Analysis of a Stochastic Predator-Prey Model in Polluted Environments

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***Abstract*—In this paper, a stochastic predator-prey populations model in polluted environments is proposed and investigated. We first study the existence, uniqueness and boundedness of the global positive solution. Then we establish the sufficient conditions for extinction, non-persistence in the mean and weak persistence in the mean of the predator and prey populations. The threshold between weak persistence in the mean and extinction for each species is obtained. Finally, we study the global asymptotic stability of the solution. Our results reveal that the more the number of random noises, the easier the species go to extinction.**

***Index Terms*—environmental pollution, stochastic noises, persistence, extinction.**

I. INTRODUCTION

ENVIRONMENTAL pollution by modern industry, agriculture, and other human activities is one of the most important socio-ecological problems in the world today. The presence of toxicant in the environment is a great threat to the survival of the exposed living beings. This motivates scholars to analyze the survival of populations in polluted environments and to establish the persistence-extinction thresholds of the populations.

In recent years, many scholars have studied the survival of populations with toxicants effect by establishing mathematical models. Hallam and his colleagues did pioneering work in [1], [2], [3], where the authors studied some deterministic population systems with toxins effect and established the theoretical persistence-extinction thresholds for their models. From then on, many deterministic models in polluted environmrnts were proposed and analyzed. For example, Hallam and Ma [4], Ma et al. [5], [6], Freedman and Shukla [7], Wang and Ma [8], Buonomo et al. [9], Srinivasu [10] and He and Wang [11] proposed some single-species population models in polluted environments and established the persistence-and-extinction thresholds for their models. Liu and Ma [12] studied the persistence-and-extinction thresholds for two-species Lotka-Volterra models with toxins effect. Ma et al. [13] and Pan et al. [14] extended the threshold results in [12] to n-dimensional food chain model and n-dimensional factualistic system, respectively. Liu at al. [15], [16], Jiao et al. [17] and Li and Chen [18] considered the population models in polluted environments with pulse input of environmental toxins.

However, in the nature world the growth of population is inevitably affected by the random interference factors ([19], [20]). Thus it is important to study stochastic population models in polluted environments and to reveal the effects of random noises on the dynamics of populations. In this area, Gard did pioneering work in [21], where he first proposed a stochastic single-species model with toxins effect and investigated the dynamics of the model by supposing that the concentration of toxicant in the organism is a constant. Besides, Liu and Wang [22] obtained the persistence-extinction threshold for a stochastic logistic model in polluted environments. From then on, stochastic population models in in polluted environments have received great attention and have been studied extensively owing to their theoretical and practical significance (see e.g., [23]-[29]). Especially, taking into account the fact that predator-prey model is one of the most important models in biomathematics and ecology, Wang [24] has investigated the following stochastic predator-prey model in polluted environments:

$$
\begin{cases}
dx_1 = x_1[r_{10} - r_{11}C_0(t) - a_{11}x_1 - a_{12}x_2]dt \\
\qquad\qquad + \alpha_1 x_1 dB_1(t), \\
dx_2 = x_2[-r_{20} - r_{21}C_0(t) + a_{21}x_1 - a_{22}x_2]dt \\
\qquad\qquad + \alpha_2 x_2 dB_2(t), \\
\dfrac{dC_0(t)}{dt} = a_1 C_e(t) + d_1\theta\beta/a_1 - (l_1 + l_2)C_0(t), \\
\dfrac{dC_e(t)}{dt} = -hC_e(t) + u(t),
\end{cases}
\tag{1}
$$

where $x_1(t)$ represents the size of the prey population at time t; $x_2(t)$ stands for the size of the predator population at time t; $r_{i0} > 0$ is the growth rate of the species i; $r_{i1} \geq 0$ denotes the species i's dose-response parameters for toxicant concentration in the body; $C_0(t)$ represents the concentration of toxicant in the organism at time t; $a_{ii} > 0$ is the intra-specific competition coefficients of species i; $a_{12} > 0$ stands for the capture rate; $a_{21} > 0$ measures the efficiency of food conversion; $B_1(t)$ and $B_2(t)$ are two independent standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a filtration $\{\mathcal{F}_t\}_{t\in R_+}$ satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets); $\alpha_i (i = 1, 2)$ stands for the intensity of the random noises; $C_e(t)$ is the concentration of toxicant in the environment at time t; $a_1 C_e(t)$ represents the organism's net absorption amounts of toxicant from the environment; $d_1\theta\beta/a_1$ is the organism's net absorption amounts of toxicant from the food; $(l_1 + l_2)C_0(t)$ represents

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the reduction of poison due to the metabolism and excretion; Parameters $a_1, d_1(\leq a_1), \theta, \beta, l_1$ and l_2 are positive constants; a_1 represents the per unit mass organism's absorption rate of toxicant from the environment; d_1 stands for the per unit mass organism's rate of toxicant from the food; θ represents the concentration of toxicant in the resources; β is the per unit mass organism's intake rate of toxicant from the food; l_1 and l_2 are the decomposition and emission rates of the toxicant in the organism, respectively; $h > 0$ is the environment's ability to clean up poison; $u(t) \leq U_2$ represents emission rate of environmental toxicant. Wang [24] has obtained the persistence-extinction threshold for model (1).

Based on the study [24], we find some interesting problems:

(Q1) Model (1) assumes that the parameters r_{10} and r_{20} are affected by independent random noises. However, in the nature world, the random noises on r_{10} and r_{20} may or may not correlate to each other ([19]). For example, rain may affect both x_1 and x_2. Thus what happens if both r_{10} and r_{20} are affected by correlated random noises?

(Q2) Boundedness is an important properties for population models, which was not investigated in [24]. Then when the solution of the model is bounded?

(Q3) In the study of population models, the global asymptotic stability of the solution is one of the most interesting topics. However, [24] did not consider global asymptotic stability of the solution.

The aims of this paper are to study the above problems. Suppose that r_{i0} affected by n independent standard Brownian motions, then we obtain the following stochastic model:

$$\begin{cases} dx_1 = x_1[r_{10} - r_{11}C_0(t) - a_{11}x_1 - a_{12}x_2]dt \\ \qquad + x_1 \sum_{i=1}^{n} \alpha_{1i} dB_i(t) \\ dx_2 = x_2[-r_{20} - r_{21}C_0(t) + a_{21}x_1 - a_{22}x_2]dt \\ \qquad + x_2 \sum_{i=1}^{n} \alpha_{2i} dB_i(t) \\ \dfrac{dC_0(t)}{dt} = a_1 C_e(t) + d_1\theta\beta/a_1 - (l_1 + l_2)C_0(t) \\ \dfrac{dC_e(t)}{dt} = -hC_e(t) + u(t) \end{cases} \tag{2}$$

with initial data

$$x_i(0) > 0, C_0(0) = C_e(0) = 0$$

where α_{1i}, α_{2i} are constants, $B_i(t)$, $(1 \leq i \leq n)$, are independent standard Brownian motions defined on $(\Omega, \mathcal{F}, \mathcal{P})$. Clearly, if $\alpha_{11} \neq 0$, $\alpha_{1i} = 0$, $2 \leq i \leq n$, $\alpha_{21} \neq 0$ and $\alpha_{2i} = 0$, $2 \leq i \leq n$, then model (2) becomes model (1).

The rest of the paper is arranged as follows. In Section 2, we show that for any given initial data, model (2) has a unique global positive solution. Then in Section 3, we establish the sufficient conditions for stochastic boundedness of the solution. We carry out the survival analysis for model (2) in Section 4. Sufficient conditions for extinction, non-persistence in the mean and weak persistence in the mean of the species are established. The threshold between extinction and weak persistence in the mean is obtained for each species. Afterwards, we investigate the global asymptotic stability of model (2) in Section 5. In the last section, we give some conclusions.

II. Existence and Uniqueness of the Solution

$C_0(t)$ and $C_e(t)$ are the concentrations of toxicant, hence we must give some conditions under which $0 \leq C_0(t) < 1$, $0 \leq C_e(t) < 1$. In fact, the last two equations in model (2) are linear with respect to $C_0(t)$ and $C_e(t)$, it is easy to obtain their explicit solutions, so we have

Lemma 1. *For model (2), if $0 < a_1 + d_1\theta\beta/a_1 < l_1 + l_2$, $U_2 \leq h$, then $0 \leq C_0(t) < 1, 0 \leq C_e(t) < 1$ for all $t \in R_+$ a.s.*

From now on, we always assume that $0 < a_1 + d_1\theta\beta/a_1 < l_1 + l_2$, $U_2 \leq h$. We concentrate on the following subsystem of model (2):

$$\begin{cases} dx_1 = x_1[r_{10} - r_{11}C_0(t) - a_{11}x_1 - a_{12}x_2]dt \\ \qquad + x_1 \sum_{i=1}^{n} \alpha_{1i} dB_i(t) \\ dx_2 = x_2[-r_{20} - r_{21}C_0(t) + a_{21}x_1 - a_{22}x_2]dt \\ \qquad + x_2 \sum_{i=1}^{n} \alpha_{2i} dB_i(t). \end{cases} \tag{3}$$

System (3) is a population model, so we should first give some conditions under which (2) has a global positive solution.

Theorem 1. *For model (3), if $a_{ij} > 0$, then for any given positive initial value $x(0) = (x_1(0), x_2(0)) \in R_+^2$, there exists a unique solution $x(t) = (x_1(t), x_2(t))$ to model (3) a.s. (almost surely) and this solution does not leave R_+^2 with probability 1.a.s.*

Proof: Since the coefficients of system (3) satisfy the local Lipschitz condition, so for any given initial conditions $x(0) = (x_1(0), x_2(0)) \in R_+^2$, there exists a unique local saturated solution $x(t) = (x_1(t), x_2(t))$ defined on $t \in [0, \tau_e]$, where τ_e is the time of the explosion ([30]). In order to prove this is a general solution, we only need to prove $\tau_e = \infty$. Let $n_0 > 0$ be sufficiently large such that all components of $x(0)$ are on $[1/n_0, n_0]$. For every integer $n > n_0$, define the stopping time

$$\tau_n = \inf\{t \in [0, \tau_e] : x_i(t) \leq \frac{1}{n} \ or \ x_i(t) \geq n\},$$

where we always set $\inf \varnothing = \infty$. Obviously, τ_n increases monotonically with respect to n. Let $\tau_\infty = \lim_{t \to +\infty} \tau_n$, hence $\tau_\infty \leq \tau_e$. Now we need to prove $\tau_\infty = \infty$. If it is false, we can find a positive constant $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$\mathcal{P}\{\tau_\infty < \infty\} > \varepsilon.$$

Then there exists an integer $n_1 \geq n_0$ satisfying

$$\mathcal{P}\{\tau_n < T\} > \varepsilon \ , n > n_1 \tag{4}$$

Define a function $V(x)$ which is from R_+^2 to R_+ as follows:

$$V(x) = a_{21}(x_1 - 1 - \ln x_1) + a_{12}(x_2 - 1 - \ln x_2).$$

This function is non-negative because

$$u - 1 - \ln u \geq 0, \ u > 0.$$

According to Itô's formula, we can see that

$$\begin{aligned}
&dV(x(t)) = a_{21}(x_1 - 1)\Bigg[[r_{10} - r_{11}C_0(t) - a_{11}x_1 \\
&-a_{12}x_2]dt + \sum_{i=1}^{n} \alpha_{1i} dB_i(t)\Bigg] + 0.5a_{21}\sum_{i=1}^{n}\alpha_{1i}^2 dt \\
&+a_{12}(x_2 - 1)\Bigg[[-r_{20} - r_{21}C_0(t) + a_{21}x_1 \\
&-a_{22}x_2]dt + \sum_{i=1}^{n} \alpha_{2i} dB_i(t)\Bigg] + 0.5a_{12}\sum_{i=1}^{n}\alpha_{2i}^2 dt \\
&= \Bigg[0.5a_{21}\sum_{i=1}^{n} a_{1i}^2 + 0.5a_{12}\sum_{i=1}^{n}\alpha_{2i}^2 - r_{10}a_{21} + r_{20}a_{12} \\
&+a_{21}r_{11}C_0(t) + a_{12}r_{21}C_0(t) + [r_{10}a_{21} + a_{11}a_{21} \\
&-a_{12}a_{21} - a_{21}r_{11}C_0(t)]x_1 + [-r_{20}a_{12} + a_{12}a_{21} \\
&+a_{12}a_{22} - a_{12}r_{21}C_0(t)]x_2 - a_{11}a_{21}x_1^2 - a_{12}a_{22}x_2^2\Bigg]dt \\
&+a_{21}(x_1 - 1)\sum_{i=1}^{n} a_{1i}dB_i(t) + a_{12}(x_2 - 1)\sum_{i=1}^{n} a_{2i}dB_i(t) \\
&= G(x)dt + a_{21}(x_1 - 1)\sum_{i=1}^{n} a_{1i}dB_i(t) \\
&+a_{12}(x_2 - 1)\sum_{i=1}^{n} a_{2i}dB_i(t),
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
G(x) \quad &= 0.5a_{21}\sum_{i=1}^{n} a_{1i}^2 + 0.5a_{12}\sum_{i=1}^{n}\alpha_{2i}^2 - r_{10}a_{21} \\
&+r_{20}a_{12} + +a_{21}r_{11}C_0(t) + a_{12}r_{21}C_0(t) \\
&+[r_{10}a_{21} + a_{11}a_{21} - a_{12}a_{21} - a_{21}r_{11}C_0(t)]x_1 \\
&+[-r_{20}a_{12} + a_{12}a_{21} + a_{12}a_{22} \\
&-a_{12}r_{21}C_0(t)]x_2 - a_{11}a_{21}x_1^2 - a_{12}a_{22}x_2^2.
\end{aligned}$$

Obviously, there is a positive constant $G_1 > 0$ such that $G(x) < G_1$. Substituting this inequality into (5) gives

$$\begin{aligned}
dV(x(t)) \quad &\leq G_1 dt + a_{21}(x_1 - 1)\sum_{i=1}^{n} a_{1i}dB_i(t) \\
&+a_{12}(x_2 - 1)\sum_{i=1}^{n} a_{2i}dB_i(t).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^{\tau_n \bigcap T} dV(x(t)) \leq \int_0^{\tau_n \bigcap T} G_1 dt \\
&+ \int_0^{\tau_n \bigcap T} \Bigg[a_{21}(x_1 - 1)\sum_{i=1}^{n} a_{1i}dB_i(t) \\
&+a_{12}(x_2 - 1)\sum_{i=1}^{n} a_{2i}dB_i(t)\Bigg].
\end{aligned}$$

Taking the expectation on the both sides, we have

$$\begin{aligned}
\mathbb{E}(x(\tau_n \textstyle\bigcap T)) &\leq V(x(0)) + G_1\mathbb{E}(\tau_n \textstyle\bigcap T) \\
&\leq V(x(0)) + G_1 T.
\end{aligned} \tag{6}$$

Let $\Omega_n = \{\tau_n \leq T\}$, then by (16), we have

$$\mathcal{P}(\Omega_n) \geq \varepsilon.$$

Note that for any $\omega \in \Omega_n$, there is a i such that $x_i(\tau_n, \omega) = n$ or $x_i(\tau_n, \omega) = 1/n$. Therefore, $V(x(\tau_n, \omega))$ does not less than

$$\begin{aligned}
\min\Bigg\{ \quad &a_{21}(n - 1 - \ln n), a_{12}(n - 1 - \ln n), \\
&a_{21}\left(\frac{1}{n} - 1 + \ln n\right), a_{12}\left(\frac{1}{n} - 1 + \ln n\right)\Bigg\}.
\end{aligned}$$

That is to say,

$$\begin{aligned}
&V(x(0)) + G_1 T \geq \mathbb{E}[1_{\Omega_n}(\omega)V(x(\tau_n)))] \\
&\geq \varepsilon \min\Bigg\{a_{21}(n - 1 - \ln n), a_{12}(n - 1 - \ln n), \\
&a_{21}\left(\frac{1}{n} - 1 + \ln n\right), a_{12}\left(\frac{1}{n} - 1 + \ln n\right)\Bigg\},
\end{aligned}$$

where 1_{Ω_n} is the index function of Ω_n. Letting $n \to \infty$ gives contradictory.

$$\infty > V(x(0)) + G_1 T = \infty.$$

This completes the proof. ■

III. Boundedness of the solution

In the previous section, we have shown that model (3) has a unique global positive solution. Now let us show that the solution is stochastically bounded.

Definition 1. *Model (3) is said to be stochastically bounded, if for $\forall\varepsilon > 0$, there is a positive constant K such that*

$$\liminf_{t \to +\infty} P\{x_i(t) \leq K\} \geq 1 - \varepsilon, \ i = 1, 2.$$

Theorem 2. *Let $(x_1(t), x_2(t))$ be a solution to (3) with initial value $(x_1(0), x_2(0)) \in R_+^2$. If $a_{22} > a_{21}$, then model (3) is stochastically bounded.*

Proof: To begin with, let us show that for any $p > 1$, there exists $G_i(p)$ such that

$$\mathbb{E}[x_i^p(t)] \leq G_i(p), \ i = 1, 2.$$

Define

$$V(x) = x_1^p,$$

where $p > 1$. Applying Itô's formula leads to

$$\begin{aligned}
dV(x) \quad &= px_1^p\Bigg[r_{10} - r_{11}C_0(t) - a_{11}x_1 - a_{12}x_2 \\
&+0.5(p-1)\sum_{i=1}^{n}\alpha_{1i}^2\Bigg]dt + px_1^p\sum_{i=1}^{n}\alpha_{1i}dB_i(t).
\end{aligned}$$

Making use of Itô's formula again to $e^t V(x)$ results in

$$\begin{aligned}
&d[e^t V(x)] = e^t V(x)dt + e^t dV(x) \\
&= e^t x_1^p dt + e^t px_1^p\Bigg[r_{10} - r_{11}C_0(t) - a_{11}x_1 - a_{12}x_2 \\
&+0.5(p-1)\sum_{i=1}^{n}\alpha_{1i}^2\Bigg]dt + pe^t x_1^p\sum_{i=1}^{n}\alpha_{1i}dB_i(t).
\end{aligned}$$

Taking expectations on both sides, we can obtain that

$$\begin{aligned}
\mathbb{E}[e^t x_1^p] \quad &\leq x_1^p(0) + p\mathbb{E}\int_0^t e^s x_1^p(s)\Bigg[1/p + r_{10} \\
&+0.5p\sum_{i=1}^{n}\alpha_{1i}^2 - a_{11}x_1(s)\Bigg]ds \\
&\leq x_1^p(0) + \int_0^t e^s L_1(p)ds \\
&= x_1^p(0) + L_1(p)(e^t - 1),
\end{aligned}$$

where

$$L_1(p) = \frac{\left[1 + pr_{10} + 0.5p^2 \sum_{i=1}^{n} \alpha_{1i}^2\right]^{p+1}}{(p+1)^{p+1} a_{11}^p}.$$

Thus there exists a $T > 0$ such that

$$\mathbb{E}[x_1^p(t)] \le 1.5L_1(p)$$

for all $t > T$. At the same time, an application of the continuity of $\mathbb{E}[x_1^p(t)]$ results in that there exist $\tilde{L}_1(p) > 0$ such that $\mathbb{E}[x_1^p(t)] \le \tilde{L}_1(p)$ for $t \le T$. Let

$$G_1(p) = \max\{1.5L_1(p), \tilde{L}_1(p)\},$$

then for $t \ge 0$, we have

$$\mathbb{E}[x_1^p(t)] \le G_1(p).$$

On the other hand, similarly, we can show that

$$\begin{aligned} d[e^t x_2^p] &= e^t x_2^p dt + e^t p x_2^p \bigg[- r_{20} - r_{21} C_0(t) \\ &+ a_{21} x_1 - a_{22} x_2 + 0.5(p-1) \sum_{i=1}^{n} \alpha_{2i}^2 \bigg] dt \\ &+ p e^t x_2^p \sum_{i=1}^{n} \alpha_{2i} dB_i(t). \end{aligned}$$

Taking expectations on both sides results in

$$\begin{aligned} \mathbb{E}[e^t x_2^p] &\le x_2^p(0) + p\mathbb{E}\int_0^t e^s x_2^p(s) \bigg[1/p + a_{21} x_1(s) \\ &+ 0.5p \sum_{i=1}^{n} \alpha_{2i}^2 - a_{22} x_2(s)\bigg] ds \\ &\le x_2^p(0) + p\int_0^t e^s \bigg\{ \mathbb{E}\bigg[x_2^p(s)\bigg[1/p - a_{22} x_2(s) \\ &+ 0.5p \sum_{i=1}^{n} \alpha_{2i}^2\bigg]\bigg] + a_{21}\mathbb{E}[x_2^p(s) x_1(s)] \bigg\} ds \\ &\le x_2^p(0) + p\int_0^t e^s \mathbb{E}\bigg[x_2^p(s)\bigg[1/p - a_{22} x_2(s) \\ &+ 0.5p \sum_{i=1}^{n} \alpha_{2i}^2\bigg]\bigg] ds + pa_{21}\int_0^t e^s \mathbb{E}\bigg[x_2^{p+1}(s)\bigg] ds \\ &+ \frac{pa_{21}}{p+1}\int_0^t e^s \mathbb{E}\bigg[x_1^{p+1}(s)\bigg] ds \\ &= x_2^p(0) + p\mathbb{E}\int_0^t e^s x_2^p(s)\bigg[1/p + 0.5p \sum_{i=1}^{n} \alpha_{2i}^2 \\ &- (a_{22} - a_{21}) x_2(s)\bigg] ds \\ &+ \frac{pa_{21}}{p+1}\int_0^t e^s \mathbb{E}\bigg[x_1^{p+1}(s)\bigg] ds \\ &\le x_2^p(0) + \int_0^t e^s L_2(p) ds \\ &+ \frac{pa_{21}}{p+1} G_1(p+1)\int_0^t e^s ds \\ &= x_2^p(0) + \bigg[L_2(p) + \frac{pa_{21}}{p+1} G_1(p+1)\bigg](e^t - 1), \end{aligned}$$

where

$$L_2(p) = \frac{\left[1 + 0.5p^2 \sum_{i=1}^{n} \alpha_{2i}^2\right]^{p+1}}{(p+1)^{p+1}(a_{22} - a_{21})^p}.$$

The third inequality follows from the Yong inequality: for $\forall a, b \in R$ and $\forall p, q, \varepsilon > o$

$$|a|^p |b|^q \le |a|^{p+q} + \frac{q}{p+q}\left[\frac{p}{\varepsilon(p+q)}\right]^{p/q} |b|^{p+q}.$$

Thus we get

$$\limsup_{t\to+\infty} \mathbb{E}[x_2^p(t)] \le L_2(p) + \frac{pa_{21} G_1(p+1)}{p+1} = L_3(p).$$

Then there exists a $T > 0$ such that

$$\mathbb{E}[x_2^p(t)] \le 1.5L_3(p)$$

for all $t > T$. There also exists $\tilde{L}_3(p) > 0$ such that

$$\mathbb{E}[x_2^p(t)] \le \tilde{L}_3(p)$$

for $t \le T$. Let

$$G_2(p) = \max\left\{1.5L_3(p), \tilde{L}_3(p)\right\},$$

then for $t \ge 0$, we have

$$\mathbb{E}[x_2^p(t)] \le G_2(p).$$

Now we are in the position to show the stochastic boundedness of model (3). For $\forall \varepsilon > 0$, let $K = \sqrt{\max\{G_1(2), G_2(2)\}/\varepsilon}$, then by Chebyshev's inequality, we have

$$\mathcal{P}\Big\{x_i(t) < K\Big\} \le \frac{E\big[x_i^2(t)\big]}{K^2} = K^{-2} E\big[x_i^2(t)\big].$$

Therefore,

$$\liminf_{t\to+\infty} \mathcal{P}\Big\{x_i(t) \le K\Big\} \le K^{-2} G_i(2) \le \varepsilon.$$

This completes the proof. ∎

Theorem 3. *The solution of model (3) has the property that*

$$\limsup_{t\to+\infty} \frac{\ln x_i(t)}{\ln t} \le 1, \quad a.s., \ i = 1, 2. \tag{7}$$

Proof: Define

$$W(x) = a_{21} x_1 + a_{12} x_2.$$

According to Itô's formula, we have

$$\begin{aligned} & e^t \ln\Big(a_{21} x_1 + a_{12} x_2\Big) - \ln\Big(a_{21} x_1(0) + a_{12} x_2(0)\Big) \\ &= \int_0^t e^s \bigg\{ \ln W(x(s)) + \frac{1}{W(x(s))}\bigg[a_{21} x_1(s) \\ &\times \Big(r_{10} - r_{11} C_0(s) - a_{11} x_1(s) - a_{12} x_2(s)\Big) \\ &+ a_{12} x_2(s)\Big(r_{20} - r_{21} C_0(s) + a_{21} x_1 - a_{22} x_2\Big)\bigg]\bigg\} ds \\ &- \int_0^t \frac{e^s}{2W^2(x(s))} \sum_{i=1}^{n} \alpha_{1i}^2 a_{21}^2 x_1^2(s) ds \\ &- \int_0^t \frac{e^s}{2W^2(x(s))} \sum_{i=1}^{n} \alpha_{2i}^2 a_{12}^2 x_2^2(s) ds \\ &+ \sum_{i=1}^{n} N_{i1}(t) + \sum_{i=1}^{n} N_{i2}(t), \end{aligned} \tag{8}$$

where

$$N_{i1}(t) = \int_0^t \frac{e^s}{W(x(s))} \alpha_{1i} a_{21} x_1(s) dB_i(s),$$

$$N_{i2}(t)=\int_0^t \frac{e^s}{W(x(s))}\alpha_{2i}a_{12}x_2(s)dB_i(s).$$

Let

$$N(t)=\sum_{i=1}^n (N_{i1}(t)+N_{i2}(t)).$$

Clearly, $N(t)$ is a local martingale with quadratic variation:

$$\langle N,N\rangle=\int_0^t \frac{e^{2s}}{W^2(x(s))}\sum_{i=1}^n\Big[\alpha_{1i}^2a_{21}^2x_1^2(s)+\alpha_{2i}^2a_{12}^2x_2^2(s)\Big]ds.$$

It then follows from the exponential martingale inequality that

$$\mathcal{P}\Big\{\sup_{0\le t\le \mu k}\Big[N(t)-0.5e^{-\mu k}\langle N,N\rangle\Big]>\rho e^{\mu k}\ln k\Big\}\le k^{-\rho},$$

where $\rho>1$ and $\mu>0$ is arbitrary. In view of the Borel-Cantelli lemma, for almost all $\omega\in\Omega$, there exists a $k_0(\omega)$ such that for every $k\ge k_0(\omega)$,

$$N(t)\le 0.5e^{-\mu k}\langle N(t),N(t)\rangle+\rho e^{\mu k}\ln k,\quad 0\le t\le \mu k.$$

Substituting this inequality into (8), we can observe that

$$\begin{aligned}
&e^t\ln W(t)-\ln W(0)\\
&=\int_0^t e^s\Big\{\ln W(x(s))+\frac{1}{W(x(s))}\Big[a_{21}x_1(s)\\
&\times\Big(r_{10}-r_{11}C_0(s)-a_{11}x_1(s)-a_{12}x_2(s)\Big)\\
&+a_{12}x_2(s)\Big(r_{20}-r_{21}C_0(s)+a_{21}x_1-a_{22}x_2\Big)\Big]\Big\}ds\\
&-\int_0^t\frac{e^s}{2W^2(x(s))}\sum_{i=1}^n\alpha_{1i}^2a_{21}^2x_1^2(s)ds\\
&-\int_0^t\frac{e^s}{2W^2(x(s))}\sum_{i=1}^n\alpha_{2i}^2a_{12}^2x_2^2(s)ds\\
&+\rho e^{\mu k}\ln k+0.5e^{-\mu k}\int_0^t\frac{e^{2s}}{W^2(x(s))}\\
&\times\sum_{i=1}^n\Big[\alpha_{1i}^2a_{21}^2x_1^2(s)+\alpha_{2i}^2a_{12}^2x_2^2(s)\Big]ds\\
&\le\int_0^t e^s\Big\{\ln W(x(s))+\frac{1}{W(x(s))}\Big[a_{21}x_1(s)\\
&\times\Big(r_{10}-a_{11}x_1(s)\Big)+a_{12}x_2(s)\Big(r_{20}-a_{22}x_2(s)\Big)\Big]\\
&-\frac{1}{2W^2(x(s))}\sum_{i=1}^n\alpha_{1i}^2a_{21}^2x_1^2(s)(1-e^{s-\mu k})\\
&-\frac{1}{2W^2(x(s))}\sum_{i=1}^n\alpha_{2i}^2a_{12}^2x_2^2(s)(1-e^{s-\mu k})\Big\}ds\\
&+\rho e^{\mu k}\ln k.
\end{aligned}$$

It is easy to see that for arbitrary $0\le t\le\mu k$, there exists a constant C independent of k such that

$$\begin{aligned}
&\ln W(x)+\frac{1}{W(x)}\Big[a_{21}x_1\Big(r_{10}-a_{11}x_1\Big)\\
&+a_{12}x_2\Big(r_{20}-a_{22}x_2\Big)\Big]\\
&-\frac{1}{2W^2(x)}\sum_{i=1}^n\alpha_{1i}^2a_{21}^2x_1^2(1-e^{t-\mu k})\\
&-\frac{1}{2W^2(x)}\sum_{i=1}^n\alpha_{2i}^2a_{12}^2x_2^2(1-e^{t-\mu k})ds\le C.
\end{aligned}$$

In other words, for arbitrary $0\le t\le\mu k$, one can obtain

$$\begin{aligned}
&e^t\ln\Big(a_{21}x_1(t)+a_{12}x_2(t)\Big)\\
&\le C[e^t-1]+\rho e^{\mu k}\ln k+\ln\Big(a_{21}x_1(0)+a_{12}x_2(0)\Big).
\end{aligned}$$

Consequently, if $\mu(k-1)\le t\le\mu k$ and $k\ge k_0(\omega)$, then

$$\begin{aligned}
&\frac{\ln(a_{21}x_1(t)+a_{12}x_2)}{\ln t}\le e^{-t}\frac{\ln(a_{21}x_1(0)+a_{12}x_2(0))}{\ln t}\\
&+\frac{C[1-e^{-t}]}{\ln t}+\frac{\rho e^{-\mu(k-1)}e^{\mu k}\ln k}{\ln t}.
\end{aligned}$$

Letting $k\to+\infty$ results in

$$\limsup_{t\to+\infty}\frac{\ln(a_{21}x_1(t)+a_{12}x_2(t))}{\ln t}\le\rho e^{\mu}.$$

Letting $\rho\to1$ and $\mu\to0$ yields the desired assertion. ■

IV. Persistence and extinction

In this section we shall consider the persistence and extinction of x_1 and x_2. To this end, let us introduce some notations and recall an useful lemma. Set

$$y^*=\limsup_{t\to+\infty}y(t),y_*=\liminf_{t\to+\infty}y(t),\langle y(t)\rangle=\frac{1}{t}\int_0^t y(s)ds,$$

$$\Delta=a_{11}a_{22}+a_{12}a_{21},$$

$$\Delta_2=a_{21}\Big[r_{10}-0.5\sum_{i=1}^n\alpha_{1i}^2\Big]-a_{11}\Big[r_{20}+0.5\sum_{i=1}^n\alpha_{2i}^2\Big],$$

$$\Phi=a_{11}r_{21}+a_{21}r_{11}.$$

Lemma 2. *([25]) Suppose that $x(t)\in C[\Omega\times R_+,R_+^0]$, where $R_+^0=\{a|a>0,a\in R\}$.*
(i) If there exist positive numbers λ_0,T and $\lambda\ge0$ such that

$$\ln x(t)\le\lambda t-\lambda_0\int_0^t x(s)ds+\sum_{i=1}^n\beta_iB_i(t)$$

for all $t\ge T$, where $B_i(t)(i=1,2)$ are independent standard Brownian motions, $\beta_i(1\le i\le n)$ are constants, then

$$\langle x\rangle^*\le\lambda/\lambda_0,\ a.s.$$

(ii) If there exist positive numbers λ_0,T and $\lambda\ge0$ such that

$$\ln x(t)\ge\lambda t-\lambda_0\int_0^t x(s)ds+\sum_{i=1}^n\beta_iB_i(t)$$

for all $t\ge T$, where $B_i(t)(i=1,2)$ are independent standard Brownian motions, $\beta_i(1\le i\le n)$ are constants, then

$$\langle x\rangle_*\ge\lambda/\lambda_0,\ a.s.$$

Definition 2. *(i) $x(t)$ is said to go to extinction if $\lim_{t\to+\infty}x(t)=0$.*
(ii) $x(t)$ is said to be non-persistent in the mean if $\langle x\rangle^*=0$
(iii) $x(t)$ is said to be weakly persistent in the mean if $\langle x\rangle^*>0$.

Lemma 3. *For model (3), if $\Delta_2>0$, then for $i=1,2$,*

$$[\ln x_i(t)/t]^*\le0,a.s.$$

Proof: Suppose that $y_1(t)$ is the solution of the following equation:

$$dy_1 = y_1[r_{10} - a_{11}y_1]dt + y_1\sum_{i=1}^{n}\alpha_{1i}dB_i(t). \tag{9}$$

$y_2(t)$ is the solution of the following equation:

$$dy_2 = y_2[-r_{20} + a_{21}y_1 - a_{22}y_2]dt + y_2\sum_{i=1}^{n}\alpha_{2i}dB_i(t). \tag{10}$$

By the stochastic comparison theorem ([31]), we get

$$x_1(t) \le y_1(t),\ x_2(t) \le y_2(t).$$

Now we are in the position to prove

$$[\ln y_1/t]^* \le 0,\ [\ln y_2/t]^* \le 0.$$

By Lemma A.1 in [32], we have

$$\lim_{t\to+\infty}[\ln y_1(t)/t] = 0. \tag{11}$$

Consequently,

$$\left[\frac{\ln x_1(t)}{t}\right]^* \le \limsup_{t\to+\infty}\left[\frac{\ln y_1(t)}{t}\right] = 0.$$

According to Itô's formula, one can observe that

$$\ln\left[\frac{y_1(t)}{y_1(0)}\right] = \left(r_{10} - 0.5\sum_{i=1}^{n}\alpha_{1i}^2\right)t - a_{11}\int_0^t y_1(s)ds + \sum_{i=1}^{n}\alpha_{1i}B_i(t),$$

and,

$$\ln\left[\frac{y_2(t)}{y_2(0)}\right] = \left(-r_{20} - 0.5\sum_{i=1}^{n}\alpha_{2i}^2\right)t + a_{21}\int_0^t y_1(s)ds - a_{22}\int_0^t y_2(s)ds + \sum_{i=1}^{n}\alpha_{2i}B_i(t).$$

Hence

$$a_{21}\ln\left[\frac{y_1(t)}{y_1(0)}\right] + a_{11}\ln\left[\frac{y_2(t)}{y_2(0)}\right] = \Delta_2 t - a_{11}a_{22}\int_0^t y_2(s)ds + a_{21}\sum_{i=1}^{n}\alpha_{1i}B_i(t) + a_{11}\sum_{i=1}^{n}\alpha_{2i}B_i. \tag{12}$$

For arbitrarily given $\varepsilon > 0$, by (11), we can find a positive constant T such that

$$a_{21}\frac{\ln[y_1(t)/y_1(0)]}{t} < \varepsilon$$

for $t > T$. Substituting this inequality into (12) gives

$$\begin{aligned} a_{11}\ln\left[\frac{y_2(t)}{y_2(0)}\right] &> (\Delta_2 - \varepsilon)t - a_{11}a_{22}\int_0^t y_2(s)ds \\ &+ a_{21}\sum_{i=1}^{n}\alpha_{1i}B_i(t) + a_{11}\sum_{i=1}^{n}\alpha_{2i}B_i(t) \\ &> (\Delta_2 - \varepsilon)t - a_{11}a_{22}\int_0^t y_2(s)ds \\ &+ a_{21}\sum_{i=1}^{n}\alpha_{1i}B_i(t). \end{aligned}$$

Since $\Delta_2 > 0$, we can use Lemma 2, hence

$$\langle y_2\rangle_* \ge \frac{\Delta_2 - \varepsilon}{a_{11}a_{22}}.$$

By the arbitrariness of ε, we can see that

$$\langle y_2\rangle_* \ge \frac{\Delta_2}{a_{11}a_{22}}.$$

So for arbitrarily given $\varepsilon > 0$, we can find a positive constant T_1 such that

$$a_{11}a_{22}\langle y_2(t)\rangle > a_{11}a_{22}\langle y_2(t)\rangle_* - \varepsilon \ge \Delta_2 - \varepsilon,\ t > T.$$

On the other hand, compute that

$$\begin{aligned} &a_{21}\frac{\ln[y_1(t)/y_1(0)]}{t} + a_{11}\frac{\ln[y_2(t)/y_2(0)]}{t} \\ &= \Delta_2 - a_{11}a_{22}\langle y_2(t)\rangle + a_{21}\sum_{i=1}^{n}\alpha_{1i}B_i(t)/t \\ &+ a_{11}\sum_{i=1}^{n}\alpha_{2i}B_i(t)/t. \end{aligned}$$

Hence

$$\begin{aligned} &a_{21}\frac{\ln[y_1(t)/y_1(0)]}{t} + a_{11}\frac{\ln[y_2(t)/y_2(0)]}{t} \\ &< \varepsilon + a_{21}\sum_{i=1}^{n}\alpha_{1i}B_i(t)/t + a_{11}\sum_{i=1}^{n}\alpha_{2i}B_i(t)/t \end{aligned}$$

Taking the upper limit on both sides, we get

$$[\ln y_2(t)/t]^* \le \varepsilon.$$

Therefore

$$[\ln y_2(t)/t]^* \le 0.$$

This completes the proof. ■

Theorem 4. *For the prey populations x_1,*

(i) if $r_{10} - 0.5\sum_{i=1}^{n}\alpha_{1i}^2 - r_{11}\langle C_0\rangle_* < 0$, then $x_1(t)$ goes to extinction a.s.;

(ii) If $r_{10} - 0.5\sum_{i=1}^{n}\alpha_{1i}^2 - r_{11}\langle C_0\rangle_* = 0$, then $x_1(t)$ is non-persistent in the mean;

(iii) If $r_{10} - 0.5\sum_{i=1}^{n}\alpha_{1i}^2 - r_{11}\langle C_0\rangle_* > 0$, then $x_1(t)$ is weakly persistent in the mean a.s.

Proof: (i) According to Itô's formula,

$$\begin{aligned} \ln(x_1(t)/x_1(0))/t &= r_{10} - 0.5\sum_{i=1}^{n}\alpha_{1i}^2 - r_{11}\langle C_0(t)\rangle \\ &- a_{11}\langle x_1(t)\rangle - a_{12}\langle x_2(t)\rangle + \sum_{i=1}^{n}\alpha_{1i}B_i(t)/t, \end{aligned} \tag{13}$$

$$\begin{aligned} \ln(x_2(t)/x_2(0))/t &= -r_{20} - 0.5\sum_{i=1}^{n}\alpha_{2i}^2 - r_{21}\langle C_0(t)\rangle \\ &+ a_{21}\langle x_1(t)\rangle - a_{22}\langle x_2(t)\rangle + \sum_{i=1}^{n}\alpha_{2i}B_i(t)/t. \end{aligned} \tag{14}$$

Taking the upper limit on the both sides of (13) gives

$$\begin{aligned} [\ln x_1(t)/t]^* &= r_{10} - 0.5\sum_{i=1}^{n}\alpha_{1i}^2 - r_{11}\langle C_0\rangle_* - a_{11}\langle x_1\rangle_* \\ &- a_{12}\langle x_2\rangle_* < 0. \end{aligned}$$

Therefore $\lim_{t\to+\infty} x_1(t) = 0\ a.s.$

(ii) For arbitrarily given $\varepsilon > 0$, we can find a positive constant $T_1 > 0$ such that

$$r_{11}\langle C_0(t)\rangle > r_{11}\langle C_0\rangle_* - \varepsilon, \quad \forall t > T.$$

Substituting this inequality into (13) gives

$$\ln(x_1(t)/x_1(0))/t \le r_{10} - 0.5\sum_{i=1}^{n}\alpha_{1i}^2 - r_{11}\langle C_0\rangle_* + \varepsilon - a_{11}\langle x_1(t)\rangle + \sum_{i=1}^{n}\alpha_{1i}B_i(t)/t.$$

Then by Lemma 2, we get

$$\langle x_1\rangle^* \le [r_{10} - 0.5\sum_{i=1}^{n}\alpha_{1i}^2 - r_{11}\langle C_0\rangle_* + \varepsilon]/a_{11}.$$

By the arbitrariness of ε, one can see that

$$\langle x_1\rangle^* \le [r_{10} - 0.5\sum_{i=1}^{n}\alpha_{1i}^2 - r_{11}\langle C_0\rangle_*]/a_{11}. \tag{15}$$

Notice that

$$r_{10} - 0.5\sum_{i=1}^{n}\alpha_{1i}^2 = r_{11}\langle C_0\rangle_*$$

and $\langle x_1\rangle^* \ge 0$, therefore $\langle x_1\rangle^* = 0,\ a.s.$

(iii) By (13), we get

$$a_{11}\langle x_1\rangle^* + a_{12}\langle x_2\rangle^* \ge r_{10} - 0.5\sum_{i=1}^{n}\alpha_{1i}^2 - r_{11}\langle C_0\rangle_* > 0.$$

That is to say $\langle x_1\rangle^* > 0$ a.s. In fact, for $\forall\omega \in \{\langle x_1\rangle^* = 0\}$, we have $\langle x_2(\omega)\rangle^* > 0$. Taking the upper limit on both sides of (14), we have

$$[\ln(x_2(t,\omega))/t]^* \le -r_{20} - 0.5\sum_{i=1}^{n}\alpha_{2i}^2 - r_{21}\langle C_0\rangle_* - a_{22}\langle x_2(\omega)\rangle_* < 0.$$

which is contradicted with

$$\langle x_2(\omega)\rangle^* > 0.$$

Therefore $\langle x_1\rangle^* > 0$
This completes the proof. ■

Theorem 5. *For the predator populations x_2,*
(i)if $\Delta_2 - \Phi\langle C_0\rangle_* < 0$,then $x_2(t)$ goes to extinction a.s.;
(ii) If $\Delta_2 - \Phi\langle C_0\rangle_* = 0$,then $x_2(t)$ is non-persistent in the mean a.s.;
(iii) If $\Delta_2 - \Phi\langle C_0\rangle_* > 0$,then $x_2(t)$ is weakly persistent a.s.

Proof: (i) Clearly, if

$$\frac{\Delta_2}{\Phi} < \langle C_0\rangle_* < \frac{r_{10} - 0.5\sum_{i=1}^{n}\alpha_{1i}^2}{r_{11}},$$

then

$$\frac{\Delta_2}{\Phi} < \frac{r_{10} - 0.5\sum_{i=1}^{n}\alpha_{1i}^2}{r_{11}}.$$

By (14) and (15), it is easy to see that

$$[t^{-1}\ln x_2(t)]^* \le a_{11}^{-1}[\Delta_2 - \Phi\langle C_0\rangle_*] - a_{22}\langle x_2\rangle_* < 0.$$

That is to say, $\lim_{t\to+\infty} x_2(t) = 0, a.s.$

If $r_{10} - 0.5\sum_{i=1}^{n}\alpha_{1i}^2 - r_{11}\langle C_0\rangle_* < 0$, then by Theorem 4, we have $\langle x_1\rangle^* = 0$. Hence according to (14),

$$\begin{aligned}[t^{-1}\ln x_2(t)]^* &\le -r_{20} - 0.5\sum_{i=1}^{n}\alpha_{2i}^2 - r_{21}\langle C_0\rangle_* + a_{21}\langle x_1\rangle^* - a_{22}\langle x_2\rangle_* \\ &= -r_{20} - 0.5\sum_{i=1}^{n}\alpha_{2i}^2 - r_{21}\langle C_0\rangle_* - a_{22}\langle x_2\rangle_* \\ &< 0\end{aligned}$$

That is to say,

$$\lim_{t\to+\infty} x_2(t) = 0,\ a.s.$$

(ii) Here we use reductio ad absurdum to prove (ii). If $\langle x_2\rangle^* > 0$, then by Lemma 3, we get $[t^{-1}\ln x_2(t)]^* = 0$. By (14), we have

$$-r_{20} - 0.5\sum_{i=1}^{n}\alpha_{2i}^2 - r_{21}\langle C_0\rangle_* + a_{21}\langle x_1\rangle^* \ge a_{22}\langle x_2\rangle_* \ge 0.$$

For arbitrarily given $\varepsilon > 0$, we can find a positive constant $T > 0$ such that

$$r_{21}\langle C_0(t)\rangle > r_{21}\langle C_0\rangle_* - \varepsilon.$$

and

$$a_{21}\langle x_1\rangle < a_{21}\langle x_1\rangle^* + \varepsilon$$

for all $t > T$. Substituting this inequality into (14) gives

$$\ln(x_2(t)/x_2(0))/t \le -r_{20} - 0.5\sum_{i=1}^{n}\alpha_{2i}^2 - r_{21}\langle C_0\rangle_* + 2\varepsilon + a_{21}\langle x_1(t)\rangle^* - a_{22}\langle x_2(t)\rangle + \sum_{i=1}^{n}\alpha_{2i}B_i(t)/t.$$

Then by Lemma 2, we get

$$\langle x_2\rangle^* \le \frac{-r_{20} - 0.5\sum_{i=1}^{n}\alpha_{2i}^2 - r_{21}\langle C_0\rangle_* + a_{21}\langle x_1\rangle^* + 2\varepsilon}{a_{22}}.$$

By the arbitrariness of ε, we have

$$\langle x_2\rangle^* \le \frac{-r_{20} - 0.5\sum_{i=1}^{n}\alpha_{2i}^2 + a_{21}\langle x_1\rangle^* - r_{21}\langle C_0\rangle_*}{a_{22}}.$$

When (15) is used in this inequality, one can see that

$$\langle x_2\rangle^* \le \frac{1}{a_{11}a_{22}}[\Delta_2 - \Phi\langle C_0\rangle_*] = 0.$$

This is a contradiction, so we have $\langle x_2\rangle^* = 0$.

(iii) Clearly,

$$\begin{aligned}&a_{21}t^{-1}\ln(x_1(t)/x_1(0)) + a_{11}t^{-1}\ln(x_2(t)/x_2(0)) \\ &= \Delta_2 - \Phi\langle C_0(t)\rangle - \Delta\langle x_2(t)\rangle \\ &+ a_{21}\sum_{i=1}^{n}\alpha_{1i}B_i(t)/t + a_{11}\sum_{i=1}^{n}\alpha_{2i}B_i(t)/t.\end{aligned}$$

We take upper limit on both sides of the equation. Since $\Delta_2 > 0$, then by Lemma 2, we have

$$\langle x_2\rangle^* \ge \frac{1}{\Delta}[\Delta_2 - \phi\langle C_0\rangle_*] > 0.$$

This completes the proof. ∎

Theorem 6. *For model (2) we have*

(a) *For prey populations* x_1, *set* $b_1 = r_{10} - 0.5\sum_{i=1}^{n}\alpha_{1i}^2$.

(i) *if*

$$b_1 - r_{11}\left[\frac{a_1}{(l_1+l_2)h}\langle u\rangle_* + \frac{d_1\theta\beta}{(l_1+l_2)a_1}\right] < 0,$$

then $x_1(t)$ *goes to extinction a.s.;*

(ii) *If*

$$b_1 - r_{11}\left[\frac{a_1}{(l_1+l_2)h}\langle u\rangle_* + \frac{d_1\theta\beta}{(l_1+l_2)a_1}\right] = 0,$$

then $x_1(t)$ *is non-persistent in the mean a.s.;*

(iii) *If*

$$b_1 - r_{11}\left[\frac{a_1}{(l_1+l_2)h}\langle u\rangle_* + \frac{d_1\theta\beta}{(l_1+l_2)a_1}\right] > 0,$$

then $x_1(t)$ *is weakly persistent in the mean a.s.*

(b) *For predator populations* x_2,

(iv) *if*

$$\Delta_2 - \Phi\left[\frac{a_1}{(l_1+l_2)h}\langle u\rangle_* + \frac{d_1\theta\beta}{(l_1+l_2)a_1}\right] < 0,$$

then $x_2(t)$ *goes to extinction a.s.;*

(v) *If*

$$\Delta_2 - \Phi\left[\frac{a_1}{(l_1+l_2)h}\langle u\rangle_* + \frac{d_1\theta\beta}{(l_1+l_2)a_1}\right] = 0,$$

then $x_2(t)$ *is non-persistent in the mean a.s.;*

(vi) *If*

$$\Delta_2 - \Phi\left[\frac{a_1}{(l_1+l_2)h}\langle u\rangle_* + \frac{d_1\theta\beta}{(l_1+l_2)a_1}\right] > 0,$$

then $x_2(t)$ *is weakly persistent in the mean a.s.*

Remark 1.From Theorem 4,one can observe that x_1 is gong to extinction if and only if $r_{11}\langle C_0\rangle_* + 0.5\sum_{i=1}^{n}\alpha_{1i}^2 > r_{10}$;$x_1$ is weekly persistent if and only if $r_{11}\langle C_0\rangle_* + 0.5\sum_{i=1}^{n}\alpha_{1i}^2 < r_{10}$,That is to say $r_{10} - r_{11}\langle C_0\rangle_* - 0.5\sum_{i=1}^{n}\alpha_{1i}^2$ is the threshold between weak persistence and extinction of x_1. Similarly,from Theorem 5,we can observe that x_2 is gong to extinction if and only if $\Phi\langle C_0\rangle_* > \Delta_2$;$x_2$ is weekly persistent if and only if $\Phi\langle C_0\rangle_* < \Delta_2$.In other words, $\Delta_2 - \Phi\langle C_0\rangle_*$ is the threshold between weak persistence and extinction of x_2.

V. Global Asymptotic Stability

Definition 3. *System (3) is said to be globally asymptotically stable if*

$$\lim_{t\to+\infty}|x_{11}(t) - x_{12}(t)| = \lim_{t\to+\infty}|x_{21}(t) - x_{22}(t)| = 0$$

for any two positive solution $(x_{11}(t), x_{21}(t))$ *and* $(x_{12}(t), x_{22}(t))$ *of system (3).*

Lemma 4. *([33]) Suppose that an* $n-$*dimensional stochastic process* $X(t)$ *on* $t \geq 0$ *satisfies the condition*

$$\mathbb{E}|X(t) - X(s)|^{\alpha_1} \leq c|t-s|^{1+\alpha_2}, 0 \leq s, t < \infty$$

for some positive constants α_1, α_2 *and* c. *Then there exists a continuous modification* $\tilde{X}(t)$ *of* $X(t)$ *which has the property that for every* $\theta \in (0, \alpha_2/\alpha_1)$ *there is a positive random variable* $h(\omega)$ *such that*

$$\mathcal{P}\left\{\sup_{0<|t-s|<h(\omega), 0\leq s,t<\infty}\frac{X(t)-X(s)}{|t-s|^{\theta}} \leq \frac{2}{1-2^{-\theta}}\right\} = 1$$

In other words, almost every sample path of $\tilde{X}(t)$ *is locally but uniformly Hölder continuous with exponent* θ.

Lemma 5. *Let* $(x_1(t), x_2(t))^T$ *be a solution of (3) on* $t \geq 0$. *If* $a_{22} > a_{21}$, *then almost every sample path of* $x_i(t)$ *is uniformly continuous,* $i = 1, 2$.

Proof: The first equation in model (3) is equivalent to the following stochastic integral equation

$$\begin{aligned} x_1(t) = x_1(0) &+ \int_0^t x_1\Big[r_{10} - r_{11}C_0(t) - a_{11}x_1 - a_{12}x_2\Big]ds \\ &+ \int_0^t x_1\sum_{i=1}^{n}\alpha_{1i}dB_i(s). \end{aligned}$$

Notice that

$$\begin{aligned} &\mathbb{E}\Big|x_1\Big[r_{10} - r_{11}C_0(t) - a_{11}x_1 - a_{12}x_2\Big]\Big|^p \\ &= \mathbb{E}\Big[|x_1|^p\Big|r_{10} - r_{11}C_0(t) - a_{11}x_1 - a_{12}x_2\Big|^p\Big] \\ &\leq 0.5\mathbb{E}|x_1|^{2p} + 0.5\mathbb{E}\Big|r_{10} - r_{11}C_0(t) - a_{11}x_1 - a_{12}x_2\Big|^{2p} \\ &\leq 0.5\Big\{G_1(2p) + 3^{2p-1}\Big[|r_{10}|^{2p} + a_{11}\mathbb{E}|x_1(t)|^{2p} \\ &\quad + a_{12}\mathbb{E}|x_2(t)|^{2p}\Big]\Big\} \\ &\leq 0.5\Big\{G_1(2p) + 3^{2p-1}\Big[|r_{10}|^{2p} + a_{11}G_1(2p) + a_{12}G_2(2p)\Big]\Big\} \\ &= K_2(p). \end{aligned}$$

Moreover, in view of the moment inequality for stochastic integrals one can obtain that for $0 \leq t_1 \leq t_2$ and $p > 2$,

$$\begin{aligned} &\mathbb{E}\Big|\int_{t_1}^{t_2}\sum_{i=1}^{n}\alpha_{1i}x_1(s)dB_i(s)\Big|^p \\ &\leq n^{p-1}\sum_{i=1}^{n}\mathbb{E}\Big|\int_{t_1}^{t_2}\alpha_{1i}x_1(s)dB_i(s)\Big|^p \\ &\leq n^{p-1}\sum_{i=1}^{n}[\alpha_{1i}^2]^p\left[\frac{p(p-1)}{2}\right]^{p/2}(t_2-t_1)^{\frac{p-2}{2}}\int_{t_1}^{t_2}\mathbb{E}|x_1|^p ds \\ &\leq n^{p-1}\sum_{i=1}^{n}[\alpha_{1i}^2]^p\left[\frac{p(p-1)}{2}\right]^{p/2}(t_2-t_1)^{\frac{p}{2}}G_1(p). \end{aligned}$$

Then for $0 < t_1 < t_2 < \infty, t_2 - t_1 \leq 1, 1/p + 1/q = 1$, we have

$$\mathbb{E}(|x_1(t_2)-x_1(t_1)|^p)=\mathbb{E}\Big|\int_{t_1}^{t_2}x_1\Big[r_{10}-r_{11}C_0(t)$$
$$-a_{11}x_1-a_{12}x_2\Big]ds+\int_{t_1}^{t_2}x_1\sum_{i=1}^{n}\alpha_{1i}dB_i(s)\Big|^p$$
$$\le 2^{p-1}\mathbb{E}\Big|\int_{t_1}^{t_2}x_1\Big[r_{10}-r_{11}C_0(t)-a_{11}x_1-a_{12}x_2\Big]ds\Big|^p$$
$$+2^{p-1}\mathbb{E}\Big|\int_{t_1}^{t_2}x_1\sum_{i=1}^{n}\alpha_{1i}dB_i(s)\Big|^p$$
$$\le 2^{p-1}(t_2-t_1)^{p/q}\int_{t_1}^{t_2}\mathbb{E}\Big|x_1\Big[r_{10}-r_{11}C_0(t)-a_{11}x_1$$
$$-a_{12}x_2\Big]\Big|^p ds$$
$$+2^{p-1}n^{p-1}\sum_{i=1}^{n}[\alpha_{1i}^2]^p\left[\frac{p(p-1)}{2}\right]^{p/2}(t_2-t_1)^{p/2}G_1(p)$$
$$\le 2^{p-1}(t_2-t_1)^{p/q+1}K_2(p)$$
$$+2^{p-1}n^{p-1}\sum_{i=1}^{n}[\alpha_{1i}^2]^p\left[\frac{p(p-1)}{2}\right]^{p/2}(t_2-t_1)^{p/2}G_1(p)$$
$$\le 2^{p-1}(t_2-t_1)^{p/2}\Big[(t_2-t_1)^{p/2}+(\tfrac{p(p-1)}{2})^{p/2}\Big]K_3(p)$$
$$\le 2^{p-1}(t_2-t_1)^{p/2}\Big[(1+(\tfrac{p(p-1)}{2})^{p/2}\Big]K_3(p),$$

where

$$K_3(p)=\max\{K_2(p),n^{p-1}\sum_{i=1}^{n}[\alpha_{1i}^2]^pG_1(p)\}.$$

Then it follows from Lemma 4 that almost every sample path of $x_1(t)$ is locally but uniformly Hölder-continuous with exponent θ for every $\theta\in(0,\frac{p-2}{2p})$. Therefore almost every sample path of $x_1(t)$ is uniformly continuous on $t\ge 0$. In the same way we can demonstrate that almost every sample path of $x_2(t)$ is uniformly continuous. ■

Lemma 6. *([34]) Let f be a non-negative function defined on R_+ such that f is integrable and is uniformly continuous. Then $\lim_{t\to+\infty}f(t)=0$.*

Theorem 7. *If*

$$a_{11}-a_{21}>0,\ a_{22}-a_{12}>0 \tag{16}$$

then system (3) is globally asymptotically stable.

Proof: Define

$$V(t)=|\ln x_{11}(t)-\ln x_{12}(t)|+|\ln x_{21}(t)-\ln x_{22}(t)|,$$

then $V(t)$ is a continuous and positive function on $t\ge 0$. A direct calculation of the right differential $d^+V(t)$ of $V(t)$, and then applying Itô's formula yields

$$d^+V(t)$$
$$=sgn(x_{11}-x_{12})\Big\{\Big[\frac{dx_{11}}{x_{11}}-\frac{(dx_{11})^2}{2x_{11}^2}\Big]$$
$$-\Big[\frac{dx_{12}}{x_{12}}-\frac{(dx_{12})^2}{2x_{12}^2}\Big]\Big\}$$
$$+sgn(x_{21}-x_{22})\Big\{\Big[\frac{dx_{21}}{x_{21}}-\frac{(dx_{21})^2}{2x_{21}^2}\Big]$$
$$-\Big[\frac{dx_{22}}{x_{22}}-\frac{(dx_{22})^2}{2x_{22}^2}\Big]\Big\}$$
$$=sgn(x_{11}-x_{12})\Big\{-a_{11}[x_{11}-x_{12}]-a_{12}[x_{21}-x_{22}]\Big\}dt$$
$$+sgn(x_{21}-x_{22})\Big\{a_{21}[x_{11}-x_{12}]-a_{22}[x_{21}-x_{22}]\Big\}dt$$
$$\le\Big\{-a_{11}|x_{11}-x_{12}|+a_{12}|x_{21}-x_{22}|-a_{22}(t)|x_{21}-x_{22}|$$
$$+a_{21}|x_{11}-x_{12}|\Big\}dt$$
$$=-\Big\{(a_{11}-a_{21})|x_{11}-x_{12}|+(a_{22}-a_{12})|x_{21}-x_{22}|\Big\}dt.$$

Integrating both sides leads to

$$V(t)\le V(0)-\int_0^t\Big[(a_{11}-a_{21})|x_{11}(s)-x_{12}(s)|$$
$$+(a_{22}-a_{12})|x_{21}(s)-x_{22}(s)\Big]ds.$$

Consequently

$$V(t)+\int_0^t\Big[(a_{11}-a_{21})|x_{11}(s)-x_{12}(s)|+(a_{22}-a_{12})|x_{21}(s)$$
$$-x_{22}(s)\Big]ds\le V(0)<\infty.$$

It then follow from $V(t)\ge 0$ and (16) that

$$|x_{11}(t)-x_{12}(t)|\in L^1[0,\infty),|x_{21}(t)-x_{22}(t)|\in L^1[0,\infty)$$

Then the desired assertion follows from Lemmas 5 and 6 immediately. ■

VI. Numerical simulations

In this section, let us introduced some numerical simulations to illustrate the main results by using the methods mentioned in [35], [36]. For the sake of simplicity, we choose $n=2$ and consider the following discretization equation:

$$\begin{cases} x_1^{(k+1)} &= x_1^{(k)}+x_1^{(k)}\Big[r_{10}-r_{11}C_0(k\Delta t)-a_{11}x_1^{(k)} \\ & -a_{12}x_2^{(k)}\Big]\Delta t+\Big\{x_1^{(k)}\alpha_{11}\xi_{11}^{(k)} \\ & +x_1^{(k)}\alpha_{21}\xi_{21}^{(k)}+0.5\alpha_{11}x_1^{(k)}\Big((\xi_{11}^{(k)})^2-1\Big) \\ & +0.5\alpha_{21}x_1^{(k)}((\xi_{21}^{(k)})^2-1)\Big\}\sqrt{\Delta t}, \\ x_2^{(k+1)} &= x_2^{(k)}+x_2^{(k)}\Big[-r_{20}-r_{21}C_0(k\Delta t)+a_{21}x_1^{(k)} \\ & -a_{22}x_2^{(k)}\Big]\Delta t+\Big\{x_2^{(k)}\alpha_{12}\xi_{12}^{(k)} \\ & +x_2^{(k)}\alpha_{22}\xi_{22}^{(k)}+0.5\alpha_{12}x_2^{(k)}\Big((\xi_{12}^{(k)})^2-1\Big) \\ & +0.5\alpha_{22}x_1^{(k)}((\xi_{22}^{(k)})^2-1)\Big\}\sqrt{\Delta t}, \end{cases}$$

where $\xi_{11}^{(k)}$, $\xi_{21}^{(k)}$, $\xi_{12}^{(k)}$ and $\xi_{22}^{(k)}$, $k=1,2,...,n$, are Gaussian random variables. In the following figures, we always choose $r_{10}=0.8$, $r_{20}=0.1$, $r_{11}=r_{21}=1$, $C_0(t)=0.1+0.05\sin t$, $a_{11}=0.5$, $a_{12}=0.4$, $a_{21}=0.3$, $a_{22}=0.5$.

In Fig.1, we choose $\alpha_{11}^2=0.2$, $\alpha_{21}^2=0$, $\alpha_{12}^2=\alpha_{22}^2=0.1$. Then according to Theorem 2, the model is stochastically bounded. See Fig.1.

In Fig.2, the values of parameters are the same with these in Fig.1. Then according to Theorem 3, (7) holds. See Fig.2.

In Fig.3(a), we choose $\alpha_{21}^2=1.4$, then by Theorems 4 and 5, both x_1 and x_2 are extinct. See Fig.3(a). In Fig.3(b), we choose $\alpha_{21}^2=0.4$, then in view of Theorems 4 and 5, x_1 is weakly persistent in the mean and x_2 is extinct. See Fig.3(b). In Fig.3(c), we choose $\alpha_{21}^2=0.04$, then according to Theorems 4 and 5, both x_1 and x_2 are weakly persistent in the mean. See Fig.3(c). By comparing Fig.1 and Fig.3(a), we can see that the more the random noises, the easier the species go to extinction.

In Fig.4, the values of parameters are the same with these in Fig.1. Then according to Theorem 6, the model is globally asymptotically stable, see Fig.4.

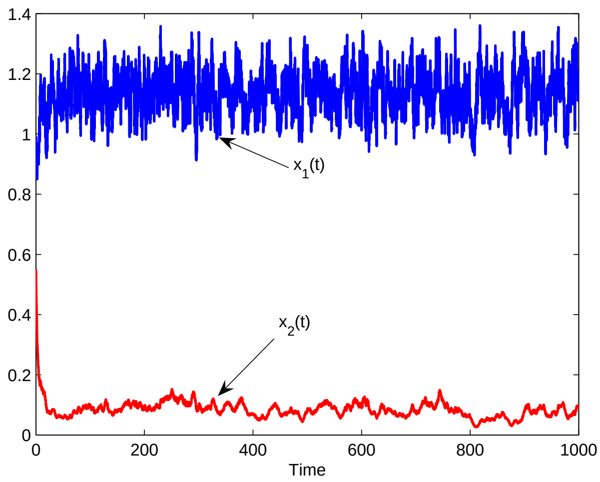


Fig. 1: Trajectories for model (3) with $\alpha_{11}^2 = 0.2$, $\alpha_{21}^2 = 0$, $\alpha_{12}^2 = \alpha_{22}^2 = 0.1$. This figure shows that the model is stochastically bounded.

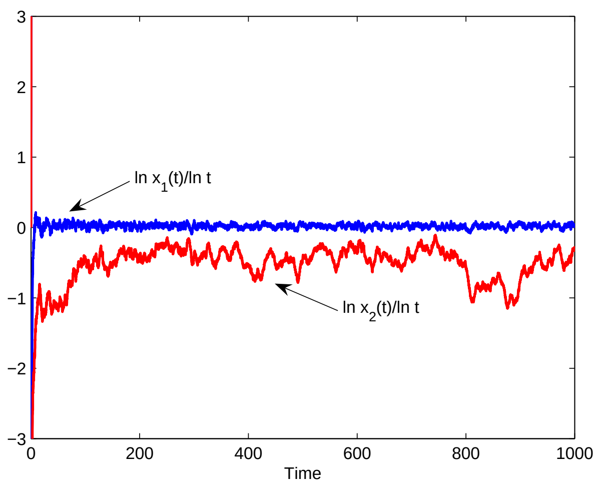
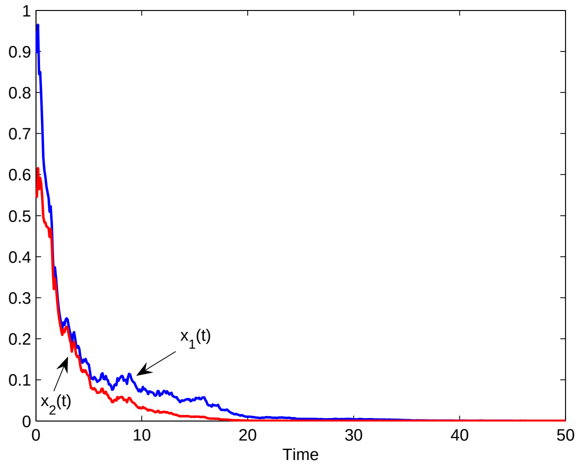


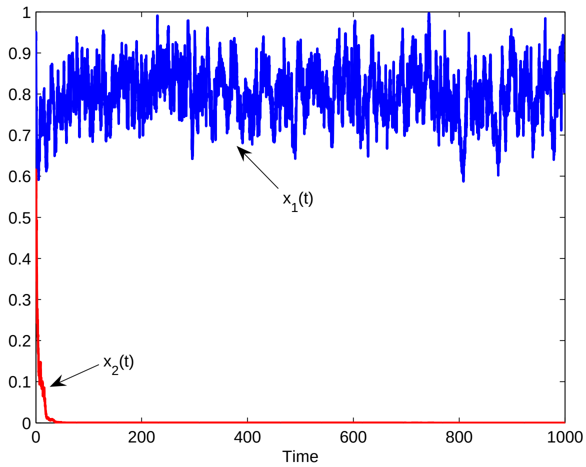
Fig. 2: Trajectories for model (3) with $\alpha_{11}^2 = 0.2$, $\alpha_{21}^2 = 0$, $\alpha_{12}^2 = \alpha_{22}^2 = 0.1$. This figure shows that (7) holds.

VII. Conclusions

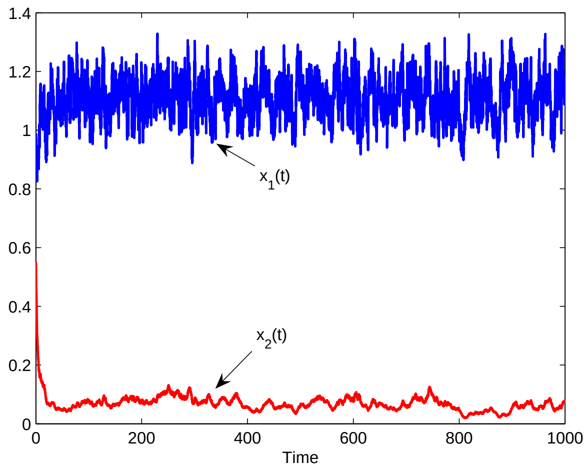
In this paper, under the assumptions that r_{10} and r_{20} are affected by n independent standard Brownian motions, we have proposed and investigated a stochastic predator-prey populations model in polluted environments. We have established the existence, uniqueness and boundedness of the global positive solution. Sufficient conditions for extinction, non-persistence in the mean, weak persistence in the mean of the predator and prey populations have been established. The threshold between weak persistence in the mean and



(a)



(b)



(c)

Fig. 3: Solution of model (3). (a) shows that both x_1 and x_2 are extinct ($\alpha_{21}^2 = 1.4$); (b) shows that x_1 is weakly persistent in the mean and x_2 is extinct ($\alpha_{21}^2 = 0.4$); (c) shows that both x_1 and x_2 are weakly persistent in the mean ($\alpha_{21}^2 = 0.04$).

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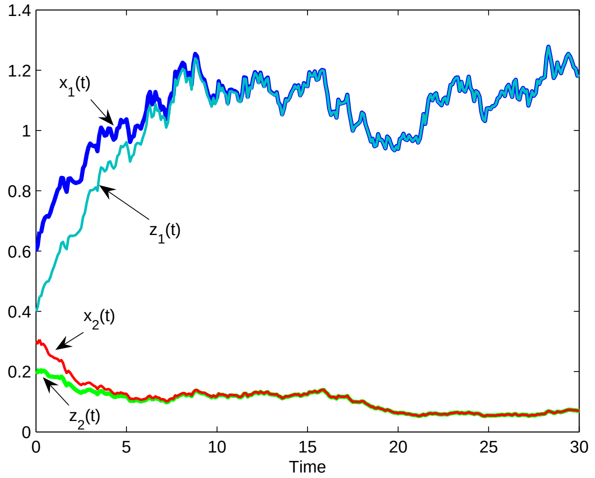

Fig. 4: Plot of two solution trajectories for model (3) with two sets of initial conditions $x_1(0) = 0.6,\ x_2(0) = 0.3$ and $z_1(0) = 0.4,\ z_2(0) = 0.2$. This figure shows that model (1) is globally asymptotically stable.

extinction for each species has been obtained. We have also studied the global asymptotic stability of the solution.

Our results indicate that the random interference of the prey populations x_1 is neither conducive to the survival of x_1 nor unfavorable to x_2. However the random interference of the predator populations x_2 is only not conducive to the survival of x_2.

Our Theorems give some important and interesting biological meanings. From Theorem 5 one can observe that if the two species have the same concentration of toxicant in the body, the ability for x_1 to resist the toxicant is stronger than that of x_2. Theorem 5 shows that if the average growth rate $r_{10} - r_{11}\langle C_0(t)\rangle$ of prey populations is less than certain negative value for sufficiently large t, then both predator and prey populations are going to extinction. If the average natural mortality rate $r_{20} + 0.5\sum_{i=1}^{n}\alpha_{2i}^2 + r_{21}\langle C_0(t)\rangle$ of predator populations is larger than the maximum number of average ingestion rate of prey populations $[r_{10} - 0.5\sum_{i=1}^{n}\alpha_{1i}^2 - r_{11}\langle C_0\rangle]/a_{11}$, then predator populations are going to extinction. Our results also reveal that the more the random noises, the easier the species go to extinction. So in order to conserve biological diversity, we have the following solutions:

(i) To reduce the intensity of the the random noises.
(ii) To reduce the number of the random noises.
(ii) To reduce the input of the toxicant.

Some interesting problems deserve further investigation. In Theorem 2 and Theorem 7, the conditions have some limitations on a_{ij}. It is interesting to study whether these conditions can be dropped. It is also of interest to investigate other multi-species systems (see e.g. [37], [38]).

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