Orlicz Mixed Geominimal Surface Area

Yuanyuan Guo, Tongyi Ma*, Li Gao

Abstract—**In this paper, we deal with the Orlicz geominimal surface area and give an integral representation by the Orlicz-Petty body. The notion of Orlicz mixed geominimal surface area will be introduced as an extension of the Orlicz geominimal surface area. Furthermore, some related inequalities are established, including Alexandrov-Fenchel type inequality, analogous cyclic inequality, Blaschke-Santaló type inequality, and affine isoperimetric inequality.**

Index Terms—**Convex bodies, Orlicz geominimal surface area, Orlicz mixed geominimal surface area.**

I. Introduction

LET $\mathcal{K}^n$ denote the class of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean n-space $\mathbf{R}^n$. For the class of convex bodies containing the origin in their interiors and the class of origin-symmetric convex bodies in $\mathbf{R}^n$, we write $\mathcal{K}_o^n$ and $\mathcal{K}_c^n$, respectively. $\mathcal{S}_o^n$ denotes the class of star bodies (about the origin) in $\mathbf{R}^n$. Write S^{n-1} and B for the unit sphere and the standard Euclidean unit ball in $\mathbf{R}^n$, respectively. Besides, we use $V(K)$ to denote the n-dimensional volume of a body K, and write $\omega_n = V(B)$ for the n-dimensional volume of B.

The classical geominimal surface area was firstly introduced by Petty [1], which serves as a bridge connecting many areas of geometry: affine differential geometry, relative differential geometry, and Minkowskian geometry. For $K \in \mathcal{K}_o^n$, the geominimal surface area, $G(K)$, of K is defined by (see [1])

$$\omega_n^{1/n} G(K) = \inf\{nV_1(K,Q)V(Q^*)^{1/n} : Q \in \mathcal{K}_o^n\}, \quad (1)$$

where Q^* denotes the polar of convex body Q, and $V_1(K,Q)$ is the mixed volume of $K, Q \in \mathcal{K}_o^n$ (see [2]).

The development of L_p-space appeared in the early 1960s (see [3]), and started to make rapidly progress from the initial Lutwak's contributions ([4], [5]) in the mid 1990s. In his seminal paper [5], Lutwak extended the classical geominimal surface area to L_p-version and obtained related inequalities. Ma et al. also studied this topic of L_p-space. For instance, Ma et al. [6] defined the concept of ith L_p-mixed affine surface areas and established related monotonic inequality. In [7], Ma et al. obtained some Brunn-Minkowski type inequalities of L_p-geominimal surface area. There are many papers on L_p-space, see e.g., [8], [9], [10], [11], [12], [13], [14], [15], [16], [17].

Manuscript received January 01, 2016; revised March 11, 2016. This work was supported by the National Natural Science Foundation of China under Grant No.11561020, and was supported by the Science and Technology Plan of the Gansu Province under Grant No.145RJZG227.

*T. Y. Ma is with the College of Mathematics and Statistics, Hexi University, Zhangye, 734000, China. And is with the College of Mathematics and Statistics, Northwest Normal University, Lanzhou, 730070, China. e-mail: matongyi@126.com.

Y. Y. Guo is with the College of Mathematics and Statistics, Northwest Normal University, Lanzhou, 730070, China.

L. Gao is with the College of Mathematics and Statistics, Northwest Normal University, Lanzhou, 730070, China.

Lutwak [5] defined the L_p-geominimal surface area $G_p(K)$ of $K \in \mathcal{K}_o^n$ as follows: For $p \geq 1$,

$$\omega_n^{p/n} G_p(K) = \inf\{nV_p(K,Q)V(Q^*)^{p/n} : Q \in \mathcal{K}_o^n\}, \quad (2)$$

where $V_p(K,Q)$ denotes the L_p-mixed volume of $K, Q \in \mathcal{K}_o^n$ (see [5]). When $p = 1$, $G_1(K)$ is just classical geominimal surface area $G(K)$.

Based on the homogeneous of volume and L_p-mixed volume, the L_p-geominimal surface area can be defined by

$$G_p(K) = \inf\{nV_p(K,Q) : Q \in \mathcal{K}_o^n \text{ and } V(Q^*) = \omega_n\}. \quad (3)$$

In recent years, the Orlicz Brunn-Minkowski theory has aroused increasing attention, which plays such a significant role that it is undeniably applied to a large number of areas of geometry. The beautiful Orlicz Brunn-Minkowski theory, a new extension of L_p-Brunn-Minkowski theory, originated from Lutwak, Yang and Zhang (see [18], [19]). In these papers, the affine isoperimetric inequalities for L_p-projection bodies and L_p-centroid bodies were expanded to Orlicz space. However, because of lacking homogeneity for nonhomogeneous function $\phi(t)$, the way of defining Orlicz addition is nontrivial extremely to be found appropriately. Fortunately, in the groundbreaking paper [20], Gardner, Hug and Weil have gotten over the difficulty. They introduced the definition of Orlicz addition and Orlicz mixed volume. On the basis of the linear Orlicz addition for convex bodies, they also established the new Orlicz Brunn-Minkowski inequality and the Orlicz Minkowski mixed volume inequality. Their classical counterparts are the Brunn-Minkowski inequality and Minkowski inequality, which have been applied in many fields. The more development of the Orlicz Brunn-Minkowski theory, see, for example, [21], [22], [23], [24], [25], [26], [27], [28] among others.

More recently, Yuan et al. introduced the Orlicz geominimal surface area $G_\phi(K)$ of $K \in \mathcal{K}_o^n$ (see [29]). Let Φ denote the set of convex functions $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(0) = 0$ and $\phi(1) = 1$. For $K \in \mathcal{K}_o^n$, and $\phi \in \Phi$,

$$G_\phi(K) = \inf\{nV_\phi(K,Q) : Q \in \mathcal{K}_o^n \text{ and } V(Q^*) = \omega_n\}, \quad (4)$$

where $V_\phi(K,Q)$ denotes Orlicz mixed volume of K, Q (see [25]).

Yuan et al. [29] have proved the existence property of Orlicz geominimal surface area. In this paper, the first goal is to establish the uniqueness property of Orlicz geominimal surface area. Then we can give the integral representation of Orlicz geominimal surface area by the Orlicz-Petty body (see Section 3.1).

Motivated by the work of Zhu et al. [30], we introduce Orlicz geominimal surface area and extend the related theory to Orlicz version. We define the Orlicz mixed geominimal surface area (see Section 3.2), which extends the concept of Orlicz geominimal surface area. In addition, we devote to the general ith Orlicz mixed geominimal surface area. Some related inequalities for Orlicz mixed geominimal surface are established, including Alexandrov-Fenchel type inequality, analogous cyclic inequality, Blaschke-Santaló type inequality, and isoperimetric inequality (see Section 3.3). These inequalities are natural extensions of inequalities for L_p-geominimal surface area.

Our paper is organized as follows. Firstly, in Section II, we provide some preliminaries, including definitions and known results we will use. Then, in Section III, we give our main results and proofs.

II. Preliminaries

For a compact convex set $K \in \mathcal{K}_o^n$, its support function, $h_K = h(K,\cdot) : S^{n-1} \to \mathbf{R}$ is defined by,

$$h_K(u) = \max\{u \cdot x : x \in K\}, \quad u \in S^{n-1},$$

where $u \cdot x$ denotes the standard inner product of u and x in $\mathbf{R}^n$. Note that the compact convex set K is uniquely determined by its support function h_K.

Let $\mathrm{GL}(n)$ denote the group of linear transformations. If $A \in \mathrm{GL}(n)$, then

$$h_{AK}(u) = h_K(A^t u),$$

where A^t denotes the transpose of A (see [31]). For $K, L \in \mathcal{K}_o^n$, the Hausdorff metric is defined by

$$\delta(K,L) = \sup_{u \in S^{n-1}} |h(K,u) - h(L,u)|.$$

A set $K \subset \mathbf{R}^n$ is said to be a star body (about the origin), if the line segment from the origin to any point $x \in K$ is contained in K and K has continuous and positive radial function $\rho_K(\cdot)$. Here, the radial function of K, $\rho_K = \rho(K,\cdot) : S^{n-1} \to [0,\infty)$, is defined by

$$\rho_K(u) = \max\{\lambda : \lambda u \in K\},$$

and it uniquely determines the compact convex set K. Two star bodies K, L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

For $K \in \mathcal{K}_o^n$, then K^*, the polar body of K is defined by (see [31], [32])

$$K^* = \{x \in \mathbf{R}^n : x \cdot y \le 1, \forall y \in K\}.$$

When $K \in \mathcal{K}_o^n$, it can be easily proved that $(K^*)^* = K$.

From the definitions it follows obviously that for each convex body $K \in \mathcal{K}_o^n$, we easily get

$$h_{K^*}(u) = \frac{1}{\rho_K(u)} \text{ and } \rho_{K^*}(u) = \frac{1}{h_K(u)}, \text{ for all } u \in S^{n-1}.$$

For $K \in \mathcal{K}_o^n$, the Blaschke-Santaló inequality (see [33], [34]) states as follows: If $K \in \mathcal{K}_c^n$ then

$$V(K)V(K^*) \le \omega_n^2, \tag{5}$$

with equality if and only if K is an ellipsoid.

For $K, L \in \mathcal{K}^n$, and $\lambda, \mu \ge 0$ (not both zero), the Minkowski linear combination $\lambda K + \mu L \in \mathcal{K}^n$ is defined by (see [5])

$$h(\lambda K + \mu L, \cdot) = \lambda h(K,\cdot) + \mu h(L,\cdot).$$

The classical Brunn-Minkowski inequality states that for $K, L \in \mathcal{K}^n$ and $\lambda, \mu \ge 0$ (not both zero), the volume of the bodies and of their Minkowski linear combination $\lambda K + \mu L \in \mathcal{K}^n$ are related by (see [35])

$$V(\lambda K + \mu L)^{\frac{1}{n}} \ge \lambda V(K)^{\frac{1}{n}} + \mu V(L)^{\frac{1}{n}},$$

with equality if and only if K and L are homothetic.

For real $p \ge 1$, $K, L \in \mathcal{K}_o^n$, and $\lambda, \mu \ge 0$ (not both zero), the Firey linear combination $\lambda \cdot K +_p \mu \cdot L$, is defined by (see [3])

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot) = \lambda h(K,\cdot)^p + \mu h(L,\cdot)^p,$$

where "$\cdot$" in $\lambda \cdot K$ denotes the Firey scalar multiplication.

After that Firey [3] established the L_p-Brunn-Minkowski inequality. If $p > 1$, $\lambda, \mu \ge 0$ (not both zero), and $K, L \in \mathcal{K}_o^n$, then

$$V(\lambda \cdot K +_p \mu \cdot L)^{\frac{p}{n}} \ge \lambda V(K)^{\frac{p}{n}} + \mu V(L)^{\frac{p}{n}},$$

with equality if and only if K and L are dilates.

In [5], Lutwak defined the harmonic L_p-combination $\lambda \circ K \hat{+}_p \mu \circ L$, as follows: For $K, L \in \mathcal{S}_o^n$ and $\lambda, \mu \ge 0$ (not both zero),

$$\rho(\lambda \circ K \hat{+}_p \mu \circ L, \cdot)^{-p} = \lambda \rho(K,\cdot)^{-p} + \mu \rho(L,\cdot)^{-p}.$$

If $K, L \in \mathcal{K}_o^n$ (rather than being in $\mathcal{S}_o^n$), then

$$\lambda \circ K \hat{+}_p \mu \circ L = (\lambda \cdot K^* +_p \mu \cdot L^*)^*.$$

Further Lutwak established the L_p Brunn-Minkowski inequality. If $p \ge 1$, $\lambda, \mu \ge 0$ (not both zero), and $K, L \in \mathcal{K}_o^n$, then

$$V(\lambda \circ K \hat{+}_p \mu \circ L)^{-\frac{p}{n}} \ge \lambda V(K)^{-\frac{p}{n}} + \mu V(L)^{-\frac{p}{n}}, \tag{6}$$

with equality if and only if K and L are dilates (see [5]).

In [20], Gardner, Hug and Weil introduced the definition of Orlicz mixed volume. For $\phi \in \Phi$, $K, L \in \mathcal{K}_o^n$, the Orlicz mixed volume $V_\phi(K,L)$ of K, L is defined by

$$V_\phi(K,L) = \frac{1}{n} \int_{S^{n-1}} \phi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u) \mathrm{d}S_K(u), \tag{7}$$

where $S_K(\cdot)$ is the surface area measure of K.

Apparently, we have

$$V_\phi(K,K) = V(K). \tag{8}$$

For $\phi(t) = t^p$ with $p \ge 1$, the Orlicz mixed volume $V_\phi(K,L)$ reduces to L_p-mixed volume $V_p(K,L)$ of K, L (see [5]):

$$V_p(K,L) = \frac{1}{n} \int_{S^{n-1}} h(L,u)^p \mathrm{d}S_p(K,u),$$

where $S_p(K,\cdot)$ is the L_p-surface area measure of K.

Xi, Jin and Leng [25] established the Orlicz Minkowski inequality: Let $\phi \in \Phi$, if $K, L \in \mathcal{K}_o^n$, then

$$V_\phi(K,L) \ge V(K)\phi\left(\left(\frac{V(L)}{V(K)}\right)^{\frac{1}{n}}\right), \tag{9}$$

with equality if K and L are dilates. When ϕ is strictly convex, equality in (9) holds if and only if K and L are dilates. When $\phi(t) = t^p (p \ge 1)$, the corresponding results of above reduces to the L_p-Brunn-Minkowski inequality.

The following result provides an Orlicz geominimal surface area inequality by Yuan et al. [29]. If $K \in \mathcal{K}_c^n$ and $\phi \in \Phi$, then

$$G_\phi(K) \le nV(K)\phi\left(\left(\frac{\omega_n}{V(K)}\right)^{\frac{1}{n}}\right), \tag{10}$$

with equality only if K is an ellipsoid.

III. Main Results and Proofs

A. Orlicz geominimal surface area

At first, we prove the uniqueness of Orlicz geominimal surface area. Then combining with the existence of Orlicz geominimal surface area proved by Yuan et al. [29], we give the integral representation of Orlicz geominimal surface area.

Theorem 3.1. If $K \in \mathcal{K}_o^n$ and $\phi \in \Phi$, then there exists a unique body $\bar{K} \in \mathcal{K}_o^n$ such that

$$G_\phi(K) = nV_\phi(K, \bar{K}) \quad \text{and} \quad V(\bar{K}^*) = \omega_n.$$

Proof. In [29], the existence property of Theorem 3.1 has been proven, now we just prove the uniqueness.

Suppose $L_1, L_2 \in \mathcal{K}_o^n$ with $L_1 \neq L_2$, such that $V(L_1^*) = \omega_n = V(L_2^*)$, and

$$V_\phi(K, L_1) = V_\phi(K, L_2).$$

Defined $L \in \mathcal{K}_o^n$, by

$$L = \frac{1}{2} \cdot L_1 + \frac{1}{2} \cdot L_2.$$

Since obviously,

$$L^* = \frac{1}{2} \circ L_1^* \hat{+} \frac{1}{2} \circ L_2^*,$$

and $V(L_1^*) = \omega_n = V(L_2^*)$, it follows from the (6) that

$$V(L^*) \leq \omega_n,$$

with equality if and only if $L_1 = L_2$.

By the definition (7) of Orlicz mixed volume, together with the convexity of ϕ, we have

$$\begin{aligned} & V_\phi(K, L) \\ = \ & \frac{1}{n} \int_{S^{n-1}} \phi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u) \mathrm{d}S_K(u) \\ = \ & \frac{1}{n} \int_{S^{n-1}} \phi\left(\frac{h_{\frac{1}{2}\cdot L_1 + \frac{1}{2}\cdot L_2}(u)}{h_K(u)}\right) h_K(u) \mathrm{d}S_K(u) \\ = \ & \frac{1}{n} \int_{S^{n-1}} \phi\left(\frac{1}{2}\frac{h_{L_1}(u)}{h_K(u)} + \frac{1}{2}\frac{h_{L_2}(u)}{h_K(u)}\right) h_K(u) \mathrm{d}S_K(u) \\ \leq \ & \frac{1}{2n} \int_{S^{n-1}} \phi\left(\frac{h_{L_1}(u)}{h_K(u)}\right) h_K(u) \mathrm{d}S_K(u) \\ + \ & \frac{1}{2n} \int_{S^{n-1}} \phi\left(\frac{h_{L_2}(u)}{h_K(u)}\right) h_K(u) \mathrm{d}S_K(u) \\ = \ & \frac{1}{2} V_\phi(K, L_1) + \frac{1}{2} V_\phi(K, L_2) \\ = \ & V_\phi(K, L_1) \\ = \ & V_\phi(K, L_2), \end{aligned}$$

with equality if and only if $L_1 = L_2$. Thus

$$V_\phi(K, L) < V_\phi(K, L_1) = V_\phi(K, L_2),$$

is the contradiction that would arise if it were the case that $L_1 \neq L_2$.

The unique convex body whose existence is guaranteed by Theorem 3.1 can be denoted by $T_\phi K$ (Orlicz-Petty body of K). We use $T_\phi^* K$ to denote the polar body of $T_\phi K$. Thus, for $K \in \mathcal{K}_o^n$, the body $T_\phi K$ is defined by

$$G_\phi(K) = nV_\phi(K, T_\phi K) \text{ and } V(T_\phi^* K) = \omega_n.$$

Let

$$T^n = \{\bar{K} \in \mathcal{K}_o^n : G_\phi(K) = nV_\phi(K, \bar{K}) \text{ and } V(\bar{K}^*) = \omega_n\}.$$

Lemma 3.1. (and Definition) For $K \in \mathcal{K}_o^n$ and $\phi \in \Phi$, there exists a unique convex body $T_\phi K \in T^n$ with

$$G_\phi(K) = nV_\phi(K, T_\phi K).$$

By Lemma 3.1 and (7), we get the following integral representation of $G_\phi(K)$.

Theorem 3.2. For $K \in \mathcal{K}_o^n$ and $\phi \in \Phi$, there exists a unique convex body $T_\phi K \in T^n$ with

$$G_\phi(K) = \int_{S^{n-1}} \phi\left(\frac{h_{T_\phi K}(u)}{h_K(u)}\right) h_K(u) \mathrm{d}S(u).$$

B. Orlicz mixed geominimal surface area

We now define a new concept: the Orlicz mixed geominimal surface area, $G_\phi(K_1, \cdots, K_n)$, of $K_1, \cdots, K_n \in \mathcal{K}_o^n$ as follow:

Definition 3.1. For each $K_i \in \mathcal{K}_o^n$, there exists a unique convex body (Orlicz-Petty body of K_i) $T_\phi K_i \in T^n (i = 1, \cdots, n)$ with

$$\begin{aligned} & G_\phi(K_1, \cdots, K_n) \\ = \ & \int_{S^{n-1}} \left[\phi\left(\frac{h_{T_\phi K_1}(u)}{h_{K_1}(u)}\right) h_{K_1}(u) \phi\left(\frac{h_{T_\phi K_2}(u)}{h_{K_2}(u)}\right) h_{K_2}(u) \right. \\ & \left. \cdots \phi\left(\frac{h_{T_\phi K_n}(u)}{h_{K_n}(u)}\right) h_{K_n}(u)\right]^{\frac{1}{n}} \mathrm{d}S(u). \end{aligned}$$

Let $g_\phi(K_i, u) = \phi\left(\frac{h_{T_\phi K_i}(u)}{h_{K_i}(u)}\right) h_{K_i}(u)$, then the above formula can be expressed as follows:

$$G_\phi(K_1, \cdots, K_n) = \int_{S^{n-1}} [g_\phi(K_1, u) \cdots g_\phi(K_n, u)]^{\frac{1}{n}} \mathrm{d}S(u). \tag{11}$$

Lemma 3.2. (Hölder's integral inequality, see [36], [37]) Let $f_0, f_1, \cdots, f_k$ be Borel measurable functions on X. Suppose that $p_0, p_1, \cdots, p_k$ are nonzero real numbers with $\sum_{i=1}^k \frac{1}{p_i} = 1$. Then

$$\int_X f_0(u) f_1(u) \cdots f_k(u) \mathrm{d}u \leq \prod_{i=1}^k \left(\int_X f_0(u) f_i(u)^{p_i} \mathrm{d}u\right)^{\frac{1}{p_i}},$$

with equality if and only if either (a) there are constants $b_1, b_2, \cdots, b_k$ not all zero, such that $b_1|f_1(u)|^{p_1} = b_2|f_2(u)|^{p_2} = \cdots = b_k|f_k(u)|^{p_k}$, or (b) one of the functions is null.

The classical Alexandrov-Fenchel inequality for mixed volume (see [32], [38]) is one of important inequalities in convex geometry. It states that

$$\begin{aligned} & \prod_{i=0}^{m-1} V(K_1, \cdots, K_{n-m}, \underbrace{K_{n-i}, \cdots, K_{n-i}}_{m}) \\ & \leq V(K_1, \cdots, K_n)^m. \end{aligned}$$

We then prove the following Alexandrov-Fenchel type inequality for Orlicz mixed geominimal surface area.

Theorem 3.3. If $K_1, \cdots, K_n \in \mathcal{K}_o^n$, then for $1 \leq m < n$,

$$\begin{aligned} & G_\phi(K_1, \cdots, K_n)^m \\ \leq \ & \prod_{i=0}^{m-1} G_\phi(K_1, \cdots, K_{n-m}, \underbrace{K_{n-i}, \cdots, K_{n-i}}_{m}), \end{aligned}$$

with equality if the K_j are dilates of each other for $j = n - m + 1, \cdots, n$. If $m = 1$ equality holds trivially.

In particular, if $m = n$, then

$$G_\phi(K_1, \cdots, K_n)^n \le G_\phi(K_1) \cdots G_\phi(K_n), \qquad (12)$$

equality holds if the K_i are dilates of each other.

Proof. Let $H_0(u) = [g_\phi(K_1, u) \cdots g_\phi(K_{n-m}, u)]^{\frac{1}{n}}$ and $H_{i+1}(u) = [g_\phi(K_{n-i}, u)]^{\frac{1}{n}}$ for $i = 0, \ldots, m-1$. From (11) and Lemma 3.2, we get

$$\begin{aligned} & G_\phi(K_1, \cdots, K_n) \\ = & \int_{S^{n-1}} [g_\phi(K_1, u) \cdots g_\phi(K_n, u)]^{\frac{1}{n}} \mathrm{d}S(u) \\ = & \int_{S^{n-1}} H_0(u) H_1(u) \cdots H_m(u) \mathrm{d}S(u) \\ \le & \prod_{i=0}^{m-1} \left(\int_{S^{n-1}} H_0(u) H_{i+1}(u)^m \mathrm{d}S(u) \right)^{\frac{1}{m}} \\ = & \prod_{i=0}^{m-1} G_\phi^{\frac{1}{m}}(K_1, \ldots, K_{n-m}, \underbrace{K_{n-i}, \ldots, K_{n-i}}_{m}). \end{aligned}$$

As can be seen from Lemma 3.2, the equality of above inequality holds if and only if $H_0(u) H_{i+1}^m(u) = c_{ij}^m H_0(u) H_{j+1}^m(u)$ for some $c_{ij} > 0$ and all $0 \le i \ne j \le m-1$. This is equivalent to

$$\phi\left(\frac{h_{T_\phi K_{n-i}}(u)}{h_{K_{n-i}}(u)}\right) h_{K_{n-i}}(u) = c_{ij} \phi\left(\frac{h_{T_\phi K_{n-j}}(u)}{h_{K_{n-j}}(u)}\right) h_{K_{n-j}}(u),$$

which can observe the equality holds if K_{n-i} and K_{n-j} are dilates.

For $\phi \in \Phi$, by the definition of Orlicz mixed volume $V_\phi(K, L)$ of $K, L \in \mathcal{K}_o^n$, taking $L = B$, then we immediately get

$$V_\phi(K, B) = \frac{1}{n} \int_{S^{n-1}} \phi\left(\frac{h_B(u)}{h_K(u)}\right) h_K(u) \mathrm{d}S_K(u).$$

Now we define the Orlicz geominimal surface area $S_\phi(K)$, namely, $S_\phi(K) = nV_\phi(K, B)$. The special case of $\phi(t) = t^p$ with $p \ge 1$ is L_p-geominimal surface area. Then we can prove the analogous isoperimetric inequality for Orlicz mixed geominimal surface area.

Theorem 3.4. Let $K_i \in \mathcal{K}_o^n$, $1 \le i \le n$, then

$$\left(\frac{G_\phi(K_1, \cdots, K_n)}{G_\phi(B, \cdots, B)}\right)^n \le \frac{S_\phi(K_1) \cdots S_\phi(K_n)}{nV(B) \cdots nV(B)}. \qquad (13)$$

When ϕ is strictly convex, equality holds if and only if the K_i are ellipsoids with dilates of each other.

Proof. By inequality (10), we have $G_\phi(B) = nV(B) = n\omega_n$, then $G_\phi(B, \cdots, B) = G_\phi(B) = n\omega_n$. By inequalities (12), (10) and (9), we get

$$\begin{aligned} & \left(\frac{G_\phi(K_1, \cdots, K_n)}{G_\phi(B, \cdots, B)}\right)^n \\ \le & \frac{G_\phi(K_1) \cdots G_\phi(K_n)}{G_\phi(B) \cdots G_\phi(B)} \\ \le & \frac{nV(K_1)\phi\left(\left(\frac{\omega_n}{V(K_1)}\right)^{\frac{1}{n}}\right) \cdots nV(K_n)\phi\left(\left(\frac{\omega_n}{V(K_n)}\right)^{\frac{1}{n}}\right)}{nV(B) \cdots nV(B)} \\ \le & \frac{nV_\phi(K_1, B) \cdots nV_\phi(K_n, B)}{nV(B) \cdots nV(B)} \\ = & \frac{S_\phi(K_1) \cdots S_\phi(K_n)}{nV(B) \cdots nV(B)}. \end{aligned}$$

With the existence and uniqueness of $T_\phi K$, equality holds in (10) if and only if K is an ellipsoid. Combining with the equality condition of (9), we see that when ϕ is strictly convex, equality holds in (13) if and only if the K_i are ellipsoids with dilates of each other.

C. The *i*th Orlicz mixed geominimal surface area

In this section, we introduce the concept of ith Orlicz mixed geominimal surface area.

For $K, L \in \mathcal{K}_o^n$, and $i \in \mathbf{R}$, the ith Orlicz mixed geominimal surface area, $G_{\phi,i}(K, L)$, of K, L is defined by

$$G_{\phi,i}(K, L) = \int_{S^{n-1}} g_\phi(K, u)^{\frac{n-i}{n}} g_\phi(L, u)^{\frac{i}{n}} \mathrm{d}S(u). \qquad (14)$$

By the Lemma 3.1, we get

$$G_\phi(B) = nV_\phi(B, T_\phi B), \qquad (15)$$

since

$$G_\phi(B) = n\omega_n = nV_\phi(B, B), \qquad (16)$$

combining (15), (16) and the uniqueness of Lemma 3.1, we have

$$T_\phi B = B.$$

Let $L = B$ and write

$$G_{\phi,i}(K, B) = G_{\phi,i}(K). \qquad (17)$$

Combining (14), (17) and $h_{T_\phi B} = h_B = 1$, we have

$$G_{\phi,i}(K) = \int_{S^{n-1}} g_\phi(K, u)^{\frac{n-i}{n}} \mathrm{d}S(u).$$

By (11), (14) and (17), we easily get

$$G_{\phi,0}(K, B) = G_\phi(K), \quad G_{\phi,i}(K, K) = G_\phi(K), \qquad (18)$$

$$G_{\phi,0}(K, L) = G_\phi(K), \quad G_{\phi,n}(K, L) = G_\phi(L). \qquad (19)$$

The following Theorem deals with the cyclic inequality for the ith Orlicz mixed geominimal surface.

Theorem 3.5. Let $K, L \in \mathcal{K}_o^n$, $i, j, k \in \mathbf{R}$, and $i < j < k$, then

$$G_{\phi,i}(K, L)^{k-j} G_{\phi,k}(K, L)^{j-i} \ge G_{\phi,j}(K, L)^{k-i}, \qquad (20)$$

equality holds if K and L are dilates.

Proof. From definition (14) and Hölder's integral inequality (see [36]), we get

$$\begin{aligned} & G_{\phi,i}(K, L)^{\frac{k-j}{k-i}} G_{\phi,k}(K, L)^{\frac{j-i}{k-i}} \\ = & \left[\int_{S^{n-1}} g_\phi(K, u)^{\frac{n-i}{n}} g_\phi(L, u)^{\frac{i}{n}} \mathrm{d}S(u)\right]^{\frac{k-j}{k-i}} \\ \times & \left[\int_{S^{n-1}} g_\phi(K, u)^{\frac{n-k}{n}} g_\phi(L, u)^{\frac{k}{n}} \mathrm{d}S(u)\right]^{\frac{j-i}{k-i}} \\ = & \left\{\int_{S^{n-1}} \left[g_\phi(K, u)^{\alpha_1} g_\phi(L, u)^{\alpha_2}\right]^{\frac{k-i}{k-j}} \mathrm{d}S(u)\right\}^{\frac{k-j}{k-i}} \\ \times & \left\{\int_{S^{n-1}} \left[g_\phi(K, u)^{\beta_1} g_\phi(L, u)^{\beta_2}\right]^{\frac{k-i}{j-i}} \mathrm{d}S(u)\right\}^{\frac{j-i}{k-i}} \\ \ge & \int_{S^{n-1}} g_\phi(K, u)^{\frac{n-j}{n}} g_\phi(L, u)^{\frac{j}{n}} \mathrm{d}S(u) \\ = & G_{\phi,j}(K, L), \end{aligned}$$

where $\alpha_1 = \frac{(n-i)(k-j)}{n(k-i)}$, $\alpha_2 = \frac{i(k-j)}{n(k-i)}$, $\beta_1 = \frac{(n-k)(j-i)}{n(k-i)}$ and $\beta_2 = \frac{k(j-i)}{n(k-i)}$.

We prove inequality (20). According to the equality condition of Hölder's integral inequality, the equality in (20) holds if and only if for any $u \in S^{n-1}$,

$$\frac{g_\phi(K,u)^{\frac{n-i}{n}} g_\phi(L,u)^{\frac{i}{n}}}{g_\phi(K,u)^{\frac{n-k}{n}} g_\phi(L,u)^{\frac{k}{n}}}$$

is a constant. Namely, for any $u \in S^{n-1}$, $g_\phi(K,u)/g_\phi(L,u)$ is a constant. By the similar to the proof of Theorem 3.3, we conclude that equality in (20) holds if K and L are dilates of one another.

Taking $L = B$ in Theorem 3.5 and using (17), we immediately obtain:

Corollary 3.1. If $K, L \in \mathcal{K}_o^n$, $i, j, k \in \mathbf{R}$, and $i < j < k$, then

$$G_{\phi,i}(K)^{k-j} G_{\phi,k}(K)^{j-i} \geq G_{\phi,j}(K)^{k-i},$$

equality holds if K is a ball with centroid at the origin.

The following inequalities are the Minkowski inequalities for the ith Orlicz mixed geominimal surface area.

Theorem 3.6. For $K, L \in \mathcal{K}_o^n, i \in \mathbf{R}$, then for $i < 0$ or $i > n$,

$$G_{\phi,i}(K,L)^n \geq G_\phi(K,L)^{n-i} G_\phi(L)^i, \tag{21}$$

for $0 < i < n$,

$$G_{\phi,i}(K,L)^n \leq G_\phi(K,L)^{n-i} G_\phi(L)^i. \tag{22}$$

Equality of each inequality holds if K and L are dilates. For $i = 0$ or $i = n$, above inequalities are identical.

Proof. (i) For $i < 0$, let $(i, j, k) = (i, 0, n)$ in Theorem 3.5, we get

$$G_{\phi,i}(K,L)^n G_{\phi,n}(K,L)^{-i} \geq G_{\phi,0}(K,L)^{n-i},$$

equality holds if K and L are dilates.

From (19), we have

$$G_{\phi,i}(K,L)^n G_\phi(L)^{-i} \geq G_\phi(K)^{n-i},$$

i.e.,

$$G_{\phi,i}(K,L)^n \geq G_\phi(K)^{n-i} G_\phi(L)^i,$$

equality holds if K and L are dilates.

(ii) For $i > n$, let $(i, j, k) = (0, n, i)$ in Theorem 3.5, we get

$$G_{\phi,0}(K,L)^{i-n} G_{\phi,i}(K,L)^n \geq G_{\phi,n}(K,L)^i,$$

equality holds if K and L are dilates.

From (19), we have

$$G_\phi(K)^{i-n} G_{\phi,i}(K,L)^n \geq G_\phi(L)^i,$$

i.e.,

$$G_{\phi,i}(K,L)^n \geq G_\phi(K)^{n-i} G_\phi(L)^i,$$

equality holds if K and L are dilates.

(iii) For $0 < i < n$, let $(i, j, k) = (0, i, n)$ in Theorem 3.5, we get

$$G_{\phi,0}(K,L)^{n-i} G_{\phi,n}(K,L)^i \geq G_{\phi,i}(K,L)^n,$$

equality holds if K and L are dilates.

From (19), we have inequality (22).

(iv) For $i = 0$ (or $i = n$), by (19), inequality (21) (or (22)) is identical.

Taking $L = B$ in Theorem 3.6, using (17) and $G_\phi(B) = n\omega_n$, we immediately obtain:

Corollary 3.2. For $K, L \in \mathcal{K}_o^n$, $i \in \mathbf{R}$, then for $i < 0$ or $i > n$,

$$G_{\phi,i}(K)^n \geq (n\omega_n)^i G_\phi(K)^{n-i}, \tag{23}$$

for $0 < i < n$,

$$G_{\phi,i}(K)^n \leq (n\omega_n)^i G_\phi(K)^{n-i}. \tag{24}$$

Equality of each inequality holds if K is a ball with centroid at the origin. For $i = 0$ or $i = n$, above inequalities are identical.

Zhu, Li and Zhou [39] established the Blaschke-Santaló type inequality for L_p-geominimal surface area. It states that if $K \in \mathcal{K}_o^n$, $p \geq 1$, then

$$G_p(K) G_p(K^*) \leq (n\omega_n)^2, \tag{25}$$

with equality if and only if K is an ellipsoid.

Then, we can prove the following Blaschke-Santaló type inequality for Orlicz geominimal surface area.

Theorem 3.7. Let $K \in \mathcal{K}_c^n$ and $\phi \in \Phi$, then

$$G_\phi(K) G_\phi(K^*) \leq (n\omega_n)^2.$$

When ϕ is strictly convex, equality holds if and only if K is an elliposoid.

Proof. From the definition of Orlicz geominimal surface area (4) and Orlicz Minkowski inequality (9), we have

$$\begin{aligned} & \omega_n G_\phi(K) \phi\left(\left(\frac{V(K^*)}{V(Q^*)}\right)^{\frac{1}{n}}\right) \\ \leq \;& nV_\phi(K,Q)\phi\left(\left(\frac{V(K^*)}{V(Q^*)}\right)^{\frac{1}{n}}\right) V(Q^*) \\ \leq \;& nV_\phi(K,Q) V_\phi(Q^*, K^*). \end{aligned}$$

Since $K \in \mathcal{K}_o^n$, taking $Q = K$, and together with the equation (8) and the Blaschke-Santaló inequality (5), we get

$$\omega_n G_\phi(K) \leq nV(K)V(K^*) \leq n\omega_n^2,$$

i.e.,

$$G_\phi(K) \leq n\omega_n. \tag{26}$$

Similarly,

$$G_\phi(K^*) \leq n\omega_n. \tag{27}$$

Combining (26) and (27), we get

$$G_\phi(K) G_\phi(K^*) \leq (n\omega_n)^2. \tag{28}$$

Suppose that ϕ is strictly convex. By the equality conditions of (9) and (5), equality holds in (28) if and only if K is an ellipsoid.

When $\phi(t) = t^p$ with $p \geq 1$, the above Blaschke-Santaló type inequality (28) for Orlicz geominimal surface area reduces to the Blaschke-Santaló type inequality (25) for L_p-geominimal surface area.

Then the following similar results of the ith Orlicz mixed geominimal surface area can be established.

Theorem 3.8. If $K, L \in \mathcal{K}_c^n$, and $0 \leq i \leq n$, then

$$G_{\phi,i}(K,L) G_{\phi,i}(K^*, L^*) \leq (n\omega_n)^2.$$

For $0 < i < n$, equality holds if K and L are ellipsoids with dilates of each other. For $i = 0$ (or $i = n$), the equality of above inequality holds if K (or L) is an ellipsoid.

Proof. For $0 < i < n$, by (22) and Theorem 3.7, we get

$$\begin{aligned} & G_{\phi,i}(K,L)^n G_{\phi,i}(K^*,L^*)^n \\ \le \; & [G_\phi(K)G_\phi(K^*)]^{n-i}[G_\phi(L)G_\phi(L^*)]^i \\ = \; & (n\omega_n)^{2n}. \end{aligned}$$

That is,

$$G_{\phi,i}(K,L)G_{\phi,i}(K^*,L^*) \le (n\omega_n)^2,$$

equality holds if K and L are ellipsoids with dilates of each other.

Based on (19) and (22), we can see Theorem 3.8 is obviously for $i = 0$ (or $i = n$), and equality holds if K (or L) is an ellipsoid.

Theorem 3.9. If $K, L \in K_c^n$, then
(i) $0 \le i \le n$,

$$G_{\phi,i}(K)G_{\phi,i}(K^*) \le (n\omega_n)^2,$$

equality holds if K is a ball.
(ii) $i \ge n$,

$$G_{\phi,i}(K)G_{\phi,i}(K^*) \ge (n\omega_n)^2,$$

equality holds if K is a ball.

Proof. (i) By Theorem 3.8, letting $L = B$, we get

$$G_{\phi,i}(K,B)G_{\phi,i}(K^*,B^*) \le (n\omega_n)^2.$$

By (17), we get

$$G_{\phi,i}(K)G_{\phi,i}(K^*) \le (n\omega_n)^2,$$

equality holds if K is a ball.

(ii) For all $i \ge n$, by inequality (23), we have

$$\left(\frac{G_{\phi,i}(K)}{G_{\phi,i}(B)}\right)^n \ge \left(\frac{G_\phi(K)}{G_\phi(B)}\right)^{n-i}.$$

From Theorem 3.7 and $G_\phi(B) = n\omega_n$, we obtain

$$\left(\frac{G_{\phi,i}(K)G_{\phi,i}(K^*)}{G_{\phi,i}(B)^2}\right)^n \ge \left(\frac{G_\phi(K)G_\phi(K^*)}{G_\phi(B)^2}\right)^{n-i} \ge 1.$$

That is

$$G_{\phi,i}(K)G_{\phi,i}(K^*) \ge (n\omega_n)^2,$$

equality holds if K is a ball.

Now we establish the generalized isoperimetric inequalities for $G_{\phi,i}(K)$.

Theorem 3.10. If $K, L \in \mathcal{K}_c^n$, then
(i) $0 \le i \le n$,

$$\frac{G_{\phi,i}(K)}{G_{\phi,i}(B)} \le \left(\frac{V(K)}{V(B)}\phi\left(\left(\frac{\omega_n}{V(K)}\right)^{\frac{1}{n}}\right)\right)^{\frac{n-i}{n}},$$

equality holds if K is a ball.
(ii) $i \ge n$,

$$\frac{G_{\phi,i}(K)}{G_{\phi,i}(B)} \ge \left(\frac{V(K)}{V(B)}\phi\left(\left(\frac{\omega_n}{V(K)}\right)^{\frac{1}{n}}\right)\right)^{\frac{n-i}{n}},$$

equality holds if K is a ball.

Proof. (i) For $i = 0$, by (18) and (10), we have

$$\begin{aligned} \frac{G_\phi(K)}{G_\phi(B)} &\le \frac{nV(K)\phi\left(\left(\frac{\omega_n}{V(K)}\right)^{\frac{1}{n}}\right)}{nV(B)\phi\left(\left(\frac{\omega_n}{V(B)}\right)^{\frac{1}{n}}\right)} \\ &= \frac{V(K)}{V(B)}\phi\left(\left(\frac{\omega_n}{V(K)}\right)^{\frac{1}{n}}\right), \end{aligned}$$

equality holds if K is a ball.

For $i = n$, by (17), (18), (19), the equality holds trivially.

For $0 < i < n$, by (24), we get

$$\begin{aligned} \left(\frac{G_{\phi,i}(K)}{G_{\phi,i}(B)}\right)^n &\le \left(\frac{G_\phi(K)}{G_\phi(B)}\right)^{n-i} \\ &\le \left(\frac{V(K)}{V(B)}\phi\left(\left(\frac{\omega_n}{V(K)}\right)^{\frac{1}{n}}\right)\right)^{n-i}, \end{aligned}$$

equality holds if K is a ball.

(ii) For $i = n$, by (17), (18), (19), the equality holds trivially. For $i > n$, by (23), we get

$$\begin{aligned} \left(\frac{G_{\phi,i}(K)}{G_{\phi,i}(B)}\right)^n &\ge \left(\frac{G_\phi(K)}{G_\phi(B)}\right)^{n-i} \\ &\ge \left(\frac{V(K)}{V(B)}\phi\left(\left(\frac{\omega_n}{V(K)}\right)^{\frac{1}{n}}\right)\right)^{n-i}. \end{aligned}$$

We complete the proof.

ACKNOWLEDGMENT

The referee of this paper proposed many very valuable comments and suggestions to improve the accuracy and readability of the original manuscript. We would like to express our most sincere thanks to the anonymous referee.

REFERENCES

[1] C. M. Petty, "Geominimal surface area," *Geometry Dedicata*, vol. 3, no. 1, pp. 77-97, 1974.

[2] E. Lutwak, "Volume of mixed bodies," *Transactions of the American Mathematical Society*, Vol. 294, no. 2, pp. 487-500, 1986.

[3] W. J. Firey, "p-means of convex bodies," *Mathematica Scandinavica*, vol. 10, no. 1, pp. 17-24, 1962.

[4] E. Lutwak, "The Brunn-Minkowski-Firey theory I: Mixed volumes and the Minkowski problem," *Journal Difierential Geometry*, vol. 38, no. 1, pp. 131-150, 1993.

[5] E. Lutwak, "The Brunn-Minkowski-Firey theory II: Affine and geominimal surface areas," *Advances in Mathematics*, vol. 118, no. 2, pp. 244-294, 1996.

[6] T. Y. Ma and W. D. Wang, "Some Inequalities for Generalized -mixed Affine Surface Areas," *IAENG International Journal of Applied Mathematics*, vol. 45, no. 4, pp. 321-326, 2015.

[7] T. Y. Ma and Y. B. Feng, "Some Inequalities for p-Geominimal Surface Area and Related Results," *IAENG International Journal of Applied Mathematics*, vol. 46, no. 1, pp. 92-96, 2016.

[8] E. Lutwak, D. Yang and G. Zhang, "L_p affine isoperimetric inequalities," *Journal Difierential Geometry*, vol. 56, no. 1, pp. 111-132, 2000.

[9] W. D. Wang and Y. B. Feng, "A general L_p-version of Prttys affine projection inequality," *Taiwanese Journal of Mathematics*, vol. 17, no. 2, pp. 517-528, 2013.

[10] W. D. Wang and G. S. Leng, "L_p-mixed affine surface areas," *Journal of Mathematical Analysis and Applications*, vol. 335, no. 1, pp. 341-354, 2007.

[11] E. Werner, "On L_p-affine surface areas," *Indiana University Mathematics Journal*, vol. 56, no. 5, pp. 2305-2324, 2007.

[12] E. Werner and D. Ye, "New L_p-affine isoperimetric inequalities," *Advances in Mathematics*, vol. 218, no. 6, pp. 762-780, 2008.

[13] E. Werner and D. Ye, "Inequalities for mixed p-affine surface area," *Mathematische Annalen*, vol. 347, no.3, pp. 703-737, 2010.

[14] J. Yuan, S. J. Lv and G. S. Leng, "The p-affine surface area," *Mathematical Inequalities Applications*, vol. 10, no. 3, pp. 693-702, 2007.
[15] D. Ye, "L_p-geominimal surface area and related inequalities," *International Mathematics Research Notices*, vol. 2015, no. 9, pp. 2465-2498, 2014.
[16] D. Ye, B. C. Zhu and J. Z. Zhou, "The mixed L_p-geominimal surface areas for multiple convex bodies," to be published.
[17] B. C. Zhu, J. Z. Zhou and W. X. Xu, "Affine isoperimetric inequalities for L_p-geominimal surface area," in *Proceedings of ICMSC 2014, Springer*, 2014.
[18] E. Lutwak, D. Yang and G. Zhang, "Orlicz projection bodies," *Advances in Mathematics*, vol. 223, no. 1, pp. 220-242, 2010.
[19] E. Lutwak, D. Yang and G. Zhang, "Orlicz centroid bodies," *Journal Difierential Geometry*, vol. 84, no. 2, pp. 365-387, 2010.
[20] R. J. Gardner, D. Hug and W. Weil, "The Orlicz-Brunn-Minkowski theory: a general framework, additions, and inequalities," *Journal of Differential Geometry*, vol. 97, no. 3, pp. 427-476, 2014.
[21] F. W. Chen, J. Z. Zhou and C. L. Yang, "On the reverse Orlicz Busemann-Petty centroid inequality," *Advances in Applied Mathematics*, vol. 47, no. 4, pp. 820-828, 2011.
[22] C. Haberl, E. Lutwak, D. Yang and G. Zhang, "The even Orlicz Minkowski problem," *Advances in Mathematics*, Vol. 224, no. 6, pp. 2485-2510, 2010.
[23] Q. Z. Huang and B. W. He, "On the Orlicz Minkowski problem for polytopes," *Discrete and Computational Geometry*, Vol. 48, no. 2, pp. 281-297, 2012.
[24] A. J. Li and G. S. Leng, "A new proof of the Orlicz Busemann-Petty centroid inequality," *Proceedings of the American Mathematical Society*, Vol. 139, no. 4, pp. 1473-1481, 2011.
[25] D. M. Xi, H. L. Jin and G. S. Leng, "The Orlicz Brunn-Minkowski Theory," *Advances in Mathematics*, vol. 260, no. 1, pp. 350-374, 2014.
[26] D. Ye, "New Orlicz Affine Isoperimetric Inequalities," *Journal of Mathematical Analysis and Applications*, vol. 427, no. 2, pp. 905-929, 2015.
[27] G. X. Zhu, "The Orlicz centroid inequality for star bodies," *Advances in Applied Mathematics*, vol. 48, no. 2, pp. 432-445, 2012.
[28] D. Zou, G. Xiong, "Orlicz-Legendre ellipsoids," *The Journal of Geometric Analysis*, pp. 1-29, 2015.
[29] S. F. Yuan, H. L. Jin and G. S. Leng, "Orlicz geominimal surface areas," *Mathematical Inequalities and Applications*, vol. 18, no. 1, pp. 353-362, 2015.
[30] B. C. Zhu, J. Z. Zhou and W. X Xu, "L_p-mixed geominimal surface area," *Journal of Mathematical Analysis and Applications*, vol. 422, no. 2, pp. 1247-1263, 2015.
[31] R. J. Gardner, *"Geometric tomography, second edition"*, Cambridge University Press, New York, 2006.
[32] R. Schneider, *"Convex bodies: The Brunn-Minkowski theory"*, Cambridge University Press, Cambridge, 1993.
[33] M. Meyer and A. Pajor, "On Santalós inequality," in *"Gemetric aspects of functional analysis (Joram Lindenstrauss and Vitali D. Milman, Eds.), Springer Lecture Notes in Mathematics*, vol. 1376, pp. 261-263, Springer-Verlag, New York/Berlin, 1989.
[34] M. Meyer and A. Pajor, "On the Blaschke-Santalós inequality," *Archiv Der Mathematik*, vol. 55, no. 1, pp. 82-93, 1990.
[35] R. J. Gardner, "The Brunn-Minkowski inequality," *Bulletin of the American Mathematical Society*, vol. 39, no. 3, pp. 355-405, 2002.
[36] H. Federer, *"Geometric Measure Theory"*, Springer-Verlag, New York, 1969.
[37] G. H. Hardy, J. E. Littlewood and G. Pólya, *"inequalities"*, Cambridge University Press, London, 1934.
[38] M. Ludwing, C. Schütt and E. Werner, "Approximation of the Euclidean ball by polytopes," *Studia Mathematica*, Vol. 173, no. 1, pp. 1-18, 2006.
[39] B. C. Zhu, N. Li and J. Z. Zhou, "Isoperimetric inequalities for L_p-geominimal surface area," *Glasgow Mathematical Journal*, vol. 53, no. 3, pp. 717-726, 2011.