{\mu}_\ell - \lambda) - \boldsymbol{E}[\bar{\boldsymbol{c}}_\ell]\boldsymbol{x} - d_\ell(\boldsymbol{x}, \gamma) \\ \geq \; & \Phi^{-1}(\mu_{\hat{p}_\ell}^{-1}(\hat{\mu}_\ell - \lambda)) \cdot \sqrt{\boldsymbol{x}^T V_\ell \boldsymbol{x}}, \ell = 1, \cdots, k \end{aligned} \tag{36}$$

Then, MINMAX3($\hat{\boldsymbol{\mu}}, \gamma$) can be equivalently reduced to the following problem.

[MINMAX4($\hat{\boldsymbol{\mu}}, \gamma$)]

$$\min_{\boldsymbol{x} \in X, \lambda \in \Lambda} \lambda \tag{37}$$

subject to

$$\begin{aligned} & \mu_{f_\ell}^{-1}(\hat{\mu}_\ell - \lambda) - \boldsymbol{E}[\bar{\boldsymbol{c}}_\ell]\boldsymbol{x} - d_\ell(\boldsymbol{x}, \gamma) \\ \geq \; & \Phi^{-1}(\mu_{\hat{p}_\ell}^{-1}(\hat{\mu}_\ell - \lambda)) \cdot \sqrt{\boldsymbol{x}^T V_\ell \boldsymbol{x}}, \ell = 1, \cdots, k \end{aligned} \tag{38}$$

In order to solve MINMAX4($\hat{\boldsymbol{\mu}}, \gamma$), which is a nonlinear programming problem, we first define the following function for the constraints (38).

$$\begin{aligned} g_\ell(\boldsymbol{x}, \gamma, \lambda) \stackrel{\text{def}}{=} \; & \mu_{\hat{f}_\ell}^{-1}(\hat{\mu}_\ell - \lambda) - \boldsymbol{E}[\bar{\boldsymbol{c}}_\ell]\boldsymbol{x} - d_\ell(\boldsymbol{x}, \gamma) \\ & -\Phi^{-1}(\mu_{p_\ell}^{-1}(\hat{\mu}_\ell - \lambda)) \cdot \sqrt{\boldsymbol{x}^T V_\ell \boldsymbol{x}}, \\ & \ell = 1, \cdots, k \end{aligned} \tag{39}$$

It should be noted here that $g_\ell(\boldsymbol{x}, \gamma, \lambda), \ell = 1, \cdots, k$ are concave functions for any fixed $\lambda \in \Lambda$. Under Assumption 3, it is clear that the constraint set $G(\lambda, \gamma)$ is a convex set for any $\lambda \in [0, 1]$, which is defined as follows.

$$G(\lambda, \gamma) \stackrel{\text{def}}{=} \{\boldsymbol{x} \in X \mid g_\ell(\boldsymbol{x}, \gamma, \lambda) \geq 0, \ell = 1, \cdots, k\} \tag{40}$$

The constraint set $G(\lambda, \gamma)$ satisfies the following property for $\lambda \in \Lambda$.

Property 1

If $\lambda_1, \lambda_2 \in \Lambda$ and $\lambda_1 < \lambda_2$, then it holds that $G(\lambda_1) \subset G(\lambda_2)$.

(Proof)

From Assumptions 1 and 2, it holds that $\mu_{f_\ell}^{-1}(\hat{\mu}_\ell - \lambda_1) < \mu_{f_\ell}^{-1}(\hat{\mu}_\ell - \lambda_2)$, $\Phi^{-1}(\mu_{\hat{p}_\ell}^{-1}(\hat{\mu}_\ell - \lambda_1)) > \Phi^{-1}(\mu_{\hat{p}_\ell}^{-1}(\hat{\mu}_\ell - \lambda_2))$. This means that $g_\ell(\boldsymbol{x}, \gamma, \lambda_1) < g_\ell(\boldsymbol{x}, \gamma, \lambda_2)$ for any $\boldsymbol{x} \in X$. Therefore, $G(\lambda_1, \gamma) \subset G(\lambda_2, \gamma)$ for any $\lambda_1 < \lambda_2$.

From Property 1, we can easily obtain the optimal solution $(\boldsymbol{x}^*, \lambda^*)$ of MINMAX4($\hat{\boldsymbol{\mu}}, \gamma$) by using the bisection method for $\lambda \in \Lambda$, where it is assumed that $G(\lambda_{\max}, \gamma) \neq \phi, G(\lambda_{\min}, \gamma) = \phi$.

[Algorithm 3]

Step 1: Set $\lambda_0 = \lambda_{\max}, \lambda_1 = \lambda_{\min}$ and $\lambda \leftarrow (\lambda_0 + \lambda_1)/2$.

Step 2: Solve the following convex programming problem for $\lambda \in \Lambda$, and let us denote the optimal solution as $\boldsymbol{x}^* \in X$.

$$\max_{\boldsymbol{x} \in X} g_j(\boldsymbol{x}, \gamma, \lambda) \tag{41}$$

subject to

$$g_\ell(\boldsymbol{x}, \gamma, \lambda) \geq 0, \ell = 1, \cdots, k, i \neq j \tag{42}$$

Step 3: If $\mid \lambda_1 - \lambda_0 \mid < \epsilon$, then go to Step 4, where ϵ is a sufficiently small positive constant. If $g_j(\boldsymbol{x}^*, \lambda) < 0$ or there exists the index $i \neq j$ such that $g_\ell(\boldsymbol{x}^*, \lambda) < 0$, then set $\lambda_1 \leftarrow \lambda, \lambda \leftarrow (\lambda_0 + \lambda_1)/2$, and go to Step 2. Otherwise, if $g_j(\boldsymbol{x}^*, \lambda) \geq 0$ and $g_\ell(\boldsymbol{x}^*, \lambda) \geq 0$ for any $\ell = 1, \cdots, k, i \neq j$, then set $\lambda_0 \leftarrow \lambda, \lambda \leftarrow (\lambda_0 + \lambda_1)/2$, and go to Step 2.

Step 4: Set $\lambda^* \leftarrow \lambda$, and the optimal solution $(\boldsymbol{x}^*, \lambda^*)$ to MINMAX4($\hat{\boldsymbol{\mu}}, \gamma$) is obtained.

The relationship between the optimal solution $(\boldsymbol{x}^*, \lambda^*)$ of MINMAX4($\hat{\boldsymbol{\mu}}, \gamma$) and M-$\gamma$-Pareto optimal solutions can be characterized by the following theorem.

Theorem 6

(1) If $\boldsymbol{x}^* \in X, \lambda^* \in \Lambda$ is a unique optimal solution to MINMAX4($\hat{\boldsymbol{\mu}}, \gamma$), then $\boldsymbol{x}^* \in X, \hat{p}_\ell^* = \mu_{\hat{p}_\ell}^{-1}(\hat{\mu}_\ell - \lambda^*) \in P_\ell, \ell = 1, \cdots, k$ is an M-γ-Pareto optimal solution to MOP9(γ).

(2) If $\boldsymbol{x}^* \in X, \hat{p}_\ell^* \in P_\ell, \ell = 1, \cdots, k$ is an M-γ-Pareto optimal solution to MOP9(γ), then $\boldsymbol{x}^* \in X$, $\lambda^* = \hat{\mu}_\ell - \mu_{\hat{p}_\ell}(\hat{p}_\ell^*) = \hat{\mu}_\ell - \mu_{f_\ell}(f_\ell(\boldsymbol{x}^*, \hat{p}_\ell^*)), \ell = 1, \cdots, k$ is an optimal solution to MINMAX4($\hat{\boldsymbol{\mu}}, \gamma$) for some reference membership values $\hat{\boldsymbol{\mu}} = (\hat{\mu}_1, \cdots, \hat{\mu}_k)$.

Now, following the above discussions, we can present the interactive algorithm in order to derive a satisfactory solution from among an M-γ-Pareto optimal solution set.

[Algorithm 4]

Step 1: The decision maker specifies $\hat{p}_{\ell\min} > 0.5$ and $\hat{p}_{\ell\max} < 1, \ell = 1, \cdots, k$ in his/her subjective manner. On the interval $P_\ell = [\hat{p}_{\ell\min}, \hat{p}_{\ell\max}]$, the decision maker sets his/her membership functions $\mu_{\hat{p}_\ell}(\hat{p}_\ell), \ell = 1, \cdots, k$ according to Assumption 1.

Step 2: Corresponding to the interval P_ℓ, compute $f_{\ell\min}$ and $f_{\ell\max}$ by solving the problems (29) and (30). On the interval $[f_{\ell\min}, f_{\ell\max}]$, the decision maker sets his/her membership functions $\mu_{f_\ell}(f_\ell(\boldsymbol{x}, \hat{p}_\ell)), \ell = 1, \cdots, k$ according to Assumption 2.

Step 3: Set the initial reference membership values as $\hat{\mu}_\ell = 1, \ell = 1, \cdots, k$.

Step 4: Solve MINMAX4($\hat{\boldsymbol{\mu}}, \gamma$) to obtain the M-$\gamma$-Pareto optimal solution.

Step 5: If the decision maker is satisfied with the current values of the M-γ-Pareto optimal solution $\mu_{D_{f_\ell}}(\boldsymbol{x}^*, \hat{p}_\ell^*), \ell = 1, \cdots, k$, where $\hat{p}_\ell^* = \mu_{\hat{p}_\ell}^{-1}(\hat{\mu}_\ell - \lambda^*)$, then stop. Otherwise, the decision maker must update his/her reference membership values $\hat{\mu}_\ell, \ell = 1, \cdots, k$ and/or a permissible possibility level γ, and return to Step 4.

IV. A CROP PLANNING PROBLEM IN THE PHILIPPINES

In this section, we apply Algorithm 2 to a second crop planning problem of paddy fields in the Philippines [23] under the hypothetical decision maker, in which the water

TABLE I
PROFIT COEFFICIENTS $c_{tj}, t = 1, \cdots, 5, j = 1, \cdots, 7$

	j	1	2	3	4	5	6	7
t	year	c_{t1}	c_{t2}	c_{t3}	c_{t4}	c_{t5}	c_{t6}	c_{t7}
1	1989	4.5	32.6	4.0	72.6	7.3	2.7	10.9
2	1990	5.7	22.7	29.5	13.6	6.3	4.3	24.5
3	1991	3.8	26.3	42.3	42.9	5.1	1.2	13.3
4	1992	3.5	21.3	20.1	35.7	5.2	2.5	26.1
5	1993	4.4	26.2	39.3	22.5	8.4	2.2	26.6

availability constraint in the dry season is expressed as a equality constraint with a fuzzy random variable. In the model farm, only rice (x_1) is grown in the wet season (May to October), and tobacco(x_2), tomatoes(x_3), garlic(x_4), mung beans(x_5), corn(x_6) and sweet peppers(x_7) are grown in the dry season (November to April), where x_j means the cultivation area (unit: 1 ha) for each crop $j = 1, \cdots, 7$. It is assumed that the farmer has only two persons of available family labor, but he does not have access to hired labor. The farmer must decide the planting ratio among seven kinds of crops ($x_j, j = 1, \cdots, 7$) in his/her farmland to maximize his/her total income and minimize total work hours.

Table I shows the profit coefficients c_{tj} of seven crops j ($j = 1, \cdots, 7$) in each year [23]. From Table I, we can compute the expected values as

$$\begin{aligned} & (E[\bar{c}_1], E[\bar{c}_2], E[\bar{c}_3], E[\bar{c}_4], E[\bar{c}_5], E[\bar{c}_6], E[\bar{c}_7]) \\ = \; & (4.38, 25.82, 27.04, 37.46, 6.46, 2.58, 20.28), \end{aligned}$$

and the variance covariance matrix V.

$$V_1 = \begin{pmatrix} 0.717 & 0.1005 & -0.504 & -7.296 \\ 0.1005 & 19.13 & -30.13 & 79.39 \\ -0.504 & -30.13 & 242.06 & -239.13 \\ -7.296 & 79.39 & -239.13 & 515.2 \\ 0.4565 & 2.994 & -1.993 & -0.217 \\ 0.787 & -1.250 & -5.924 & -9.626 \\ 0.8745 & -26.015 & 39.27 & -143.3 \end{pmatrix}$$

$$V_2 = \begin{pmatrix} 0.4565 & 0.787 & 0.8745 \\ 2.994 & -1.250 & -26.01 \\ -1.993 & -5.924 & 39.27 \\ -0.217 & -9.626 & -143.3 \\ 1.983 & 0.2665 & 1.467 \\ 0.2665 & 1.257 & 3.225 \\ 1.467 & 3.225 & 57.08 \end{pmatrix}$$

$$V = (V_1 \mid V_2)$$

In the following, we assume that the profit coefficients for seven kinds of crops can be regarded as normal random variables :

$$\bar{c}_j \sim \mathrm{N}(E[\bar{c}_j], \sigma_{jj}), j = 1, \cdots, 7.$$

Then, the first objective function $\bar{z}_1(\boldsymbol{x})$ (total profit, unit: 1000 pesos) can be defined as follows.

$$\bar{z}_1(\boldsymbol{x}) \stackrel{\text{def}}{=} \sum_{j=1}^{7} \bar{c}_j x_j = \bar{\boldsymbol{c}}\boldsymbol{x} \sim \mathrm{N}(\boldsymbol{E}[\bar{\boldsymbol{c}}]\boldsymbol{x}, \boldsymbol{x}^T V \boldsymbol{x})$$

where $\boldsymbol{x} \stackrel{\text{def}}{=} (x_1, \cdots, x_7)$. The second objective function is total working hours. Table II shows the required working hours $L_{\ell j}$ for each crop ($j = 1, \cdots, 7$) and each period (from the middle ten days in May to the last ten days in April, $\ell = 1, \cdots, 27$) [23]. Then, the second objective function (a total number of working hours, unit: 1 hour) can be expressed as :

$$z_2(\boldsymbol{x}) \stackrel{\text{def}}{=} \sum_{\ell=1}^{27} \sum_{j=1}^{7} L_{\ell j} x_j.$$

TABLE II
THE REQUIRED WORKING HOURS FOR EACH PERIOD $L_{\ell j}$

period : ℓ	$L_{\ell 1}$	$L_{\ell 2}$	$L_{\ell 3}$	$L_{\ell 4}$	$L_{\ell 5}$	$L_{\ell 6}$	$L_{\ell 7}$
2-May : 1	26						
1-Jun : 2	16						
2-Jun : 3	160						
1-Jul : 4	16						
2-Jul : 5	6						
2-Aug : 6	8						
3-Sep : 7	140						
1-Oct : 8	32	8					
3-Oct : 9		46			6		
1-Nov : 10		36		174		8	
2-Nov : 11		100	10	44	12	54	50
3-Nov : 12			22	16	12	8	20
1-Dec : 13		8	38	16	10	16	108
2-Dec : 14		16	94	16			72
3-Dec : 15		8	32	16			24
1-Jan : 16		8	14				64
2-Jan : 17		36	14		12		16
3-Jan : 18		70	6				
1-Feb : 19		70	6	180			48
2-Feb : 20		36	14			60	56
3-Feb : 21		36	6				48
1-Mar : 22			36				56
2-Mar : 23			30		32		
3-Mar : 24			30				
1-Apr : 25			38		56		
2-Apr : 26			30				
3-Apr : 27			26				

Since the upper limit of the working hours for each period ($\ell = 1, \cdots, 27$) can be computed as 8 (hours) $\times$ 2 (persons) $\times$ 10 (days) = 160 (hours), the constraints :

$$\sum_{j=1}^{7} L_{\ell j} x_j \leq 80, \;\ell = 1, \cdots, 27$$

must be satisfied. As two land area constraints (unit: 1 ha) for the wet and dry season,

$$x_1 \leq 1, \sum_{j=2}^{7} x_j \leq 1, x_j \geq 0, j = 1, \cdots, 7$$

must be satisfied. We assume that the water availability constraint in the dry season is expressed as :

$$\sum_{j=2}^{7} w_j x_j = \tilde{\bar{d}},$$

where the water demand coefficients w_j for the crops ($j = 2, \cdots, 7$) are set as $(w_2, w_3, w_4, w_5, w_6, w_7) = (264.6, 232.3, 352.8, 88.2, 44.1, 220.5)$ [23], and the water supply possible amount is defined as a following LR-type fuzzy random variable $\tilde{\bar{d}}$ (unit : 1000 gallons).

$$\mu_{\tilde{d}(\omega)}(s) = \begin{cases} L\left(\frac{\bar{b}(\omega) - s}{\alpha}\right), & s \leq \bar{b}(\omega) \\ R\left(\frac{s - \bar{b}(\omega)}{\beta}\right), & s > \bar{b}(\omega) \end{cases}$$

where $\bar{b} \sim \mathrm{N}(300, 5^2)$, $\alpha = \beta = 30$, and $L(t) = R(t) = 1 - t, 0 \le t \le 1$.

Since the penalty cost arises only for the shortage of water resource, it is assumed that $q_1^+ = 0, q_1^- = 10$ and $q_2^+ = q_2^- = 0$. Then, for the reference objective value $\hat{z}_\ell, \ell = 1, 2$ specified by the decision maker, the corresponding γ-Pareto optimal solution is obtained by solving the following minimax problem.

[MINMAX5($\hat{\boldsymbol{f}}$, γ, $\hat{\boldsymbol{p}}$)]

$$\max_{\boldsymbol{x} \in X, \lambda \in \mathrm{R}^1} \lambda$$

subject to

$$f_1(\boldsymbol{x}, \gamma, \hat{p}) - \hat{f}_1 \le \lambda$$
$$\sum_{\ell=1}^{27} \sum_{j=1}^{7} L_{\ell j} x_j - \hat{f}_2 \le \lambda$$

where $f_1(\boldsymbol{x}, \gamma, \hat{p})$ is defined as follows.

$$\begin{aligned} f_1(\boldsymbol{x}, \gamma, \hat{p}) &\stackrel{\text{def}}{=} -\sum_{j=1}^{7} E[\bar{c}_j] x_j + \Phi_0^{-1}(\hat{p}) \cdot \sqrt{\boldsymbol{x}^T V \boldsymbol{x}} \\ &\quad + d(\boldsymbol{x}, \gamma) \\ d(\boldsymbol{x}, \gamma) &\stackrel{\text{def}}{=} q_1^- \Bigg\{ (\sum_{j=2}^{7} w_j x_j - R^{-1}(\gamma)\beta) \\ &\quad \cdot \Phi(\sum_{j=2}^{7} w_j x_j - R^{-1}(\gamma)\beta) \\ &\quad - \int_{-\infty}^{\sum_{j=2}^{7} w_j x_j - R^{-1}(\gamma)\beta} b\phi(b) db \Bigg\} \end{aligned}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ mean the probability density function and the cumulative distribution one for $\mathrm{N}(300, 5^2)$, and $\Phi_0(\cdot)$ means the cumulative distribution function for $\mathrm{N}(0, 1)$.

Under the hypothetical decision maker, we apply Algorithm 2 in section III to the crop planning problem described above. In Step 1, the decision maker sets a permissible possibility level $\gamma = 1$ and a permissible probability level $\hat{p} = 0.8$. In Step 2, the decision maker sets the initial reference objective values $(\hat{f}_1, \hat{f}_2) = (-33, 680)$. In Step 3, MINMAX5($\hat{\boldsymbol{f}}$, γ, $\hat{\boldsymbol{p}}$) is solved by using Mathematica and the corresponding (γ, $\hat{\boldsymbol{p}}$)-Pareto optimal solution is obtained. In Step 4, the decision maker is not satisfied with the current value of the (γ, $\hat{\boldsymbol{p}}$)-Pareto optimal solution, he/she updates his/her reference objective values as $(\hat{f}_1, \hat{f}_2) = (-33, 620)$, and return to Step 3. The interactive processes under the hypothetical decision maker is shown in Table III. In this example, a satisfactory solution is obtained at the third iteration, in which a permissible possibility level $\gamma = 1$ and a permissible probability level $\hat{p} = 0.8$ are fixed at each iteration. For comparison, Table IV shows the interactive processes under the same conditions except that a permissible possibility level is set as $\gamma = 0.5$. By comparing Table III for $\gamma = 1$ with Table IV for $\gamma = 0.5$, it is clear that any (γ, $\hat{\boldsymbol{p}}$)-Pareto optimal solution for $\gamma = 0.5$ is superior to (γ, $\hat{\boldsymbol{p}}$)-Pareto optimal solution for $\gamma = 1$ because of the definition of a possibility measure. In any (γ, $\hat{\boldsymbol{p}}$)-Pareto optimal solution of Table III and Table IV, only tomatoes (x_3) and garlic (x_4) in the dry season and rice (x_1) in the wet season are cultivated. The larger value of a permissible possibility level γ gives the larger planting ratio of tomatoes (x_3) and the smaller one of garlic (x_4) because of the difference of the water demand coefficients of tomatoes (x_3) and garlic (x_4).

TABLE III
INTERACTIVE PROCESSES FOR $\gamma = 1$

	1	2	3
$\hat{z}_1$	-33	-33	-30
$\hat{z}_2$	680	620	620
$\hat{z}_1$	-33	-33	-30
$\hat{z}_2$	680	620	620
$z_1(\boldsymbol{x}^*)$	-27.934	-27.238	-27.204
$z_2(\boldsymbol{x}^*)$	685.07	625.76	622.80
x_1^*	0.57343	0.42734	0.42000
x_2^*	0.000	0.000	0.000
x_3^*	0.55289	0.555327	0.55535
x_4^*	0.44465	0.44466	0.44465
x_5^*	0.000	0.000	0.000
x_6^*	0.000	0.000	0.000
x_7^*	0.00246	0.000	0.000

TABLE IV
INTERACTIVE PROCESSES FOR $\gamma = 0.5$

	1	2	3
$\hat{z}_1$	-33	-33	-30
$\hat{z}_2$	680	620	620
$z_1(\boldsymbol{x}^*)$	-28.001	-27.305	-27.270
$z_2(\boldsymbol{x}^*)$	685.00	625.70	622.73
x_1^*	0.57306	0.42628	0.41894
x_2^*	0.000	0.000	0.000
x_3^*	0.53228	0.53249	0.53250
x_4^*	0.46772	0.46751	0.46750
x_5^*	0.000	0.000	0.000
x_6^*	0.000	0.000	0.000
x_7^*	0.000	0.000	0.000

V. CONCLUSIONS

In this paper, we formulate three types of multiobjective fuzzy random simple recourse programming problems, in which fuzzy random variables coefficients are involved in equality constraints. In the proposed methods, equality constraints with fuzzy random variables are defined on the basis of a possibility measure and and a two-stage programming method. For a given permissible possibility level and reference objective values specified by the decision maker, corresponding minimax problem is solved to obtain a Pareto optimal solution. The proposed method (Algorithm 2) is applied to a farm planning problem in the Philippines, in which an amount supplied of water resource in dry season is represented as a fuzzy random variable and the profit coefficients are represented as random variables. In the near future. we will apply Algorithm 4 to a farm planning problem in the Philippines, in which permissible probability levels are automatically computed through the fuzzy decision.

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