Hyers-Ulam Stability of a Class Fractional Boundary Value Problems with Right and Left Fractional Derivatives

Xiangkui Zhao, Bo Sun, Weigao Ge

Abstract—**In this paper, we consider the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of a class differential equation boundary value problems with right and left fractional derivatives. The research results mean that the type of boundary value problem has at least one exact solution and its approximate solutions are stable. The research method is a generalized fixed point theorem. Two examples are given to illustrate our theoretical results.**

Index Terms—**Generalized fixed point theorem, Right and left fractional derivatives, Hyers-Ulam stability, Hyers-Ulam-Rassias stability.**

I. Introduction

The Hyers-Ulam stability was initiated for functional equations by Hyers and Ulam [1-2]. It is a useful tool to consider the existence of exact solutions and the errors of approximate solutions. Furthermore, M. Rassias succeeded in generalizing the Hyers-Ulam stability in 1978, the generalized Hyers-Ulam stability was called Hyers-Ulam-Rassias stability [3]. Moreover, recently, lots of interesting results have been published concerning the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of functional equations [4-9].

The study on Hyers-Ulam stability of differential equations was announced by Obloza in 1993 [10-11]. Since then, many scholars focused on the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of differential equations, see [12-23] and the references therein.

It should be pointed out that, the Hyers-Ulam stability is different from Lyapunov stability of differential equations. The Hyers-Ulam stability means that a differential equation has a close exact solution generated by the approximate solution and the error of the approximate solution can be estimated. However, the Lyapunov stability means that if a solution which starts out near an equilibrium point, it will stay near the equilibrium point forever. The Hyers-Ulam stability can be widely used to the fields where it is difficult to find the exact solutions, for example, numerical analysis, optimization, biology and economics and so on [24].

To the best knowledge of the authors, there is no work in the literature which discusses the Hyers-Ulam stability of differential equations with both right and left fractional derivatives, while it is very difficult to solve the type of equation. Fractional differential equations with right and left fractional derivatives arose naturally as the Euler-Lagrange equation in fractional derivative variational principles, and are also applicable to many fields, such as the optimal control theory for functionals involving fractional derivatives [25-26] and the references therein. Furthermore, there are many papers concerning the type of equation, see [27-31] and the references therein. Hence it is interesting and important to discuss the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the following problem

$$\begin{cases} -y'' + q(x)y + \mu D_{0+}^{\alpha} D_{1-}^{\alpha} y = F(y), x \in (0,1), \\ y(0) = y(1) = 0, \end{cases} \tag{1}$$

where D_{1-}^{α} and D_{0+}^{α} are respectively the left and right Riemann-Liouville fractional derivatives of order α, $0 < \alpha < \frac{1}{2}$, μ is a real constant.

Assume that for any function $y : [0,1] \to \mathbb{R}$ with $y(0) = y(1) = 0$ satisfying the differential inequality

$$| - y'' + q(x)y + \mu D_{0+}^{\alpha} D_{1-}^{\alpha} y - F(y)| \leq \varepsilon$$

for all $x \in (0,1)$ and some $\varepsilon > 0$, there exists a solution y_0 of (1) such that $|y(x) - y_0(x)| \leq K(\varepsilon)$ for any $x \in (0,1)$, where $K(\varepsilon)$ is an expression of ε only. Then we say that the problem (1) is Hyers-Ulam stable on $[0,1]$.

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$$| - y'' + q(x)y + \mu D_{0+}^{\alpha} D_{1-}^{\alpha} y - F(y)| \leq \varphi(x)$$

for all $x \in (0,1)$, there exists a solution y_0 of (1) such that $|y(x) - y_0(x)| \leq \phi(x)$ for any $x \in (0,1)$, where $\varphi, \phi : [0,1] \to R$ are functions not depending on y and y_0 explicitly. Then we say that the problem (1) is Hyers-Ulam-Rassias stable on $[0,1]$.

The paper is organized as following. A review of some background material is presented in Section 2. The Hyers-Ulam-Rassias stability of (1) is considered in Section 3. The Hyers-Ulam stability of (1) is considered in Section 4. We end the paper with two examples of application in Section 5.

II. Preliminary

We list here the definitions and lemmas to state the main results of this paper.

Definition 1. The left Riemann-Liouville fractional derivative of order $\alpha > 0$ for a function $y : (0,1) \to \mathbb{R}$ is defined as

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^t (t-s)^{m-\alpha-1} y(s) ds,$$

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where $m-1 \le \alpha < m$.

Definition 2. The right Riemann-Liouville fractional derivative of order $\alpha > 0$ for a function $y : (0,1) \to \mathbb{R}$ is defined as

$$D_{1-}^{\alpha} y(t) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_t^1 (t-s)^{m-\alpha-1} y(s) ds,$$

where $m-1 \le \alpha < m$.

Definition 3. [20] For a nonempty set X, a function $d : X \times X \to [0, +\infty]$ is called a generalized metric on X if and only if d satisfies
(M1) $d(x,y) = 0$ if and only if $x = y$;
(M2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(M3) $d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in X$.

The only difference between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity.

Theorem 1. [32] *Let (X,d) be a generalized complete metric space. Assume that $\Lambda : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(\Lambda^{k+1} f, \Lambda^k f) < +\infty$ for some $f \in X$, then the following is true:*
(S1) *The sequence $\{\Lambda^n f\}$ converges to a fixed point f^* of Λ;*
(S2) *f^* is the unique fixed point of Λ in $X^* = \{y \in X | d(\Lambda^k f, y) < +\infty\}$;*
(S3) *If $y \in X^*$, then $d(y, f^*) \le \frac{1}{1-L} d(\Lambda y, y)$.*

Theorem 2. *Assume that $q \in C([0,1], (0,\infty))$, then*
(i) *the boundary value problem*

$$\begin{cases} -y'' + q(x)y = 0, x \in (0,1), \\ y(0) = 0, \; y'(0) = 1 \end{cases} \tag{2}$$

has a unique solution, furthermore, the unique solution is strictly increasing on $(0,1)$;
(ii) *the boundary value problem*

$$\begin{cases} -y'' + q(x)y = 0, x \in (0,1), \\ y(0) = 1, \; y(1) = 0 \end{cases} \tag{3}$$

has a unique solution, furthermore, the unique solution is strictly decreasing on $(0,1)$.

Proof: From Lemma 2.1 and Lemma 2.2 of reference [33], we see that the boundary value problem (3) has a unique solution, furthermore, the unique solution is strictly decreasing on $(0,1)$, and the boundary value problem

$$\begin{cases} -y'' + q(x)y = 0, x \in (0,1), \\ y(0) = 0, \; y(1) = 1 \end{cases} \tag{4}$$

has a unique solution, furthermore, the unique solution is strictly increasing on $(0,1)$. Let ϕ_1 be the unique solution of (4), then $u = \frac{1}{\phi_1'(0)} \phi_1$ is the unique solution of boundary value problem (2) with u strictly increasing on $(0,1)$. ■

Let

$$Q_1 = \max_{0 \le x \le 1} q(x), \; Q_2 = \int_0^1 q(x) dx,$$

$$N_1(x) = \frac{Q_1 x^{1-\alpha}}{(1-Q_2)(1-\alpha)^2} + \frac{x^{1-2\alpha}}{(1-Q_2)(1-2\alpha)},$$

$$N_2(x) = \frac{Q_1 x^{1-\alpha}}{(1-\alpha)^2(1-Q_2)} + \frac{(1+Q_2) x^{1-2\alpha}}{(1-2\alpha)(1-Q_2)},$$

$$N_3(x,t) = \frac{x^{1-\alpha}}{(1-\alpha)(1-Q_2)t^{\alpha}},$$

$$N_4(x,t) = \frac{(1+Q_2)(1-x^{1-\alpha})}{(1-Q_2)(1-\alpha)t^{\alpha}},$$

$$N_5(t) = \frac{t^{1-2\alpha}}{(1-2\alpha)(1-Q_2)},$$

$$N_6(t) = \frac{Q_1 t^{1-\alpha}}{(1-\alpha)^2(1-Q_2)},$$

$$M(x,t) = N_1(x) + N_2(x) + N_3(x,t) + N_4(x,t) + N_5(t) + N_6(t),$$

$$N(t) = \max_{x \in [0,1]} M(x,t).$$

Theorem 3. *Assume that $q \in C([0,1], (0,\infty))$ and $\int_0^1 q(x) dx < 1$, u is the unique solution of (2), v is the unique solution of (3), then for $x \in [0,1]$, the following is true:*
(i) $0 = u(0) \le u(x) \le u(1) \le \frac{1}{1-Q_2}$,
$1 = u'(0) \le u'(x) \le u'(1) \le \frac{1}{1-Q_2}$;
(ii) $0 = v(1) \le v(x) \le v(0) = 1$,
$-(1+Q_2) \le v'(0) \le v'(x) \le v'(1) \le 0$.

Proof: By Theorem 2, we know

$$0 = u(0) \le u(x) \le u(1),$$

$$0 = v(1) \le v(x) \le v(0) = 1.$$

Together with

$$u''(x) = q(x)u(x) \ge 0,$$

$$v''(x) = q(x)v(x) \ge 0,$$

we have $u'(0) \le u'(x) \le u'(1)$, $v'(0) \le v'(x) \le v'(1)$. By computing, we have

$$u'(x) = \int_0^x q(s)u(s) ds + 1,$$

$$u(x) = \int_0^x (\int_0^t q(s)u(s) ds) dt + x.$$

Hence

$$u(1) \le u(1) \int_0^1 q(x) dx + 1,$$

$$u'(1) \le u(1) \int_0^1 q(s) ds + 1,$$

$$u(1) \le \frac{1}{1 - \int_0^1 q(x) dx} = \frac{1}{1-Q_2},$$

$$u'(1) \le \frac{\int_0^1 q(x) dx}{1 - \int_0^1 q(x) dx} + 1 = \frac{1}{1-Q_2}.$$

Similarly, we have

$$v'(x) = \int_0^x q(s)v(s) ds - 1 - \int_0^1 (\int_0^t q(s)v(s) ds) dt,$$

$$v(x) = \int_0^x (\int_0^t q(s)v(s) ds) dt - x - x \int_0^1 (\int_0^t q(s)v(s) ds) dt + 1.$$

Hence

$$v'(0) = -1 - \int_0^1 (\int_0^t q(s)v(s) ds) dt$$

$$\geq -1 - \int_0^1 q(x)v(x)dx = -(1+Q_2),$$

$$v'(1) = \int_0^1 q(s)v(s)ds - 1 - \int_0^1 (\int_0^t q(s)v(s)ds)dt \leq 0.$$

■

Theorem 4. *Assume that* $q \in C([0,1],(0,\infty))$, $h \in L^1(0,1)$, *then the boundary value problem*

$$\begin{cases} -y'' + q(x)y + \mu D_{0+}^{\alpha} D_{1-}^{\alpha} y = h(x), x \in (0,1), \\ y(0) = y(1) = 0 \end{cases} \tag{5}$$

can be rewritten as the integral equation

$$y(x) = \int_0^1 k(x,t)y(t)dt + \int_0^1 G(x,t)h(t)dt,$$

where u *is the unique solution of* (2), v *is the unique solution of* (3), $\Gamma_\alpha = \frac{1}{\Gamma(1-\alpha)}$,

$$\begin{aligned} k(x,t) = \mu\Gamma_\alpha^2 \big[& k_1(x,t) - k_2(x,t) + \textstyle\int_0^x \frac{u'(s)v(x)}{t^\alpha s^\alpha} ds \\ & + \textstyle\int_x^1 \frac{u(x)v'(s)}{t^\alpha s^\alpha} ds - \int_0^t \big(\frac{u(x)v'(1)}{(1-s)^\alpha (t-s)^\alpha} \\ & + u(x)(t-s)^{-\alpha} \textstyle\int_s^1 \frac{v''(m)}{(m-s)^\alpha} dm \big) ds \big], \end{aligned}$$

$$k_1(x,t) = \begin{cases} \int_0^t \frac{v(x)\int_s^x \frac{u''(m)}{(m-s)^\alpha}dm - \frac{u'(x)v(x)}{(x-s)^\alpha}}{(t-s)^\alpha} ds, \\ \qquad 0 \leq t \leq x \leq 1, \\ \int_0^x \frac{v(x)\int_s^x \frac{u''(m)}{(m-s)^\alpha}dm - \frac{u'(x)v(x)}{(x-s)^\alpha}}{(t-s)^\alpha} ds, \\ \qquad 0 \leq x \leq t \leq 1, \end{cases}$$

$$k_2(x,t) = \begin{cases} \int_0^t \frac{u(x)\int_s^x \frac{v''(m)}{(m-s)^\alpha}dm - \frac{u(x)v'(x)}{(x-s)^\alpha}}{(t-s)^\alpha} ds, \\ \qquad 0 \leq t \leq x \leq 1, \\ \int_0^x \frac{u(x)\int_s^x \frac{v''(m)}{(m-s)^\alpha}dm - \frac{u(x)v'(x)}{(x-s)^\alpha}}{(t-s)^\alpha} ds, \\ \qquad 0 \leq x \leq t \leq 1, \end{cases}$$

$$G(x,t) = \begin{cases} u(t)v(x), & 0 \leq t \leq x \leq 1, \\ u(x)v(t), & 0 \leq x \leq t \leq 1. \end{cases}$$

Proof: By Theorem 2, we know the boundary value problem (3) has a unique solution. Hence the homogeneous boundary value problem

$$\begin{cases} -y'' + q(x)y = 0, x \in (0,1), \\ y(1) = 0, \ y(1) = 0 \end{cases} \tag{6}$$

has only the trivial solution $y(x) \equiv 0$. Hence 0 is not an eigenvalue of

$$\begin{cases} -y'' + q(x)y = \lambda y, x \in (0,1), \\ y(1) = 0, \ y(1) = 0. \end{cases} \tag{7}$$

It follows from

$$u(0) = 0, u'(0) = 1, v(0) = 1, v(1) = 0, \tag{8}$$

that

$$u(0)v'(0) - u'(0)v(0) = -1.$$

Together with

$$\begin{aligned} & (u'(x)v(x) - u(x)v'(x))' \\ & \equiv u''(x)v(x) - u(x)v''(x) \\ & \equiv q(x)u(x)v(x) - q(x)u(x)v(x) \\ & \equiv 0 \end{aligned}$$

for $x \in [0,1]$, we get

$$u'(x)v(x) - u(x)v'(x) \equiv -1, \ x \in [0,1]. \tag{9}$$

By (6), (7) and Lemma 3.6 in reference [28], we get that y is the solution of (5) if and only if

$$y(x) = \int_0^1 G(x,t)(h(t) - \mu D_{0+}^\alpha D_{1-}^\alpha y(t))dt.$$

Similar to Lemma 3.6 in reference [28], we obtain that

$$-\mu \int_0^1 G(x,t)(D_{0+}^\alpha D_{1-}^\alpha y)(t)dt = \int_0^1 k(x,t)y(t)dt.$$

Therefore y is the solution of (5) if and only if

$$y = \int_0^1 k(x,t)y(t)dt + \int_0^1 G(x,t)h(t)dt.$$

■

Remark 1. *For* $x,t \in [0,1]$, *we have*

$$\begin{aligned} 0 \leq G(x,t) &= \begin{cases} u(t)v(x), & 0 \leq t \leq x \leq 1 \\ u(x)v(t), & 0 \leq x \leq t \leq 1 \end{cases} \\ &\leq \begin{cases} u(t)v(t), & 0 \leq t \leq x \leq 1 \\ u(t)v(t), & 0 \leq x \leq t \leq 1 \end{cases} \\ &= G(t,t). \end{aligned}$$

Theorem 5. *Assume that* $q \in C([0,1],(0,\infty))$ *and* $\int_0^1 q(x)dx < 1$, *then*

$$|k(x,t)| \leq |\mu|\Gamma_\alpha^2 M(x,t) \leq |\mu|\Gamma_\alpha^2 N(t)$$

for $x,t \in [0,1]$.

Proof: For $(x,s) \in (0,1)$, we have

$$\begin{aligned} |k_1(x,t)| &\leq \int_0^{\min\{x,t\}} \frac{\frac{Q_1}{1-Q_2}\int_s^x \frac{1}{(m-s)^\alpha}dm + \frac{1}{(1-Q_2)(x-s)^\alpha}}{(t-s)^\alpha} ds \\ &\leq N_1(x), \end{aligned}$$

$$\begin{aligned} |k_2(x,t)| &\leq \frac{1}{1-Q_2} \int_0^{\min\{x,t\}} \frac{Q_1\int_s^x \frac{1}{(m-s)^\alpha}dm + \frac{1+Q_2}{(x-s)^\alpha}}{(t-s)^\alpha} ds \\ &\leq N_2(x), \end{aligned}$$

$$\int_0^x \frac{u'(s)v(x)}{t^\alpha s^\alpha} ds \leq \frac{1}{1-Q_2} \int_0^x \frac{1}{t^\alpha s^\alpha} ds = N_3(x,t),$$

$$\int_x^1 |\frac{u(x)v'(s)}{t^\alpha s^\alpha}| ds \leq \frac{1+Q_2}{1-Q_2} \int_x^1 \frac{1}{t^\alpha s^\alpha} ds = N_4(x,t),$$

$$\begin{aligned} \int_0^t \frac{|u(x)v'(1)|}{(1-s)^\alpha (t-s)^\alpha} ds &\leq \frac{1}{1-Q_2} \int_0^t \frac{1}{(1-s)^\alpha (t-s)^\alpha} ds \\ &= N_5(t), \end{aligned}$$

$$\begin{aligned} \int_0^t (u(x)(t-s)^{-\alpha} \int_s^1 \frac{v''(m)}{(m-s)^\alpha} dm) ds &\leq \frac{Q_1 t^{1-\alpha}}{(1-\alpha)^2(1-Q_2)} \\ &= N_6(t). \end{aligned}$$

Hence

$$|k(x,t)| \leq |\mu|\Gamma_\alpha^2 M(x,t) \leq |\mu|\Gamma_\alpha^2 N(t).$$

■

III. Hyers-Ulam-Rassias stability

In this section, we consider the Hyers-Ulam-Rassias stability of the problem (1).

Theorem 6. *For positive constants K_1, K_2, L with*

$$K_1L + |\mu|\Gamma_\alpha^2 K_2 < 1,$$

assume that $q \in C([0,1],(0,\infty))$ and $\int_0^1 q(x)dx < 1$. Suppose that $F : \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies the standard Lipschits condition

$$|F(y_1) - F(y_2)| \le L|y_1 - y_2| \tag{10}$$

for all $y_1, y_2 \in \mathbb{R}$. If a second continuously differential function $y : I \to \mathbb{R}$ satisfies

$$\begin{cases} |-y'' + q(x)y + \mu D_{0+}^\alpha D_{1-}^\alpha y - F(y)| \le \varphi(x), \\ \qquad\qquad\qquad\qquad\qquad x \in (0,1), \\ y(0) = y(1) = 0, \end{cases} \tag{11}$$

where $\varphi : (0,1) \to (0,\infty)$ is a continuous function with

$$\int_0^1 G(x,t)\varphi(t)dt \le K_1\varphi(x), \tag{12}$$

$$\int_0^1 M(x,t)\varphi(t)dt \le K_2\varphi(x) \tag{13}$$

for all $x \in (0,1)$. Then there exists a unique continuous function $y_0 : (0,1) \to \mathbb{R}$ such that

$$y_0 = \int_0^1 k(x,t)y_0(t)dt + \int_0^1 G(x,t)F(y_0(t))dt \tag{14}$$

and

$$|y(x) - y_0(x)| \le \frac{K_1\varphi(x)}{1 - K_1L - \mu\Gamma_\alpha^2 K_2} \tag{15}$$

for all $x \in [0,1]$.

Proof: Let us define a set X of all continuous functions $f : (0,1) \to \mathbb{R}$ by

$$X = \{f : (0,1) \to \mathbb{R} | f \ is \ continuous\}$$

with a generalized complete metric (Similar to the Theorem 3.1 of [20])

$$d(f,g) = inf\{C \in [0,+\infty] \big| \max\{|f(x) - g(x)|\} \le C\varphi(x) \text{ for all } x \in (0,1)\}. \tag{16}$$

By Theorem 3.1 of reference [20], we get that $C(X,d)$ is complete. Now define an operator Λ on X by

$$(\Lambda f)(x) = \int_0^1 k(x,t)f(t)dt + \int_0^1 G(x,t)F(f(t))dt. \tag{17}$$

We can obtain that $\Lambda f \in X$ by the Fundamental Theorem of Calculus, since F, f are continuous functions. Furthermore, we can get the fixed point of operator Λ is the solution of equation (1) by Theorem 4. Now we prove that Λ is a strictly contractive operator on X. For any $f, g \in X$, let $C_{fg} \in [0,\infty]$ be an arbitrary constant with $d(f,g) \le C_{fg}$, it follows from (16) that

$$|f(x) - g(x)| \le C_{fg}\varphi(x)$$

for any $x \in (0,1)$. Therefore, by (10), (12), (15), (16), we obtain that

$$\begin{aligned} &|(\Lambda f)(x) - (\Lambda g)(x)| \le \textstyle\int_0^1 |k(x,t)(f(t) - g(t))|dt \\ &\quad + \textstyle\int_0^1 G(x,t)|F(f(t)) - F(g(t))|dt \\ &\le |\mu|\Gamma_\alpha^2 C_{f,g} \textstyle\int_0^1 M(x,t)\varphi(t)dt \\ &\quad + LC_{f,g}\textstyle\int_0^1 G(x,t)\varphi(t)dt \\ &\le C_{f,g}|\mu|\Gamma_\alpha^2 K_2\varphi(x) + LC_{f,g}K_1\varphi(x) \\ &< C_{f,g}\varphi(x) \end{aligned}$$

for all $x \in (0,1)$. We can conclude that $d(\Lambda f, \Lambda g) < d(f,g)$ for any $f, g \in X$.

Furthermore, it follows from (11) that

$$-\varphi(x) \le -y'' + q(x)y + \mu D_{0+}^\alpha D_{1-}^\alpha y - F(y) \le \varphi(x)$$

for all $x \in (0,1)$. Then by $G(x,t) \ge 0$, for $x, t \in [0,1]$, we have

$$\begin{aligned} &-\int_0^1 G(x,t)\varphi(t)dt \\ &\le \int_0^1 G(x,t)(-y'' + q(x)y + \mu D_{0+}^\alpha D_{1-}^\alpha y - F(y))dt \\ &\le \int_0^1 G(x,t)\varphi(t)dt. \end{aligned}$$

By Theorem 4, we see

$$-\mu\int_0^1 G(x,t)D_{0+}^\alpha D_{1-}^\alpha y(t)dt = \int_0^1 K(x,t)y(t)dt.$$

Hence

$$\begin{aligned} &-\int_0^1 G(x,t)\varphi(t)dt \\ &\le y - \int_0^1 K(x,t)y(t)dt - \int_0^1 G(x,t)F(y(t))dt \\ &\le \int_0^1 G(x,t)\varphi(t)dt, \end{aligned}$$

that is

$$-\int_0^1 G(x,t)\varphi(t)dt \le y(x) - (\Lambda y)(x) \le \int_0^1 G(x,t)\varphi(t)dt.$$

Hence

$$|(\Lambda y)(x) - y(x)| \le \int_0^1 G(x,t)\varphi(t)dt \le K_1\varphi(x)$$

for all $x \in (0,1)$. Hence $d(\Lambda y), y) \le K_1 < \infty$. By using Theorem 1, we obtain that there exists a unique function $y_0 : (0,1) \to \mathbb{R}$ such that $\Lambda^n y \to y_0$ and $\Lambda y_0 = y_0$. That is, y_0 satisfies (14) for all $x \in (0,1)$. Next we define that

$$X^* = \{f \in X | \ d(y,f) < \infty\}.$$

Therefore, by (S2) of Lemma 1, y_0 is the unique continuous function satisfying equation (14) in X^*. Now we prove that y_0 is the unique fixed point of Λ in X. If $y_1 \in X$ is another fixed point in X, we see

$$|y_0(x) - y_1(x)| = |(\Lambda y_0)(x) - (\Lambda y_1)(x)| < C_{y_0,y_1}\varphi(x),$$

by Λ is a strictly contractive operator on X. Which leads to a contradiction. Therefore y_0 is the unique fixed point of Λ in X. This means that y_0 is the unique solution of (1). Finally we can get that

$$d(y,y_0) \le \frac{d(\Lambda y, y)}{1 - K_1L - |\mu|\Gamma_\alpha^2 K_2} \le \frac{K_1}{1 - K_1L - |\mu|\Gamma_\alpha^2 K_2},$$

that is

$$|y(x) - y_0(x)| \leq \frac{K_1\varphi(x)}{1 - K_1L - |\mu|\Gamma_\alpha^2 K_2}$$

for all $x \in [0,1]$, by (S3) of Lemma 1. Hence inequality (15) holds true for all $x \in [0,1]$. ■

IV. Hyers-Ulam stability

In this section, we consider the Hyers-Ulam stability of the problem (1).

Theorem 7. *For positive constant L with*

$$|\mu|\Gamma_\alpha^2 \int_0^1 N(t)dt + L\int_0^1 G(t,t)dt < 1,$$

assume that $q \in C([0,1],(0,\infty))$ and $\int_0^1 q(x)dx < 1$. Suppose that $F : \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies the standard Lipschits condition

$$|F(y_1) - F(y_2)| \leq L|y_1 - y_2| \tag{18}$$

for all $y_1, y_2 \in \mathbb{R}$. If a second continuously differential function $y : I \to \mathbb{R}$ satisfies

$$\begin{cases} |-y'' + q(x)y + \mu D_{0+}^\alpha D_{1-}^\alpha y - F(y)| \leq \varepsilon, x \in (0,1), \\ y(0) = y(1) = 0, \end{cases} \tag{19}$$

where $\varepsilon \geq 0$. Then there exists a unique continuous function $y_0 : (0,1) \to \mathbb{R}$ such that

$$y_0 = \int_0^1 k(x,t)y_0(t)dt + \int_0^1 G(x,t)F(y_0(t))dt \tag{20}$$

and

$$|y(x) - y_0(x)| \leq \frac{\varepsilon\int_0^1 G(t,t)dt}{1 - \mu\Gamma_\alpha^2\int_0^1 N(t)dt - L\int_0^1 G(t,t)dt} \tag{21}$$

for all $x \in [0,1]$.

Proof: Let us define a set X of all continuous functions $f : (0,1) \to \mathbb{R}$ by

$$X = \{f : (0,1) \to \mathbb{R} | f \ is \ continuous\}$$

with a generalized complete metric (Similar to the Theorem 4.1 of reference [20])

$$d(f,g) = inf\{C \in [0,+\infty] \big| \max\{|f(x) - g(x)|\} \leq C \text{ for all } x \in (0,1)\}. \tag{22}$$

By Theorem 4.1 of reference [20], we get that $C(X,d)$ is complete. Consider operator Λ on X

$$(\Lambda f)(x) = \int_0^1 k(x,t)f(t)dt + \int_0^1 G(x,t)F(f(t))dt.$$

Analogously to the proof of Theorem 6, we can obtain that $\Lambda f \in X$ and the fixed point of operator Λ is the solution of problem (1).

Now we prove that Λ is a strictly contractive operator on X. For any $f,g \in X$, let $C_{fg} \in [0,\infty]$ be an arbitrary constant with $d(f,g) \leq C_{fg}$, it follows from (22) that

$$|f(t) - g(t)| \leq C_{fg}$$

for any $t \in (0,1)$. Therefore, by (18), (20), (22), we obtain that

$$\begin{aligned} |(\Lambda f)(x) - (\Lambda g)(x)| &\leq \textstyle\int_0^1 |k(x,t)(f(t) - g(t))|dt \\ &\quad + \textstyle\int_0^1 G(x,t)|F(f(t)) - F(g(t))|dt \\ &\leq C_{f,g}|\mu|\Gamma_\alpha^2 \textstyle\int_0^1 N(t)dt \\ &\quad + LC_{f,g}\textstyle\int_0^1 G(t,t)dt \\ &< C_{f,g} \end{aligned}$$

for all $x \in (0,1)$. We can conclude that $d(\Lambda f, \Lambda g) < d(f,g)$ for any $f,g \in X$.

Similar to the proof of Theorem 6, we can show that for any function y which satisfies (19), we have

$$-\varepsilon \leq -y'' + q(x)y + \mu D_{0+}^\alpha D_{1-}^\alpha y - F(y) \leq \varepsilon$$

for all $x \in (0,1)$. Then

$$\begin{aligned} &-\varepsilon\int_0^1 G(x,t)dt \\ &\leq \int_0^1 G(x,t)(-y'' + q(x)y + \mu D_{0+}^\alpha D_{1-}^\alpha y - F(y))dt \\ &\leq \varepsilon\int_0^1 G(x,t)dt. \end{aligned}$$

Hence

$$-\varepsilon\int_0^1 G(x,t)dt \leq y(x) - (\Lambda y)(x) \leq \varepsilon\int_0^1 G(x,t)dt.$$

That is

$$|(\Lambda y)(x) - y(x)| \leq \varepsilon\int_0^1 G(x,t)dt \leq \varepsilon\int_0^1 G(t,t)dt,$$

for all $x \in (0,1)$. Hence $d(\Lambda y), y) \leq m < \infty$. Similar to the proof of Theorem 6, we see that there exists a unique continuous function $y_0 : (0,1) \to \mathbb{R}$ which satisfies (20) for all $x \in (0,1)$. Finally we can get that

$$\begin{aligned} |y(x) - y_0(x)| &\leq d(y, y_0) \\ &\leq \tfrac{d(\Lambda y, y)}{1 - |\mu|\Gamma_\alpha^2\int_0^1 N(t)dt - L\int_0^1 G(t,t)dt} \\ &\leq \tfrac{\varepsilon\int_0^1 G(t,t)dt}{1 - |\mu|\Gamma_\alpha^2\int_0^1 N(t)dt - L\int_0^1 G(t,t)dt}. \end{aligned}$$

Hence inequality (21) holds true for all $x \in [0,1]$. ■

V. Illustrative examples

Example 1. We choose $\alpha = 0.25$, $q(x) = 0.25$, $\mu = 0.05$, $L = 0.2$, $\varphi(x) = \varepsilon(x^{0.5} + 1)$ with $\varepsilon > 0$. Then

$$\begin{cases} -y'' + 0.25y + 0.05D_{0+}^{0.25}D_{1-}^{0.25}y = 0.2\sin(y), x \in (0,1), \\ y(0) = y(1) = 0 \end{cases} \tag{23}$$

has the identical form with (1). It is quite difficult to find the exact solution of (23). By computing, we get that

$$u = e^{0.5x} - e^{-0.5x},$$

$$v = \frac{e^{-0.5}}{e^{-0.5} - e^{0.5}}e^{0.5x} - \frac{e^{0.5}}{e^{-0.5} - e^{0.5}}e^{-0.5x}$$

are the solutions of the boundary value problems

$$\begin{cases} -y'' + 0.25y = 0, x \in (0,1), \\ y(0) = 0, \ y'(0) = 1 \end{cases} \tag{24}$$

and

$$\begin{cases} -y'' + 0.25y = 0, x \in (0,1), \\ y(0) = 1, \ y(1) = 0 \end{cases} \tag{25}$$

respectively. Hence

$G(x,t)$

$$= \frac{1}{e^{-0.5}-e^{0.5}} \begin{cases} (e^{0.5t}-e^{-0.5t})(e^{-0.5}e^{0.5x}-e^{0.5}e^{-0.5x}), \\ \qquad 0 \le t \le x \le 1, \\ (e^{0.5x}-e^{-0.5x})(e^{-0.5}e^{0.5t}-e^{0.5}e^{-0.5t}), \\ \qquad 0 \le x \le t \le 1. \end{cases}$$

Moreover, we obtain that

$$\begin{aligned} &\int_0^1 G(x,t)\varphi(t)dt \\ &= \frac{e^{-0.5}e^{0.5x}-e^{0.5}e^{-0.5x}}{e^{-0.5}-e^{0.5}} \int_0^x (e^{0.5t}-e^{-0.5t})\varepsilon(t^{0.5}+1)dt \\ &\quad + \frac{e^{0.5x}-e^{-0.5x}}{e^{-0.5}-e^{0.5}} \int_x^1 (e^{-0.5}e^{0.5t}-e^{0.5}e^{-0.5t})\varepsilon(t^{0.5}+1)dt \\ &\le 2\varepsilon(e^{0.5}-e^{-0.5})x + 2\varepsilon(e^{0.5}-e^{-0.5}) \\ &\le 2(e^{0.5}-e^{-0.5})\varphi(x) \\ &\le K_1\varepsilon\varphi(x) \end{aligned}$$

and

$$\begin{aligned} &\int_0^1 M(x,t)\varphi(t)dt \\ &\le \varepsilon(N_1(x)+N_2(x)+\frac{x^{1-\alpha}}{(1-\alpha)^2(1-Q_2)}+\frac{(1+Q_2)(1-x^{1-\alpha})}{(1-Q_2)(1-\alpha)^2} \\ &\quad + \frac{1}{(1-2\alpha)^2(1-Q_2)}+\frac{Q_1}{(1-\alpha)^2(1-Q_2)}) \\ &\le \varepsilon(\frac{2Q_1+Q_2+3}{(1-2\alpha)(1-Q_2)}x^{0.5}+\frac{Q_2+2}{(1-2\alpha)^2(1-Q_2)}+\frac{Q_1}{(1-\alpha)^3(1-Q_2)}), \\ &\le K_2\varphi(x). \end{aligned}$$

where

$$K_1 = 2(e^{0.5}-e^{-0.5}),$$

$$K_2 = \max\left\{\frac{2Q_1+Q_2+3}{(1-2\alpha)(1-Q_2)}, \frac{Q_2+2}{(1-2\alpha)^2(1-Q_2)}+\frac{Q_1}{(1-\alpha)^3(1-Q_2)}\right\}.$$

Hence

$$K_1L+\mu\Gamma_\alpha^2K_2 < 2.1\times0.2+0.05\times0.6659\times15 < 0.9194 < 1.$$

According to Theorem 6, if there exists a function $y: (0,1)\to\mathbb{R}$ which satisfies

$$\begin{cases} |-y''+0.25y+0.05D_{0+}^{0.25}D_{1-}^{0.25}y-0.2\sin(y)| \\ \qquad \le \varepsilon(x^{0.5}+1), x\in(0,1), \\ y(0)=y(1)=0, \end{cases} \tag{26}$$

then (23) has a unique exact solution $y_0: (0,1)\to\mathbb{R}$. And we know

$$|y(x)-y_0(x)| \le \frac{K_1\varphi(x)}{1-K_1L-\mu\Gamma_\alpha^2K_2} \le 26\varphi(x)$$

for all $x\in[0,1]$. This can be used in numerical analysis of (23).

Example 2. We choose $\alpha=0.25$, $q(x)=0.25$, $\mu=0.04$, $L=0.2$ and a positive constant ε. Then

$$\begin{cases} -y''+0.25y+0.04D_{0+}^{0.25}D_{1-}^{0.25}y=0.2y,\ x\in(0,1), \\ y(0)=y(1)=0 \end{cases} \tag{27}$$

has the identical form with (1). Similar to Example 1, we have

$$\begin{aligned} &G(x,t) \\ &= \frac{1}{e^{-0.5}-e^{0.5}} \begin{cases} (e^{0.5t}-e^{-0.5t})(e^{-0.5}e^{0.5x}-e^{0.5}e^{-0.5x}), \\ \qquad 0 \le t \le x \le 1, \\ (e^{0.5x}-e^{-0.5x})(e^{-0.5}e^{0.5t}-e^{0.5}e^{-0.5t}), \\ \qquad 0 \le x \le t \le 1. \end{cases} \end{aligned}$$

Moreover, by computing, we obtain

$$\int_0^1 N(t)dt \le 18.1,$$

$$\begin{aligned} &\int_0^1 G(t,t)dt \\ &\le \frac{1}{e^{-0.5}-e^{0.5}} \int_0^1 (e^{0.5t}-e^{-0.5t})(e^{-0.5}e^{0.5t}-e^{0.5}e^{-0.5t})dt \\ &< 0.1640. \end{aligned}$$

Hence

$$0.04\Gamma_{0.25}^2 \int_0^1 N(t)dt + L\int_0^1 G(t,t)dt < 0.8101 < 1.$$

According to Theorem 7, if there exists a function $y: (0,1)\to\mathbb{R}$ which satisfies

$$\begin{cases} |-y''+0.25y+0.05D_{0+}^{0.25}D_{1-}^{0.25}y-0.2y| \\ \qquad \le \varepsilon, x\in(0,1), \\ y(0)=y(1)=0, \end{cases} \tag{28}$$

then (23) has a unique exact solution $y_0: (0,1)\to\mathbb{R}$. And we know

$$|y(x)-y_0(x)| \le 0.8636\varepsilon$$

for all $x\in[0,1]$.

VI. Conclusion

In this paper, we study the stability of the approximate solutions of a class of differential boundary value problems with both right and left fractional derivatives which are widely used in many fields such as fractional derivative variational principles and optimal control theory for functionals involving fractional derivatives. The research results guarantee that the boundary value problem (1) has a close exact solution if there exists a function y satisfying (11) or (19). It also helps to estimate the errors of numerical solutions. The work is interesting, since it is very difficult to obtain the exact solutions of the type of boundary value problems. We give two examples to illustrate our theoretical results.

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