Maximal Order Block Method For The Solution Of Second Order Ordinary Differential Equations

Oluwaseun Adeyeye and Zurni Omar

***Abstract*—Block methods have been seen to be an adequate numerical method for finding the approximate solution to second order ordinary differential equations. Thus, this article presents a block method of maximal order for the direct solution of second order initial and boundary value problems. Taylor series expansion approach is adopted for the derivation of the block methods. From the numerical results obtained, this new block method performs better than previous numerical methods in existence in terms of accuracy, when compared to the exact solution of the numerical problems considered.**

***Index Terms*—Maximal Order, Block Method, Second Order, Initial Value Problems, Boundary Value Problems.**

I. Introduction

Mathematical models have been observed to birth second order ordinary equations either as initial or boundary value problems when it comes to modelling real life situations. Some of these real life models include models for beam deflection and deformation, transmission of heat, temperature distribution across a rod, amongst others [1], [2]. The need to adopt numerical solutions for obtaining an approximate solution of these second order ordinary differential equations is expedient. This is due to the condition that sometimes these ordinary differential equations have more than one solution, or the solution may not exist.

Quite a number of scholars have proposed numerical and approximate methods for the solution of second order ordinary differential equations of the form

$$y'' = f\left(x, y, y'\right). \tag{1}$$

Some authors who have discussed finding approximate solutions to (1) with initial conditions imposed include [3], [4], [5], while the numerical solution when boundary conditions were imposed include the work of [1], [6] and [7].

However, this paper intends to explore the simultaneous solutions of both initial and boundary value problems using the same block method. Although, this approach has been explored by [7] and [8], however, none this work presented a method of maximal order $2k + 2$.

Hence, this paper presents a k-step second derivative ($k = 3$) method to numerically approximate (1) and the numerical results obtained were compared with the results from the previously existing method in literature of equal order despite being of higher step-lengths.

The sections of this paper is arranged as follows; Section 2 presents the methodology, Section 3 shows the basic properties of the method while Section 4 will display the results to the numerical problems considered, and Section 5 concludes this paper.

II. Methodology

The first step entails the derivation of the $3-$step discrete scheme and its corresponding derivatives. Consider the following expression for deriving the discrete scheme.

$$y_{n+3} = \alpha_0 y_n + \alpha_1 y_{n+1} + \sum_{j=0}^{3} \beta_j f_{n+j} + \sum_{j=0}^{3} \lambda_j f'_{n+j} \tag{2}$$

which can also be expressed as

$$\begin{aligned} y_{n+3} &= \alpha_0 y_n + \alpha_1 y_{n+1} \\ &+ \left(\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_3 f_{n+3}\right) \\ &+ \left(\lambda_0 f'_n + \lambda_1 f'_{n+1} + \lambda_2 f'_{n+2} + \lambda_3 f'_{n+3}\right) \end{aligned} \tag{3}$$

Using Taylor series expansion to expand individual terms in (3)and substituting back gives the following matrix representation $Ax = B$ as:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{(h)^2}{2!} & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{(h)^3}{3!} & 0 & h & 2h & 3h & 1 & 1 & 1 & 1 \\ 0 & \frac{(h)^4}{4!} & 0 & \frac{(h)^2}{2!} & \frac{(2h)^2}{2!} & \frac{(3h)^2}{2!} & 0 & h & 2h & 3h \\ 0 & \frac{(h)^5}{5!} & 0 & \frac{(h)^3}{3!} & \frac{(2h)^3}{3!} & \frac{(3h)^3}{3!} & 0 & \frac{(h)^2}{2!} & \frac{(2h)^2}{2!} & \frac{(3h)^2}{2!} \\ 0 & \frac{(h)^6}{6!} & 0 & \frac{(h)^4}{4!} & \frac{(2h)^4}{4!} & \frac{(3h)^4}{4!} & 0 & \frac{(h)^3}{3!} & \frac{(2h)^3}{3!} & \frac{(3h)^3}{3!} \\ 0 & \frac{(h)^7}{7!} & 0 & \frac{(h)^5}{5!} & \frac{(2h)^5}{5!} & \frac{(3h)^5}{5!} & 0 & \frac{(h)^4}{4!} & \frac{(2h)^4}{4!} & \frac{(3h)^4}{4!} \\ 0 & \frac{(h)^8}{8!} & 0 & \frac{(h)^6}{6!} & \frac{(2h)^6}{6!} & \frac{(3h)^6}{6!} & 0 & \frac{(h)^5}{5!} & \frac{(2h)^5}{5!} & \frac{(3h)^5}{5!} \\ 0 & \frac{(h)^9}{9!} & 0 & \frac{(h)^7}{7!} & \frac{(2h)^7}{7!} & \frac{(3h)^7}{7!} & 0 & \frac{(h)^6}{6!} & \frac{(2h)^6}{6!} & \frac{(3h)^6}{6!} \end{pmatrix}$$
$$\times \left(\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2, \beta_3, \lambda_0, \lambda_1, \lambda_2, \lambda_3\right)^T$$
$$= \left(1, 3h, \tfrac{(3h)^2}{2!}, \tfrac{(3h)^3}{3!}, \tfrac{(3h)^4}{4!}, \tfrac{(3h)^5}{5!}, \tfrac{(3h)^6}{6!}, \tfrac{(3h)^7}{7!}, \tfrac{(3h)^8}{8!}, \tfrac{(3h)^9}{9!}\right)^T$$

and using matrix inverse approach, the following values are obtained

$$\begin{aligned} &\left(\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2, \beta_3, \lambda_0, \lambda_1, \lambda_2, \lambda_3\right)^T = (-2, 3, \\ &\quad \frac{1961h^2}{9072}, \frac{263h^2}{168}, \frac{365h^2}{336}, \frac{599h^2}{4536}, \frac{131h^3}{3780}, -\frac{103h^3}{1680}, \\ &\quad -\frac{71h^3}{840}, -\frac{349h^3}{15120})^T \end{aligned} \tag{4}$$

which gives the following method after substituting back in (3)

$$\begin{aligned} y_{n+3} = -2y_n + 3y_{n+1} &+ \frac{h^2}{9072}(1961f_n + 14202f_{n+1} \\ + 9855f_{n+2} + 1198f_{n+3}) &+ \frac{h^3}{15120}(524f'_n - 927f'_{n+1} \\ &- 1278f'_{n+2} - 349f'_{n+3}) \end{aligned} \tag{5}$$

The next group of methods required is the additional methods for the discrete scheme and the derivatives. The same

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procedure is followed as that of (5) and the methods are given to be:

$$\begin{aligned}
y_{n+2} &= -y_n + 2y_{n+1} + \tfrac{h^2}{9072}(908f_n + 6183f_{n+1} \\
&\quad + 1836f_{n+2} + 145f_{n+3}) + \tfrac{h^3}{15120}(233f'_n \\
&\quad - 1044f'_{n+1} - 1161f'_{n+2} - 58f'_{n+3}), \\
y'_n &= \tfrac{1}{h}\left(-y_n + y_{n+1}\right) + \tfrac{h}{272160}(-78076f_n - 3512f_{n+1} \\
&\quad - 19548f_{n+2} - 3329f_{n+3}) + \tfrac{h^2}{90720}(-2597f'_n \\
&\quad + 11268f'_{n+1} + 4005f'_{n+2} + 274f'_{n+3}), \\
y'_{n+1} &= \tfrac{1}{h}\left(-y_n + y_{n+1}\right) + \tfrac{h}{272160}(25319f_n + 91638f_{n+1} \\
&\quad + 16497f_{n+2} + 2626f_{n+3}) + \tfrac{h^2}{90720}(1252f'_n \\
&\quad - 11709f'_{n+1} - 3258f'_{n+2} - 215f'_{n+3}), \\
y'_{n+2} &= \tfrac{1}{h}\left(-y_n + y_{n+1}\right) + \tfrac{h}{272160}(28964f_n + 224073f_{n+1} \\
&\quad + 148932f_{n+2} + 6271f_{n+3}) + \tfrac{h^2}{90720}(1531f'_n \\
&\quad - 2556f'_{n+1} - 12411f'_{n+2} - 494f'_{n+3}), \\
y'_{n+3} &= \tfrac{1}{h}\left(-y_n + y_{n+1}\right) + \tfrac{h}{272160}(34919f_n + 260118f_{n+1} \\
&\quad + 275697f_{n+2} + 109666f_{n+3}) + \tfrac{h^2}{90720}(2020f'_n \\
&\quad + 4707f'_{n+1} + 10566f'_{n+2} - 4343f'_{n+3})
\end{aligned} \tag{6}$$

Combining equations (5) and (6) in matrix form gives:

$$\begin{pmatrix} -3 & 0 & 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{h} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{h} & 0 & 0 & 1 & 0 & 0 \\ -\frac{1}{h} & 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{h} & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y'_{n+1} \\ y'_{n+2} \\ y'_{n+3} \end{pmatrix} = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{pmatrix} \tag{7}$$

where

$$\begin{aligned}
&A_0 = -2y_n + \tfrac{h^2}{9072}(1961f_n + 14202f_{n+1} + 9855f_{n+2} \\
&+1198f_{n+3}) + \tfrac{h^3}{15120}(524f'_n - 927f'_{n+1} - 1278f'_{n+2} \\
&-349f'_{n+3}) \\
&A_1 = -y_n + \tfrac{h^2}{9072}(908f_n + 6183f_{n+1} + 1836f_{n+2} \\
&+145f_{n+3}) + \tfrac{h^3}{15120}(233f'_n - 1044f'_{n+1} - 1161f'_{n+2} \\
&-58f'_{n+3}) \\
&A_2 = -y'_n - \tfrac{1}{h}y_n + \tfrac{h}{272160}(-78076f_n - 3512f_{n+1} \\
&-19548f_{n+2} - 3329f_{n+3}) + \tfrac{h^2}{90720}(-2597f'_n + 11268f'_{n+1} \\
&+4005f'_{n+2} + 274f'_{n+3}) \\
&A_3 = -\tfrac{1}{h}y_n + \tfrac{h}{272160}(25319f_n + 91638f_{n+1} \\
&+16497f_{n+2} + 2626f_{n+3}) + \tfrac{h^2}{90720}(1252f'_n - 11709f'_{n+1} \\
&-3258f'_{n+2} - 215f'_{n+3}) \\
&A_4 = -\tfrac{1}{h}y_n + \tfrac{h}{272160}(28964f_n + 224073f_{n+1} \\
&+148932f_{n+2} + 6271f_{n+3}) + \tfrac{h^2}{90720}(1531f'_n - 2556f'_{n+1} \\
&-12411f'_{n+2} - 494f'_{n+3}) \\
&A_5 = -\tfrac{1}{h}y_n + \tfrac{h}{272160}(34919f_n + 260118f_{n+1} \\
&+275697f_{n+2} + 109666f_{n+3}) + \tfrac{h^2}{90720}(2020f'_n \\
&+4707f'_{n+1} + 10566f'_{n+2} - 4343f'_{n+3})
\end{aligned}$$

and using matrix inverse approach again, the following expressions are obtained:

$$\begin{aligned}
y_{n+1} &= y_n + hy'_n + \tfrac{h^2}{272160}(78076f_n + 35127f_{n+1} \\
&\quad + 19548f_{n+2} + 3329f_{n+3}) + \tfrac{h^3}{90720}(2597g_n \\
&\quad - 11268g_{n+1} - 4005g_{n+2} - 274g_{n+3}), \\
y_{n+2} &= y_n + 2hy'_n + \tfrac{h^2}{8505}(5731f_n + 7992f_{n+1} \\
&\quad + 2943f_{n+2} + 344f_{n+3}) + \tfrac{h^3}{2835}(206g_n \\
&\quad - 900g_{n+1} - 468g_{n+2} - 28g_{n+3}), \\
y_{n+3} &= y_n + 3hy'_n + \tfrac{h^2}{1120}(1206f_n + 2187f_{n+1} \\
&\quad + 1458f_{n+2} + 189f_{n+3}) + \tfrac{h^3}{1120}(135g_n - 486g_{n+1} \\
&\quad - 243g_{n+2} - 36g_{n+3}), \\
y'_{n+1} &= y'_n + \tfrac{h}{18144}(6893f_n + 8451f_{n+1} + 2403f_{n+2} \\
&\quad + 397f_{n+3}) + \tfrac{h^2}{30240}(1283g_n - 7659g_{n+1} \\
&\quad - 2421g_{n+2} - 163g_{n+3}), \\
y'_{n+2} &= y'_n + \tfrac{h}{567}\left(223f_n + 540f_{n+1} + 351f_{n+2} + 20f_{n+3}\right) \\
&\quad + \tfrac{h^2}{945}\left(43g_n - 144g_{n+1} - 171g_{n+2} - 8g_{n+3}\right), \\
y'_{n+3} &= y'_n + \tfrac{h}{224}\left(93f_n + 243f_{n+1} + 243f_{n+2} + 93f_{n+3}\right) \\
&\quad + \tfrac{h^2}{1120}\left(57g_n - 81g_{n+1} + 81g_{n+2} - 57g_{n+3}\right).
\end{aligned} \tag{8}$$

Equation (8) gives the expected family of methods needed to approximate boundary value problems in the form of (1) above.

III. Properties of the Block Method

As conventionally known, a linear multistep method is convergent iff it is consistent and zero-stable [9]. Hence, considering the linear operator associated with equation (2) is defined as

$$L\left[y(x);h\right] = \sum_{j=0}^{k} \alpha_j y_{n+j} - \sum_{j=0}^{k} \beta_j f_{n+j} + \sum_{j=0}^{k} \lambda_j f'_{n+j} \tag{9}$$

Expanding y_{n+j}, f_{n+j} and f'_{n+j}, we obtain the equation of the following form

$$L\left[y(x);h\right] = C_0 y(x_n) + C_1 h y^1(x_n) + \cdots + C_p h^p y^p(x_n) + \ldots$$

The method is said to be of *order* p if $C_0 = C_1 = \cdots = C_p = C_{p+1} = 0$, $C_{p+2} \neq 0$ and C_{p+2} is the error constant, where m is the order of the differential equation under consideration.

Likewise, considering the linear k-step method, the order p is said to be maximal if $p = 2k + 2$ [10].

Hence, the block method displays uniform maximal order $p = (8, 8, 8, 8, 8, 8)^T$ with error constant $C_{10} = \left(\frac{359}{50803200}, \frac{17}{793800}, \frac{27}{627200}, \frac{313}{25401600}, \frac{13}{793800}, \frac{9}{313600}\right)^T$.

Definition 3.1 (10): Given a linear k-step method, the order p is said to be maximal if $p = 2k + 2$.

Definition 3.2 (7): A linear multistep method is *consistent* if it has order $p \geq 1$.

To analyze the method for zero stability, the block method (12) is normalized to give the first characteristic polynomial as

$$\rho(R) = det\left(RA^0 - A^1\right) = R^2(R - 1)$$

where A^0 is the identity matrix of dimension 3 and A^1 is given by

$$A^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

The roots of $\rho(R) = 0$ satisfy $|R_j| \leq 1, j = 1, 2, 3$.
Hence, the block method is convergent since it is both consistent and zero-stable.

The region of absolute stability is determined by obtaining the stability polynomial from

$$det\left[\sum_{i=0}^{k} A^i q^{k-i} + z^m \sum_{i=0}^{k} B^i q^{k-i} + z^{m+1} \sum_{i=0}^{k} C^i q^{k-i}\right] \tag{10}$$

where $z = \lambda h$ and m is the order of the differential equation. Hence, the stability polynomial for the block method is gotten as

$$\begin{aligned} R(q) = &-\frac{29q^6z^9}{235200} + \frac{3q^6z^8}{6400} + \frac{q^6z^7}{1960} - \frac{17971q^6z^6}{1693440} + \frac{q^6z^5}{32} \\ &+\frac{43q^6z^4}{2240} - \frac{9q^6z^3}{28} + \frac{649q^6z^2}{1008} + \frac{43q^3z^9}{313600} + \frac{277q^3z^8}{134400} \\ &+\frac{569q^3z^7}{31360} + \frac{197143q^3z^6}{1693440} + \frac{79q^3z^5}{1280} + \frac{3187q^3z^4}{2240} - \frac{27q^3z^3}{112} \\ &+\frac{3887q^3z^2}{1008} + q^3 \end{aligned}$$

Plotting the roots of the stability polynomial in boundary locus approach displays the region of absolute stability as shown below.

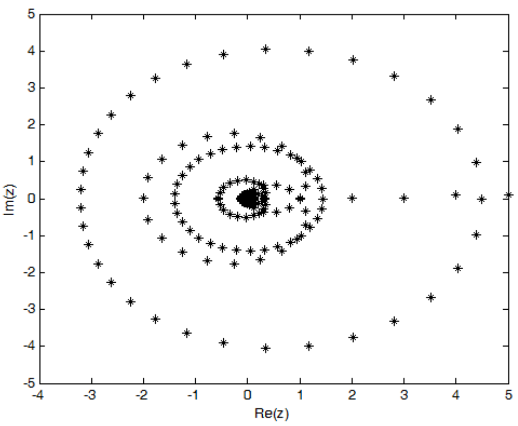

Figure 1: Region of Absolute Stability for (8)

IV. Numerical Results and Discussion

The following numerical problems are considered for the purpose of showing the accuracy of the new method when compared to previously existing methods

1) Consider the general second order initial value problem from [11]
$$y'' + \frac{6}{x}y' + \frac{4}{x^2}y = 0, y(1) = 1, y'(1) = 1, h = \frac{0.1}{32} \tag{11}$$
with exact solution
$$y(x) = \frac{5}{3x} - \frac{2}{3x^4}$$
2) Consider the special second order initial value problem from [12]
$$y'' - 100y = 0, y(0) = 1, y'(0) = -10, h = 0.01 \tag{12}$$
with exact solution
$$y(x) = e^{-10x}$$
3) Consider the non-linear second order initial value problem from [13]
$$y'' - x(y')^2 = 0, y(0) = 1, y'(0) = \frac{1}{2}, h = 0.1 \tag{13}$$
with exact solution
$$y(x) = 1 + \frac{1}{2}ln\left(\frac{2+x}{2-x}\right)$$
4) Consider the linear second order boundary value problem from [6]
$$y'' = y + \cos x, \quad y(0) = 0, y(1) = 1, h = 0.125 \tag{14}$$
with exact solution
$$\begin{aligned} y(x) = &\frac{-3\cosh 1 + 3\sinh 1 + \cos 1 + 2}{4\sinh 1}e^x + \\ &\frac{3\cosh 1 + 3\sinh 1 - \cos 1 - 2}{4\sinh 1}e^{-x} - \frac{\cos x}{2} \end{aligned}$$
5) Consider the general second order boundary value problem from [6]
$$y'' = y' - e^{(x-1)} - 1, \quad y(0) = 0, y(1) = 0, h = 0.1 \tag{15}$$
with exact solution
$$y(x) = x\left(1 - e^{(x-1)}\right)$$
6) Consider the boundary value problem from [6]
$$\begin{aligned} y'' + xy = \left(3 - x - x^2 + x^3\right)\sin x + 4x\cos x, \\ y'(0) = -1, y'(1) = 2\sin 1 \end{aligned} \tag{16}$$
with exact solution
$$y(x) = (x^2 - 1)\sin x$$
7) Consider the oscillatory nonlinear system of initial value problems studied by [14]
$$\begin{aligned} y_1'' &= -4x^2y_1 - \frac{2y_2}{\sqrt{y_1^2 + y_2^2}}, y_1(x_0) = 1, y_1'(x_0) = 0, \\ y_2'' &= -4x^2y_2 + \frac{2y_1}{\sqrt{y_1^2 + y_2^2}}, y_2(x_0) = 0, y_2'(x_0) = 0 \end{aligned} \tag{17}$$
whose exact solutions are given by $y_1(x) = cosx^2$ and $y_2(x) = sinx^2$
8) Consider the linear system of second order boundary value problems studied by [15]
$$\begin{aligned} \frac{d^2u_1}{dx^2} + (2x-1)\frac{du_1}{dx} + \cos\pi x\frac{du_2}{dx} = f_1(x) \\ \frac{d^2u_2}{dx^2} + xu_1 = f_2(x) \quad 0 \leq x \leq 1 \end{aligned} \tag{18}$$
where
$$\begin{aligned} f_1(x) &= -\pi^2\sin\pi x + (2x-1)(\pi+1)\cos\pi x \\ f_2(x) &= 2 + x\sin\pi x \end{aligned}$$
subject to boundary conditions
$$u_1(0) = u_1(1) = 0, \quad u_2(0) = u_2(1) = 0$$
whose exact solutions are
$$u_1(x) = \sin\pi x \quad \text{and} \quad u_2(x) = x^2 - x$$

9) (Bessel's IVP) Consider the Bessel differential equation that was solved by [16] and also [17]

$$x^2y'' + xy' + \left(x^2 - 0.25\right)y = 0, \quad x \in [1, 8] \quad (19)$$

subject to initial conditions

$$y(1) = \sqrt{\frac{2}{pi}}\sin 1 \simeq 0.671397071418031$$

whose exact solutions are

$$y'(1) = \frac{2\cos 1 - \sin 1}{2 * pi} \simeq 0.0954005144474746$$

The results and comparison of error of the problems considered are given in the tables below.

Table 1: Comparison of results for solving (11)

x	Exact Solution	Computed Solution	Error [11], $p = 8$	Error (New Method), $p = 8$
1.003125	1.0030765258576962262	1.0030765258576962261	8.30000E-08	1.00000E-19
1.006250	1.0060575030835162830	1.0060575030835162832	1.16000E-06	2.00000E-19
1.009375	1.0089449950888375792	1.0089449950888375790	6.63000E-06	2.00000E-19
1.012500	1.0117410181679885288	1.0117410181679885288	9.49100E-06	0.00000
1.015625	1.0144475426864138744	1.0144475426864138743	1.95350E-06	1.00000E-19
1.018750	1.0170664942356726084	1.0170664942356726084	9.41600E-06	0.00000
1.021875	1.0195997547562875920	1.0195997547562875921	4.65050E-05	1.00000E-19
1.025000	1.0220491636294317413	1.0220491636294317414	4.71220E-05	1.00000E-19
1.028125	1.0244165187384026804	1.0244165187384026804	1.86926E-04	0.00000
1.031250	1.0267035775008059839	1.0267035775008059840	4.43321E-04	1.00000E-19

Table 2: Comparison of results for solving (12)

x	Exact Solution	Computed Solution	Error [12], $p = 8$	Error (New Method), $p = 8$
0.01	0.904837418035959573	0.904837418035958956	5.744294E-13	1.211946E-16
0.02	0.818730753077981859	0.818730753077979986	1.225396E-10	1.872274E-15
0.03	0.740818220681717866	0.740818220681714096	2.179856E-10	3.769960E-15
0.04	0.670320046035639301	0.670320046035632537	3.139226E-10	6.763560E-15
0.05	0.606530659712633424	0.606530659712623130	4.196442E-10	1.023430E-14
0.06	0.548811636094026433	0.548811636094012045	5.896942E-10	1.438769E-14
0.07	0.496585303791409515	0.496585303791390105	2.036323E-10	1.941016E-14
0.08	0.449328964117221591	0.449328964117196617	1.847891E-10	2.497409E-14
0.09	0.406569659740599112	0.406569659740567982	1.677546E-10	3.112974E-14
1.00	0.367879441171442322	0.367879441171404144	1.523623E-10	3.817771E-14

Table 3: Comparison of results for solving (13)

x	Exact Solution	Computed Solution	Error (New Method), $k = 3$
0.100	1.05004172927849126820	1.05004172927829556360	1.957046E-13
0.200	1.10033534773107558060	1.10033534773047159090	6.039897E-13
0.300	1.15114043593646680530	1.15114043593520520730	1.261598E-12
0.400	1.20273255405408219100	1.20273255405036688830	3.715303E-12
0.500	1.25541281188299534160	1.25541281187507644990	7.918892E-12
0.600	1.30951960420311171550	1.30951960418894993510	1.416178E-11
0.700	1.36544375427139616910	1.36544375423523601570	3.616015E-11
0.800	1.42364893019360180680	1.42364893011887655380	7.472525E-11
0.900	1.48470027859405174160	1.48470027846053761650	1.335141E-10
1.000	1.54930614433405484570	1.54930614390236873630	4.316861E-10

Table 4: Comparison of results for solving (14)

x	Exact Solution	Computed Solution	Error [6], $k = 3$	Error (New Method), $k = 3$
0.125	0.060985349100553900	0.060985349100585467	1.140000E-07	3.156651E-14
0.250	0.138427934741475654	0.138427934741531975	2.220000E-07	5.632173E-14
0.375	0.233175541509714373	0.233175541509788966	3.200000E-07	7.469267E-14
0.500	0.346110454006479368	0.346110454006558011	3.120000E-07	7.864313E-14
0.625	0.478172624479587739	0.478172624479663170	2.900000E-07	7.543095E-14
0.750	0.630387283060996859	0.630387283061061692	2.560000E-07	6.483289E-14
0.875	0.803897221213436799	0.803897221213473963	1.300000E-07	3.716422E-14
1.000	1.000000000000000000	1.000000000000000000	N/A	0.000000

Table 5: Comparison of results for solving (15)

x	Exact Solution	Computed Solution	Error [6], $k = 3$	Error (New Method), $k = 3$
0.1	0.059343034025940089	0.059343034025920284	1.130000E-07	1.980493E-14
0.2	0.110134207176555682	0.110134207176517352	2.190000E-07	3.832952E-14
0.3	0.151024408862577146	0.151024408862521633	3.290000E-07	5.551284E-14
0.4	0.180475345562389427	0.180475345562322612	3.740000E-07	6.681520E-14
0.5	0.196734670143683288	0.196734670143608656	4.170000E-07	7.463205E-14
0.6	0.197807972378616420	0.197807972378537720	4.680000E-07	7.869950E-14
0.7	0.181427245522797494	0.181427245522724979	4.280000E-07	7.251511E-14
0.8	0.145015397537614513	0.145015397537555324	3.620000E-07	5.918955E-14
0.9	0.085646323767636384	0.085646323767598270	2.620000E-07	3.811464E-15
1.0	0.000000000000000000	0.000000000000000000	N/A	0.000000

Table 6: Comparison of results for solving (16)

h	Maximum Error (Majid et al, 2013), $k = 3$	Total Steps [6], $k = 3$	Maximum Error (New Method), $k = 3$	Total Steps (New Method), $k = 3$
$\frac{1}{8}$	3.59×10^{-5}	3	1.19×10^{-11}	3
$\frac{1}{16}$	3.49×10^{-6}	6	4.50×10^{-14}	6
$\frac{1}{32}$	1.87×10^{-7}	11	-	-
$\frac{1}{64}$	1.28×10^{-8}	22	-	-
$\frac{1}{128}$	7.76×10^{-10}	43	-	-

Table 7: Comparison of results for solving (19)

N	Maximum Error [16]	Maximum Error [17]	Maximum Error (New Method)
67	1.14×10^{-9}	7.11×10^{-7}	1.46×10^{-10}
82	3.50×10^{-10}	9.26×10^{-8}	3.30×10^{-11}
97	1.30×10^{-10}	87.8×10^{-9}	9.31×10^{-12}
112	5.50×10^{-11}	1.12×10^{-10}	3.51×10^{-12}
125	2.90×10^{-11}	2.71×10^{-11}	1.44×10^{-12}

Figure 2 shows the maximum error for Problem (17) plotted against various step sizes $h = \frac{1}{2^i}, i = 5, \ldots, 8$

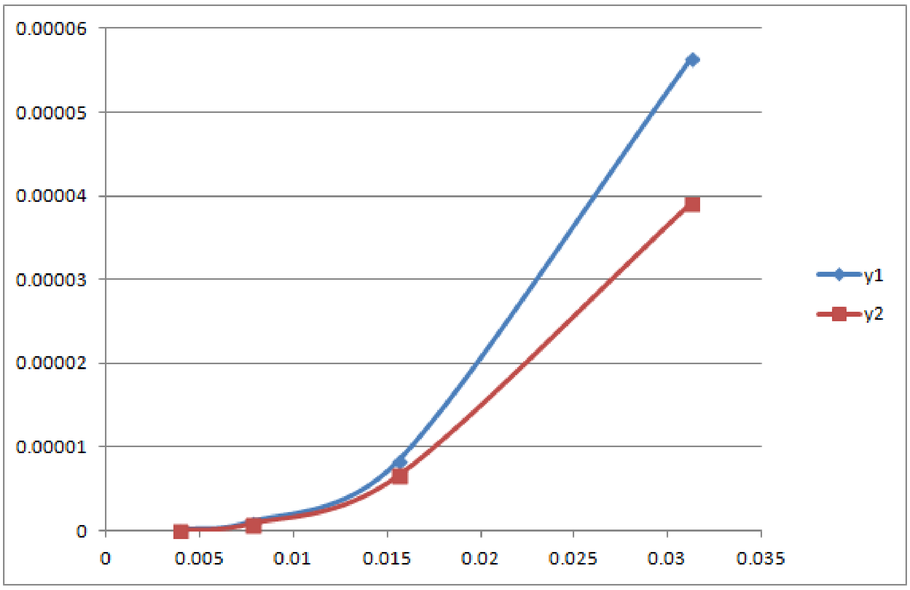

Figure 2: Step sizes versus Maximum Error for Problem (17)

Figure 3 shows the maximum error for problem (18) plotted against increasing number of iterations $N = 10, 20, 30, 40, 50$

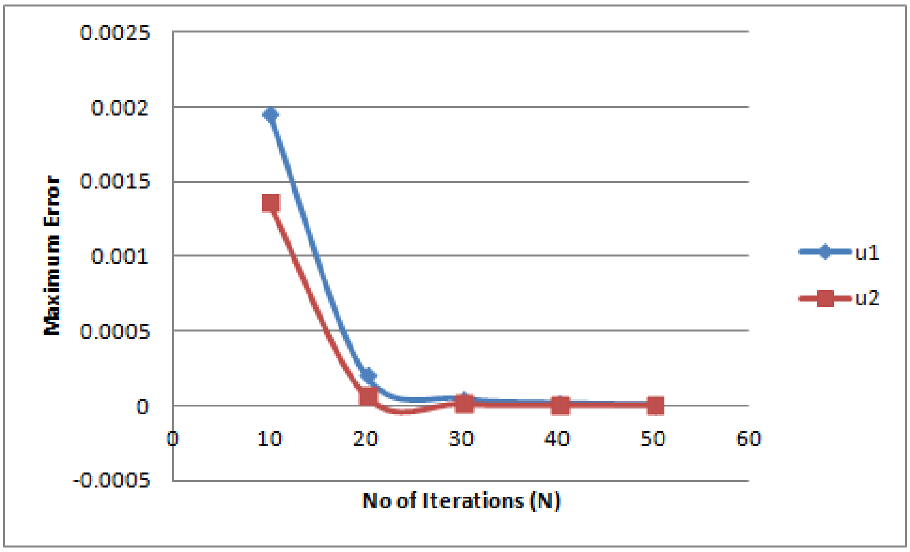

Figure 3: Number of Iterations versus Maximum Error for Problem (18)

V. Conclusion

This work presents a three-step method of maximal order for approximating the solution of second order initial and boundary value problems. To show the superiority of the maximal order block method introduced, certain numerical examples were considered. These included the solution of system of linear initial and boundary value problems, and the results were compared to previously existing methods of either equal order or equal steplength. From the results displayed in the tables above, the maximal order block method was seen to display more favourable results and also in accordance to literature, the accuracy increased as the number of iterations (N) increased and as the step-size (h) reduced when adopted for the solution of the system of ordinary differential equations. This is further justified as seen in Table 6, where the maximum error of the maximal order block method at $h-$value of $\frac{1}{16}$ is giving faster convergence than the previously existing method even at a smaller $h-$value of $\frac{1}{128}$. Therefore, the maximal order block method can be adopted for the solution of equations in the form of (1) or a system of (1) with either initial or boundary conditions imposed.

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