On Robust Exponential Stability for Grey Stochastic Time-Delay Systems

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Abstract—**In this paper, we mainly investigate the stability problem for a class of grey stochastic systems with time delays. The parameters of system are evaluated by grey numbers. Firstly, we construct a suitable Lyapunov-Krasovskii functional. Then, by using Itô's differential formulation and decomposition technique, some sufficient conditions are derived to ensure the grey system in the mean-square exponential stability and almost surely exponential stability. Finally, an example to illustrative the effectiveness of main results is also given.**

Index Terms—**Grey Stochastic Systems, Time Delays, Lyapunov-Krasovskii Functional, Decomposition Technique, Mean-Square Exponential Stability, Almost Surely Exponential Stability**

I. Introduction

Now, it has been well recognized that, time delays are unavoidably encountered in many physical systems, which are the main sources of oscillation, bifurcation, or performance degradation [1]. For example, on account of finite switching speed of neurons and amplifiers, time delays may lead to instability and oscillation in a neurons network. So, the stability problem of time-delay systems has attracted considerable attention over decades, many important results have been presented in the literature [2-11]. In [3], by using a Lyapunov-Krasovskii functional, the authors have provided novel delay-dependent conditions in terms of linear matrix inequalities for a class of neutral BAM neural networks with time-varying delays.

In addition to time-delay effects, stochastic effects are another sources of instability and uncertainties in many systems. For instance, stochastic phenomenon frequently occurs in the electrical circuit design of neural networks, and the effects of stochastic phenomenon should be taken into account [12-13]. In recent years, the stability analysis of stochastic systems has been an attractive topic for many scholars, and a large amount of results have been reported [14-19]. In [17], by using the Razumikhin-type theorem, the authors have considered the stability problem of stochastic interval systems, and several sufficient conditions have been proposed.

Also, due to slowly varying parameters, modeling errors, or unknown uncertainties, it is impossible to obtain some parameters of stochastic systems accurately. In [20], the authors have pointed out that, if the parameters are evaluated by grey numbers, the systems can become grey (uncertain) systems. So, it is significant and important to discuss the stability problem for grey stochastic systems. However, to date, there have been very few works on this problem [20-24]. In [20], the authors have investigated the exponential stability for the grey stochastic systems with distributed delays and interval parameters, and the delay-dependent criteria have been obtained to ensure the systems in p-moment exponential robust stability. In [24], the authors have studied the robust stability problem for the grey stochastic nonlinear systems with distributed delays, and have proposed several novel conditions in terms of linear matrix inequalities.

In this paper, the stability problem for a class of grey stochastic time-delay systems is studied. By constructing a suitable Lyapunov-Krasovskii functional and using Itô's differential formulation, particularly, using decomposition technique of the continuous matrix-covered sets [20-24], we will obtain several novel exponential stability criteria, which ensure the grey system in mean-square exponential stability and almost surely exponential stability. Moreover, an example is given to illustrate the effectiveness of the stability criteria.

The notations are standard. R^n and $R^{n\times n}$ denote, the n-dimensional Euclidean space and the set of real n×n matrices. The superscript "T" denotes the transpose, $\|\cdot\|$ will refer to the Euclidean norm for vector or the spectral norm of matrices. For real symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that $X-Y$ is positive semi-definite (respectively, positive definite). While, $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ is a probability space with a filtration $\{F_t\}_{t\geq 0}$, and $C([-\tau,0];R^n)$ denotes the family of all continuous R^n-valued functions φ on $[-\tau,0]$. Denote by $L^2_{F_0}([-\tau,0];R^n)$ the family of all F_0-measurable bounded $C([-\tau,0];R^n)$-valued random variables $\xi=\{\xi(\theta):-\tau\leq\theta\leq 0\}$.

II. Preliminaries and Problem formulation

In this section, we consider the following grey stochastic time-delay system:

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$$\begin{cases} dx(t) = [A(\otimes)x(t) + B(\otimes)x(t-\tau) \\ \qquad + \int_0^{\tau} G(s)x(t-s)ds]\,dt \\ \qquad + f\big(x(t), x(t-\tau), t\big)dw(t), \quad t \ge 0 \\ \\ x_0 = \xi, \ \ \xi \in L^2_{F_0}([-\tau,0]; R^n), \quad -\tau \le t \le 0 \end{cases} \quad (1)$$

where $A(\otimes)$ and $B(\otimes)$ are grey (uncertain) n×n matrices. Let $A(\otimes) = (\otimes^a_{ij})$, $B(\otimes) = (\otimes^b_{ij})$, and $\otimes^a_{ij}, \otimes^b_{ij}$ are grey elements of $A(\otimes)$ and $B(\otimes)$.

$$[L_a, U_a] = \{A(\hat{\otimes}) = (a_{ij}) : \underline{a_{ij}} \le a_{ij} \le \overline{a_{ij}}, i, j = 1,2,..n\}$$

$$[L_b, U_b] = \{B(\hat{\otimes}) = (b_{ij}) : \underline{b_{ij}} \le b_{ij} \le \overline{b_{ij}}, i, j = 1,2,..n\}$$

are the continuous matrix-covered sets of $A(\otimes)$ and $B(\otimes)$. where $A(\hat{\otimes})$ and $B(\hat{\otimes})$ are whitened (deterministic) matrices of $A(\otimes)$ and $B(\otimes)$, $[\underline{a_{ij}}, \overline{a_{ij}}]$ and $[\underline{b_{ij}}, \overline{b_{ij}}]$ are the number-covered sets of $\otimes^a_{ij}$ and $\otimes^b_{ij}$.

For system (1), the following assumptions are given:

(A1) $H : R^n \times R^n \times R_+ \to R^{n\times n}$, and satisfies the local Lipschitz condition.

(A2) Assume that there exist constants $\alpha \ge 0$, $\beta \ge 0$, for arbitrary $(x, y, t) \in H : R^n \times R^n \times R_+$, the following inequality holds:

$$Trace[f^T(x,y,t)f(x,y,t)] \le \alpha|x|^2 + \beta|y|^2$$

The following definitions and lemmas are introduced:

Definition 2.1. [20] System (1) is said to be mean-square exponentially stability, if for all $\xi \in L^2_{F_0}([-\tau,0]; R^n)$ and whitened matrices $A(\hat{\otimes}) \in [L_a, U_a]$, $B(\hat{\otimes}) \in [L_b, U_b]$, there exist constants $r > 0$ and $C > 0$, such that

$$E|x(t;\xi)|^2 \le Ce^{-rt} \sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2, \ t \ge 0$$

Definition 2.2. [20] System (1) is said to be almost surely exponential stability, if for all $\xi \in L^2_{F_0}([-\tau,0]; R^n)$ and whitened matrices $A(\hat{\otimes}) \in [L_a, U_a]$, $B(\hat{\otimes}) \in [L_b, U_b]$, there exists constant $\hat{r} > 0$, such that

$$\lim_{t\to\infty}\sup \frac{1}{t}\ln|x(t;\xi)| \le -\frac{\hat{r}}{2}, \quad a.s.$$

Lemma 2.1. [20] If $A(\otimes) = (\otimes_{ij})_{m\times n}$ is a grey matrix, $[\underline{a_{ij}}, \overline{a_{ij}}]$ is a number-covered sets of $\otimes_{ij}$, then for arbitrary whitened matrix $A(\hat{\otimes}) \in [L_a, U_a]$, we have

i) $A(\hat{\otimes}) = \dfrac{U_a + L_a}{2} + \Delta A$

ii) $0 \le \Delta A \le \dfrac{U_a - L_a}{2}$

iii) $\left\|A(\hat{\otimes})\right\| \le \left\|\dfrac{U_a + L_a}{2}\right\| + \left\|\dfrac{U_a - L_a}{2}\right\|$

where $L_a = (\underline{a_{ij}})_{m\times n}$, $U_a = (\overline{a_{ij}})_{m\times n}$, $\Delta A = (\dfrac{\overline{a_{ij}} - \underline{a_{ij}}}{2}\hat{r}_{ij})_{m\times n}$, and $\hat{r}_{ij}$ is a whitened number of γ_{ij}, $\hat{r}_{ij} \in [-1,1]$, $[-1,1]$ is a number-covered sets of γ_{ij}, and γ_{ij} is a unit grey number.

Lemma 2.2. [25] Let $x, y \in R^n$, $P \in R^{n\times n}$ is a symmetric positive definite matrix, $M, N \in R^{n\times n}$, constant $\varepsilon > 0$, we have

$$2x^T M^T PNy \le \varepsilon x^T M^T PMx + \varepsilon^{-1} y^T N^T PNy$$

Lemma 2.3. [26] (Schur complement). Given constant matrices $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$, where $S_{11} = S_{11}^T$, $S_{22} = S_{22}^T$, the following conditions are equivalent:

i) $S < 0$

ii) $S_{22} < 0$, $S_{11} - S_{12}S_{22}^{-1}S_{12}^T < 0$

III. Main Results and Proofs

In this section, several sufficient conditions are proposed to guarantee system (1) in mean-square exponential stability and almost surely exponential stability.

Theorem 3.1. System (1) is mean-square exponentially stability, if there exist symmetric matrices $P > 0$, $Q > 0$, $R > 0$ and constants $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, such that

$$\lambda_{\min}(R) \ge \varepsilon_2 \tau \sup_{0\le s\le\tau} \|G(s)\|^2 \quad (2)$$

and

$$\begin{pmatrix} \Delta_1 & P\dfrac{U_b + L_b}{2} & M \\ \dfrac{U_b^T + L_b^T}{2}P & \Delta_2 & 0 \\ M^T & 0 & -J \end{pmatrix} < 0 \quad (3)$$

where

$$\Delta_1 = Q + \tau R + P\frac{U_a + L_a}{2} + \frac{U_a^T + L_a^T}{2}P + \lambda_{\max}(P) \times [2\left\|\frac{U_a - L_a}{2}\right\| + \alpha]I_n$$

$$\Delta_2 = -Q + [\varepsilon_1 \left\| \frac{U_b - L_b}{2} \right\|^2 + \lambda_{\max}(P) \times \beta] I_n ,$$

$$M = (P\ P)\ ,\quad J = diag(\varepsilon_1 I_n \quad \varepsilon_2 I_n) .$$

Then, for all $\xi \in L^2_{F_0}([-\tau,0];R^n)$, we have

$$E|x(t,\xi)|^2$$

$$\leq \frac{\lambda_{\max}(P) + (r\tau^2 e^{r\tau} + \tau)\lambda_{\max}(Q) + (r\tau^3 e^{r\tau} + \tau^2)\lambda_{\max}(R)}{\lambda_{\min}(P)} e^{-rt}$$

$$\times \sup_{-\tau\leq\theta\leq 0} E|\xi(\theta)|^2 ,\quad t \geq 0 \tag{4}$$

Here, r is the unique positive solution of the following equation

$$r\lambda_{\max}(P) + r\tau e^{r\tau}[\lambda_{\max}(Q) + \tau\lambda_{\max}(R)] + \lambda_{\max}(\Psi) = 0 \tag{5}$$

with
$$\psi := \begin{pmatrix} \Delta_1 + \sum_{i=1}^{2} \varepsilon_i^{-1} P^2 & P\frac{U_b + L_b}{2} \\ \frac{U_b^T + L_b^T}{2} P & \Delta_2 \end{pmatrix} < 0 \tag{6}$$

and $1 + r\tau > 0$.

Proof By Lemma 2.3, it follows that

$$\begin{pmatrix} \Delta_1 & P\frac{U_b + L_b}{2} & M \\ \frac{U_b^T + L_b^T}{2} P & \Delta_2 & 0 \\ M^T & 0 & -J \end{pmatrix} < 0$$

is equivalent to

$$\psi := \begin{pmatrix} \Delta_1 + \sum_{i=1}^{2} \varepsilon_i^{-1} P^2 & P\frac{U_b + L_b}{2} \\ \frac{U_b^T + L_b^T}{2} P & \Delta_2 \end{pmatrix} < 0$$

For convenience, let $x(t,\xi) = x(t)$.

Firstly, we construct a Lyapunov-Krasovskii functional:

$$V(x(t),t) = V_1(x(t),t) + V_2(x(t),t) + V_3(x(t),t) \tag{7}$$

where

$$V_1(x(t),t) = x^T(t)Px(t)$$

$$V_2(x(t),t) = \int_0^\tau x^T(t-s)Qx(t-s)ds$$

$$V_3(x(t),t) = \int_{-\tau}^0 [\int_{t+\theta}^t x^T(s)Rx(s)dsd\theta$$

By using Itô's differential formula, we have

$$\begin{aligned} &LV(x(t),t) \\ &= x^T(t)(Q + \tau R)x(t) \\ &\quad - x^T(t-\tau)Qx(t-\tau) \\ &\quad - \int_0^\tau x^T(t-s)Rx(t-s)ds \\ &\quad + 2x^T(t)PA(\hat{\otimes})x(t) \\ &\quad + 2x^T(t)PB(\hat{\otimes})x(t-\tau) \\ &\quad + 2x^T(t)P\int_0^\tau G(s)x(t-s)ds \\ &\quad + Trace[f^T(x, x(t-\tau), t)Pf(x, x(t-\tau), t)] \end{aligned} \tag{8}$$

Obviously, we see

$$\begin{aligned} &- \int_0^\tau x^T(t-s)Rx(t-s)ds \\ &\leq -\lambda_{\min}(R)\int_0^\tau |x(t-s)|^2 ds \end{aligned} \tag{9}$$

By Lemma 2.1 and Lemma 2.2, we can get

$$\begin{aligned} &2x^T(t)PA(\hat{\otimes})x(t) \\ &= x^T(t)(P\frac{U_a + L_a}{2} + \frac{U_a^T + L_a^T}{2}P)x(t) \\ &\quad + 2x^T(t)P\Delta A x(t) \\ &\leq x^T(t)(P\frac{U_a + L_a}{2} + \frac{U_a^T + L_a^T}{2}P)x(t) \\ &\quad + \lambda_{\max}(P) \times 2\left\|\frac{U_a - L_a}{2}\right\| x^T(t)x(t) \end{aligned} \tag{10}$$

and

$$\begin{aligned} &2x^T(t)PB(\hat{\otimes})x(t-\tau) \\ &\leq x^T(t)(P\frac{U_b + L_b}{2})x(t-\tau) \\ &\quad + x^T(t-\tau)(\frac{U_b^T + L_b^T}{2}P)x(t) \\ &\quad + \varepsilon_1^{-1}x^T(t)P^2x(t) \\ &\quad + \varepsilon_1\left\|\frac{U_b - L_b}{2}\right\|^2 x^T(t-\tau)x(t-\tau) \end{aligned} \tag{11}$$

and

$$2x^T(t)P\int_0^\tau G(s)x(t-s)ds$$

$$\begin{aligned} &\leq \varepsilon_2^{-1}x^T(t)P^2x(t) \\ &\quad + \varepsilon_2(\int_0^\tau G(s)x(t-s)ds)^T(\int_0^\tau G(s)x(t-s)ds) \end{aligned} \tag{12}$$

and

$$(\int_0^\tau G(s)x(t-s)ds)^T(\int_0^\tau G(s)x(t-s)ds)$$

$$\leq \tau\int_0^{\tau}\left|G(s)x(t-s)\right|^2 ds$$

$$\leq \tau \sup_{0\leq s\leq\tau}\left\|G(s)\right\|^2\int_0^{\tau}\left|x(t-s)\right|^2 ds \quad (13)$$

By assumptions (A2), we have

$$Trace[f^T(x(t),x(t-\tau),t)Pf(x(t),x(t-\tau),t)]$$

$$\leq \lambda_{\max}(P)[\alpha x^T(t)x(t)+\beta x^T(t-\tau)x(t-\tau)] \quad (14)$$

Combining with (8)-(14), if (2) and (3) hold, we can obtain

$$LV(x(t),t)$$

$$\leq (x^T(t),x^T(t-\tau))\Psi\begin{pmatrix} x(t) \\ x(t-\tau)\end{pmatrix}$$

$$\leq \lambda_{\max}(\Psi)(|x(t)|^2+|x(t-\tau)|^2)$$

$$\leq \lambda_{\max}(\Psi)|x(t)|^2 \quad (15)$$

Now, let $\Phi = e^{rt}V(x(t),t)$ and by (15), we have

$$L\Phi(x(t),t)$$

$$= re^{rt}V(x(t),t)+e^{rt}LV(x(t),t)$$

$$\leq re^{rt}[\lambda_{\max}(P)|x(t)|^2+\lambda_{\max}(Q)\int_{t-\tau}^{t}|x(s)|^2 ds$$

$$+\tau\lambda_{\max}(R)\int_{t-\tau}^{t}|x(s)|^2 ds]+e^{rt}\lambda_{\max}(\Psi)|x(t)|^2$$

Then, it can be seen that

$$\int_0^t EL\Phi(x(s),s)ds$$

$$\leq r\lambda_{\max}(P)\int_0^t e^{rs}E|x(s)|^2 ds$$

$$+r[\lambda_{\max}(Q)+\tau\lambda_{\max}(R)]\int_0^t e^{rs}\int_{s-\tau}^{s}E|x(u)|^2 duds$$

$$+\lambda_{\max}(\Psi)\int_0^t e^{rs}E|x(s)|^2 ds$$

$$\leq [r\lambda_{\max}(P)+r\tau e^{r\tau}(\lambda_{\max}(Q)+\tau\lambda_{\max}(R))+\lambda_{\max}(\Psi)]$$
$$\times\int_0^t e^{rs}E|x(s)|^2 ds$$

$$+r\tau^2 e^{r\tau}(\lambda_{\max}(Q)+\tau\lambda_{\max}(R))\sup_{-\tau\leq\theta\leq 0}E|\xi(\theta)|^2 \quad (16)$$

Let

$$f(r)=r\lambda_{\max}(P)+r\tau e^{r\tau}[\lambda_{\max}(Q)+\tau\lambda_{\max}(R)]+\lambda_{\max}(\Psi),$$

then

$$f'(r)=\lambda_{\max}(P)+\tau[\lambda_{\max}(Q)+\tau\lambda_{\max}(R)](1+r\tau)e^{r\tau}.$$

Because

$f'(r)>0$, $f(0)=\lambda_{\max}(\Psi)<0$ and $f(+\infty)=+\infty$, equation (2) must have a uniquely positive solution r.

From (16) and noting equation (5), we can get

$$\int_0^t EL\Phi(x(s),s)ds$$

$$\leq \tau^2 e^{r\tau}(\lambda_{\max}(Q)+\tau\lambda_{\max}(R))\sup_{-\tau\leq\theta\leq 0}E|\xi(\theta)|^2 \quad (17)$$

Obviously, we have

$$E\Phi(x(0),0)$$

$$\leq [\lambda_{\max}(P)+\tau\lambda_{\max}(Q)+\tau^2\lambda_{\max}(R)]$$

$$\times\sup_{-\tau\leq\theta\leq 0}E|\xi(\theta)|^2 \quad (18)$$

Then, by (17) and (18), we get

$$E\Phi(x(t),t)$$

$$= E\Phi(x(0),0)+\int_0^t EL\Phi(x(s),s)ds$$

$$\leq [\lambda_{\max}(P)+(r\tau^2 e^{r\tau}+\tau)\lambda_{\max}(Q)+(r\tau^3 e^{r\tau}+\tau^2)\lambda_{\max}(R)]$$
$$\times\sup_{-\tau\leq\theta\leq 0}E|\xi(\theta)|^2 \quad (19)$$

On the other hand,

$$E\Phi(x(t),t)\geq E[e^{rt}x^T(t)Px(t)]\geq\lambda_{\min}(P)e^{rt}E|x(t)|^2 \quad (20)$$

By (19) and (20), we can obtain that

$$E|x(t,\xi)|^2$$

$$\leq\frac{\lambda_{\max}(P)+(r\tau^2 e^{r\tau}+\tau)\lambda_{\max}(Q)+(r\tau^3 e^{r\tau}+\tau^2)\lambda_{\max}(R)}{\lambda_{\min}(P)}e^{-rt}$$

$$\times\sup_{-\tau\leq\theta\leq 0}E|\xi(\theta)|^2,$$

which implies that system (1) is mean-square exponentially stability. This completes the proof of Theorem 3.1.

Following a similar line as the proof of Theorem 3.1, and let $P=kI_n$ $(k>0)$, $R=\varepsilon_2\tau\sup_{0\leq s\leq\tau}\|G(s)\|^2 I_n$, we can obtain another criterion.

Corollary 3.1. System (1) is mean-square exponentially stability, if there exist symmetric matrix $Q>0$, and constants $\varepsilon_1>0$, $\varepsilon_2>0$ and $k>0$, such that the following LMI holds:

$$\begin{pmatrix} \Lambda_1 & k\frac{U_b+L_b}{2} & F \\ k\frac{U_b^T+L_b^T}{2} & \Lambda_2 & 0 \\ F^T & 0 & -J_1 \end{pmatrix}<0 \quad (21)$$

where

$$\Lambda_1 = Q + \tau^2 \varepsilon_2 \sup_{0\le s\le\tau} \|G(s)\|^2 I_n$$
$$+ k(\frac{U_a + L_a}{2} + \frac{U_a^T + L_a^T}{2}) + k[2\left\|\frac{U_a - L_a}{2}\right\| + \alpha] I_n,$$
$$\Lambda_2 = -Q + (\varepsilon_1 \left\|\frac{U_b - L_b}{2}\right\|^2 + k\beta) I_n,$$
$$F = (kI_n \ kI_n), \ J_1 = diag(\varepsilon_1 I_n \ \ \varepsilon_2 I_n).$$

Then, for all $\xi \in L^2_{F_0}([-\tau,0];R^n)$, we have

$$E|x(t,\xi)|^2$$
$$\le [1 + \frac{r\tau^2 e^{r\tau} + \tau}{k} \lambda_{\max}(Q) + \varepsilon_2 \frac{r\tau^4 e^{r\tau} + \tau^3}{k} \sup_{0\le s\le\tau} \|G(s)\|^2] e^{-rt}$$
$$\times \sup_{-\tau\le\theta\le0} E|\xi(\theta)|^2, \quad t \ge 0 \tag{22}$$

here, r is the unique positive solution of the following equation

$$rk + r\tau e^{r\tau}[\lambda_{\max}(Q) + \varepsilon_2 \tau^2 \sup_{0\le s\le\tau} \|G(s)\|^2]$$
$$+ \lambda_{\max}(\Pi) = 0 \tag{23}$$

with

$$\Pi := \begin{pmatrix} \Lambda_1 + \sum_{i=1}^{2} \varepsilon_i^{-1} k^2 I_n^2 & k\frac{U_b + L_b}{2} \\ k\frac{U_b^T + L_b^T}{2} & \Lambda_2 \end{pmatrix} < 0 \tag{24}$$

and $1 + r\tau > 0$.

Remark 3.1. If $A(\otimes) \equiv A$, $B(\otimes) \equiv B$, system (1) can become a deterministic time-delay stochastic system:

$$\begin{cases} dx(t) = [Ax(t) + Bx(t-\tau) + \int_0^\tau G(s)x(t-s)ds]dt \\ \qquad + f\big(x(t), x(t-\tau), t\big)dw(t), \quad t \ge 0 \\ x_0 = \xi, \ \xi \in L^2_{F_0}([-\tau,0];R^n), \quad -\tau \le t \le 0 \end{cases} \tag{25}$$

Corollary 3.2. System (25) is mean-square exponentially stability, if there exist symmetric matrix $Q > 0$, and constants $\varepsilon > 0$, and $k > 0$, such that the following LMI holds:

$$\begin{pmatrix} \Sigma_1 & kB & kI_n \\ kB^T & \Sigma_2 & 0 \\ kI_n & 0 & -\varepsilon I_n \end{pmatrix} < 0 \tag{26}$$

where

$$\Sigma_1 = Q + \tau^2 \varepsilon \sup_{0\le s\le\tau} \|G(s)\|^2 I_n \ + k(A + A^T) + k\alpha I_n$$
$$\Sigma_2 = -Q + k\beta I_n,$$

Then, for all $\xi \in L^2_{F_0}([-\tau,0];R^n)$, we have

$$E|x(t,\xi)|^2$$
$$\le [1 + \frac{r\tau^2 e^{r\tau} + \tau}{k} \lambda_{\max}(Q) + \varepsilon \frac{r\tau^4 e^{r\tau} + \tau^3}{k} \sup_{0\le s\le\tau} \|G(s)\|^2] e^{-rt}$$
$$\times \sup_{-\tau\le\theta\le0} E|\xi(\theta)|^2, \quad t \ge 0 \tag{27}$$

here, r is the unique positive solution of the following equation

$$rk + r\tau e^{r\tau}[\lambda_{\max}(Q) + \varepsilon\tau^2 \sup_{0\le s\le\tau} \|G(s)\|^2]$$
$$+ \lambda_{\max}(\Gamma) = 0 \tag{28}$$

with $$\Gamma := \begin{pmatrix} \Sigma_1 + \varepsilon^{-1} k^2 I_n^2 & kB \\ kB^T & \Sigma_2 \end{pmatrix} < 0 \tag{29}$$

and $1 + r\tau > 0$.

Proof Now, in the proof of Corollary 3.1, let

$\varepsilon_1 = 0, \varepsilon_2 = \varepsilon$,
$L_a = U_a = A, \ L_b = U_b = B$,
$F = kI_n, \quad J_1 = \varepsilon I_n$.

We follow a similar line as the proof of Corollary 3.1, it is easy to see that system (25) is mean-square exponentially stability. The remaining details are omitted.

Theorem 3.2. Under the conditions of Theorem 3.1, system (1) is almost surely exponential stability, if for all $\xi \in L^2_{F_0}([-\tau,0];R^n)$, we have

$$\lim_{t\to\infty} \sup \frac{1}{t} \ln|x(t;\xi)| \le -\frac{r}{2}, \quad a.s. \tag{30}$$

where r is the unique positive solution of equation (5).

Proof Firstly, we have

$$E|x(t;\xi)|^2 \le K \sup_{-\tau\le\theta\le0} E|\xi(\theta)|^2 e^{-rt}$$

By Doob's martingale inequality and Cauchy inequality, for arbitrary integer $k \ge 1$ and $\delta \in (0, r)$, we get

$$P\Big(\omega : \sup_{0\le s\le\tau} |x(k\tau + s)|^2 > e^{-(r-\delta)k\tau}\Big)$$
$$\le Ke^{-r\tau} e^{-\delta k\tau} \sup_{-\tau\le\theta\le0} E|\xi(\theta)|^2$$

By Borel-Cantelli lemma, for almost all $\omega \in \Omega$ and all but finitely many k, we can obtain

$$\sup_{0\le s\le\tau}\left|x(k\tau+s)\right|^2 \le e^{-(r-\delta)k\tau}$$

Then, following a similar line as the proof of Theorem 2 in [21], we can complete the proof of Theorem 3.2, and the details are omitted.

IV. Examples

To illustrate the effectiveness of the obtained results, an example is provided as follows:

$$\begin{cases} dx(t) = [A(\otimes)x(t) + B(\otimes)x(t-0.5) \\ \qquad + \int_0^{0.5} G(s)x(t-s)ds]dt \\ \qquad + f\left(x(t), x(t-0.5), t\right)dw(t), \quad t \ge 0 \\ x_0 = \xi, \ \xi \in L^2_{F_0}([-0.5,0];R^n), \quad -0.5 \le t \le 0 \end{cases} \tag{31}$$

where

$$L_a = \begin{bmatrix} -2.95 & 0.20 \\ 0.21 & -2.19 \end{bmatrix}; \ U_a = \begin{bmatrix} -2.88 & 0.21 \\ 0.25 & -2.10 \end{bmatrix}.$$

$$L_b = \begin{bmatrix} -1.73 & 0.22 \\ 0.23 & -1.65 \end{bmatrix}; \ U_b = \begin{bmatrix} -1.70 & 0.25 \\ 0.26 & -1.58 \end{bmatrix}.$$

Here, L_a 、 U_a and L_b 、 U_b are the lower bound and upper bound matrices of $A(\otimes)$ and $B(\otimes)$.

$$f\left(x(t), x(t-0.5), t\right) = \begin{bmatrix} \frac{1}{2}x_1(t)\sin(x_2(t-0.5)) \\ \frac{1}{2}x_2(t)\sin(x_1(t-0.5)) \end{bmatrix};$$

$$G(s) = \begin{bmatrix} e^{-\frac{s+3}{6}} & 0 \\ 0 & e^{-\frac{s+2}{3}} \end{bmatrix}.$$

Obviously,

$$Trace[f^T\left(x(t), x(t-0.5), t\right) f\left(x(t), x(t-0.5), t\right)] \\ \le 0.25x^2(t)$$

and

$$\sup_{0\le s\le 0.5}\left\|G(s)\right\|^2 = e^{-1}.$$

By the programmed procedure of [20-22], we can calculate and optimize ε_1, ε_2 satisfing (2) and (3). Then, it is easy to get r=1.4283. By Theorem 3.1, we can see that system (31) is mean-square exponential stability.

V. Conclusion

In this paper, we have studied the stability problem for a class of grey stochastic time-delay systems. By using the Lyapunov stability theory, Itô's differential formula, and decomposition technique, we have proposed some novel sufficient conditions, which guarantee our considered grey system in the mean-square exponential stability and almost surely exponential stability. In addition, an example is provided to show the effectiveness of the obtained criteria.

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References

[1] M. Malek-Zavarei and M. Jamshidi, *Time-Delay Systems: Analysis, Optimation and Application,* Amsterdam, Netherlands: North-Holland, 1987.

[2] E. K. Boukas, and Z. K. Liu, *Deterministic and Stochastic Time-Delay Systems*. Boston, MA: Birkhauser, 2002.

[3] X. Lou, and B. Cui, "Exponential stability analysis for neutral BAM neural networks with time-varying delays and stochastic disturbances," *Journal of Control Theory and Applications*, vol.10, no.1, pp. 92-99, 2011.

[4] S. Arik, "Global robust stability analysis of neural networks with discrete time delays", *Chaos, Solitons and Fractals*, vol. 26, no. 5, pp. 1407-1414, 2005.

[5] S. Mohamad, K. Gopalsamy, "Exponential stability of continuous-time and discrete-time cellular neural networks with delays," *Appl. Math. Comput.*, vol.135, pp.17-38, 2003.

[6] K. Yun, J. D. Cao, and J. M. Deng, "Exponential stability and perodic solutions of fuzzy cellular neural networks with time-varying delays," *Neurocomputing*, vol.69, pp. 1619-1627, 2006.

[7] R. Rakkiyappan, and P. Balasubramaniam, "New global exponential stability results for neutral type neural networks with distributed time delays," *Neurocomputing*, vol. 171, no. 4, pp. 1039-1045, 2008.

[8] J. Cao, "New results concerning exponential stability and periodic so-lutions of delayed cellular neural networks," *Phys. Lett. A*, vol. 307, pp.136-147, 2003.

[9] Y. Glizer, V. Turetsky, and J. Shinar, "Terminal Cost Distribution in Discrete-Time Controlled System with Disturbance and Noise-Corrupted State Information," *IAENG International Journal of Applied Mathematics*, vol. 42, no. 1, pp. 52-59, 2012.

[10] L. Lin, "Stabilization of LTI Switched Systems with Input Time Delay," *Engineering Letters*, vol. 14, no. 2, pp. 117-123, 2007.

[11] T. C. Kuo, and Y. J. Huang, "Global Stabilization of Robot Control with Neural Network and Sliding Mode," *Engineering Letters*, vol. 16, no. 1, pp. 56-60, 2008.

[12] S. Blythe, X. Mao, and X. Liao, "Stability of stochastic delay neural networks," *Journal of the Franklin Institute*, vol. 338, no. 2, pp. 481 - 495, 2001.

[13] W. Chen, X. Lu, "Mean square exponential stability of uncertain stochastic delayed neural networks," *Physics Letter A*, vol. 372, no. 7, pp. 1061 - 1069, 2008.

[14] H. Yan, Q. H. Meng, X. Huang, and H. Zhang, "Robust exponential stability of stochastic time-delay systems with uncertainties and nonlinear perturbations," *International Journal of Information Acquisition*, vol. 6, no. 1, pp. 61 - 71, 2009.

[15] X. Mao. "Robustness of exponential stability of stochastic differential delay equation," *IEEE Trans.Autom. Control*, vol. 41, no. 3, pp. 442 -447, 1996.

[16] S. Y. Xu, J. Lam, and X. Mao, Y. Zou, "A new LMI condition for delay-dependent robust stability of stochastic time-delay systems," *Asian Journal of Control*, vol. 7, no. 4, pp. 419 – 423, 2005.

[17] X. Liao, and X. Mao, "Exponential stability of stochastic delay interval systems," *Systems Control Lett*, vol. 40, pp. 171–181, 2000.

[18] H. J. Chu, and L. X. Gao, "Robust exponential stability and H∞ control for jumping stochastic Cohen-Grossberg neural networks with mixed delays", *Journal of Computational Information Systems*, vol. 7, no. 3, pp. 794 -806, 2011.

[19] W. Chen, and X. Lu, "Mean square exponential stability of uncertain stochastic delayed neural networks," *Physics Letter A*, vol. 372, no. 7, pp. 1061 - 1069, 2008.

[20] C. H. Su, and S. F. Liu, "The p-moment exponential robust stability for stochastic systems with distributed delays and interval parameters," *Applied Mathematics and Mechanics*, vol. 30, no. 7, pp. 915-924, 2009.

[21] C. H. Su, and S. F. Liu, "Exponential Robust Stability of Grey Neutral Stochastic Systems with Distributed Delays," *Chinese Journal of Engineering Mathematics*, vol. 27, no. 3, pp. 403-414, 2010.
[22] J. J. Li, and C. H. Su, "Mean-square Exponential Robust Stability for a Class of Grey Stochastic Systems with Distributed Delays", *Chin. Quart. J. of Math.*, vol. 25, no. 3, pp. 451-458, 2010.
[23] C. H. Su, "Robust Stability of Grey Stochastic Delay Systems with impulsive effect", *Sys. Sci.& Math. Scis,* vol.32, no.5, pp.537-548, 2012.
[24] C. H. Su, and S. F. Liu, "Robust Stability of Grey Stochastic Nonlinear Systems with Distributed-Delays," *Mathematics in Practice and Theory*, vol. 38, no. 22, pp. 218-223, 2008.
[25] A. Friedman, *Stochastic differential equations and their applications*, New York, Academic Press, 1976.
[26] S. Boyd, L. El. Ghaoui, E. Feron and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, Philadelphia (PA), SIAM, 1994.