Notes on Distance and Similarity Measures of Dual Hesitant Fuzzy Sets

Lei Wang, Xiang Zheng*, Li Zhang, Qiang Yue, *Member, IAENG*

Abstract—**The Dual Hesitant Fuzzy Sets (DHFSs) is a useful tool to deal with vagueness and ambiguity in the multiple attribute decision making (MADM) problems. The distance and similarity measures analysis are important research topics. In this paper, we propose some new distance measures for dual hesitant fuzzy sets, and study the properties of the measures. In the end, we develop an approach for multi-criteria decision making under dual hesitant fuzzy environment, and illustrate an example to show the behavior of the proposed distance measures.**

Index Terms—**dual hesitant fuzzy set, distance measures, similarity measures, multi-criteria decision making**

I. Introduction

Zhu and Xu [1] introduced the definition of dual hesitant fuzzy set, which is a new extension of fuzzy sets (FSs) [2]. Zhu and Xu's DHFSs used the membership hesitancy function and the non-membership hesitancy function to support a more exemplary and flexible access to assign values for each element in the domain. DHFS can be regarded as a more comprehensive set, which supports a more flexible approach when the decision makers provide their judgments. The existing sets, including FSs [2], IFSs [3] and HFSs [4] can be regarded as special cases of DHFSs. When people make a decision, they are usually hesitant and irresolute for one thing or another which makes it difficult to reach a final agreement. They further indicated that DHFSs can better deal with the situations that permit both the membership and the no-membership of an element to a given set having a few different values, which can arise in a group decision making problem. For example, in the organization, some decision makers discuss the membership degree 0.6 and the non-membership 0.1 of an alternative A satisfies a criterion x. Some possibly assign (0.8, 0.2), while the others assign (0.7, 0.2). No consistency is reached among these decision makers. Accordingly, the difficulty of establishing a common membership degree and a non-membership degree is not because we have a margin of error (intuitionistic fuzzy set), or some possibility distribution values (type-2 fuzzy set), but because we have a set of possible values (hesitant fuzzy set). For such a case, the satisfactory degrees can be represented by a dual hesitant fuzzy element $\{\{0.6, 0.8, 0.7\}, \{0.1, 0.2\}\}$, which is obviously different from intuitionistic fuzzy number $(0.8, 0.2)$ or $(0.7, 0.2)$ and hesitant fuzzy number $\{0.6, 0.8, 0.7\}$.

Distance measures of FSs are an important research topic in the FS theory, which has received much attention from researchers [6-9]. Among them, the most widely used distance measures [10-12] are the Hamming distance, Euclidean distance, and Hausdorff metric. Later on, the distance measures about other extensions of fuzzy sets have also been developed. Later on, the distance measures about other extensions of fuzzy sets have also been developed. For example, Xu [13] introduced the concepts of deviation degrees and similarity degrees between two linguistic values, and between two linguistic preference relations, respectively. Li and Cheng [14] generalized the Hamming distance and the Euclidean distance by adding a parameter and gave a similarity formula for IFSs only based on the membership degrees and non-membership degrees. Hung and Yang [15] and Grzegorzewski [16] suggested a lot of similarity measures for IFSs and interval-valued fuzzy sets based on the Hausdorff metric. Xu and Xia [17] proposed a variety of distance measures for hesitant fuzzy sets. They investigated the connections of the distance measures and further developed a number of hesitant ordered weighted distance measures. Xu and Xia [18] define the distance for hesitant fuzzy information and then discuss their properties in detail. The aforementioned measures, however, cannot be used to deal with the distance measures of dual hesitant fuzzy information. However, little has been done about this issue. Thus it is very necessary to develop some theories about dual hesitant fuzzy sets. To do this, the remainder of the paper is organized as follows. Section 2 presents some basic concepts related to IFSs, HFSs and DHFSs. Section 3 aims to present the axioms for distance measures, gives some new distance measures for DHFSs. In Section 4, proposes an approach to multi-criteria decision making. Section 5 gives some conclusions.

II. Preliminaries

A. IFSs and HFSs

Definition 1 [3]. Let X be a fixed set, an intuitionistic fuzzy set (IFS) A on X is an object having the form:

$$A = \{< x, \mu_A(x), \nu_A(x) > \big| x \in X\}$$

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Lei Wang is with College of Communications Engineering, PLA University of Science and Technology, Nanjing, China. (e-mail: iponly@126.com).

Xiang Zheng is with College of Communications Engineering, PLA University of Science and Technology, Nanjing, China. (e-mail: 13851728157@139.com)

which is characterized by a membership function μ_A and a non-membership function ν_A, where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$, with the condition $0 \le \mu_A(x) + \nu_A(x) \le 1$, $\forall x \in X$. We use $< x, \mu_A, \nu_A >$ for all $x \in X$ to represent IFSs considered in the rest of the paper without explicitly mentioning it.

Hesitant fuzzy sets (HFSs) were first introduced by Torra [4] and Torra and Narukawa [5], it permits the membership degree of an element to a set to be represented as several possible values between 0 and 1:

Definition 2 [4, 5]. Let X be a fixed set, a hesitant fuzzy set (HFS) A on X is in terms of a function that when applied to X returns a subset of $[0,1]$, which can be represented as the following mathematical symbol: $A = \{< x, h_A(x) > \big| x \in X\}$,

Where $h_A(x)$ is a set of values in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set A. For convenience, we call $h_A(x)$ a hesitant fuzzy element (HFE). We use $< x, h_A >$ for all $x \in X$ to represent HFSs.

Where $l_{x_i} = \max\{l(h_A(x_i)), l(h_B(x_i))\}$ for each x_i in X, $l(h_A(x_i))$ *and* $l(h_B(x_i))$ represent the number of values in $h_A(x_i)$ *and* $h_B(x_i)$,respectively. We will talk about l_{x_i} in detail in the next section.

B. Distance and Similarity Measures of IFSs and HFSs

In intuitionistic fuzzy environments, the most widely used distance measures for two IFSs A and B on $X = \{x_1, x_2, \ldots, x_n\}$ are the following [13]:

The normalized Hamming distance:

$$d_{nh}(A,B) = \frac{1}{2n}\sum_{i=1}^{n}\begin{pmatrix} |\mu_A(x_i) - \mu_B(x_i)| \\ + |\nu_A(x_i) - \nu_B(x_i)| \end{pmatrix}; \tag{1}$$

The normalized Euclidean distance:

$$d_{ne}(A,B) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\begin{pmatrix} \left(\mu_A(x_i) - \mu_B(x_i)^2\right) + \\ \left(\nu_A(x_i) - \nu_B(x_i)^2\right) \end{pmatrix}}; \tag{2}$$

The Hausdorff metric:

$$d_{hd}(A,B) = \frac{1}{n}\sum_{i=1}^{n}\max\begin{Bmatrix} |\mu_A(x_i) - \mu_B(x_i)|, \\ |\nu_A(x_i) - \nu_B(x_i)| \end{Bmatrix} \tag{3}$$

A hesitant fuzzy set, allowing the membership of an element to be a set of several possible values, is very useful to express people's hesitancy in daily life. Xu and Xia [17] proposed a variety of distance measures for hesitant fuzzy sets.

$$d_{gnh}(A,B) = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{l_{x_i}}\begin{vmatrix} h_A^{\sigma(j)}(x_j) \\ -h_B^{\sigma(j)}(x_j) \end{vmatrix}^{\lambda}\right)\right]^{1/\lambda}, \textit{where } \lambda > 0; \tag{4}$$

III. Distance and Similarity measures for DHFES

A lot of distance and similarity measures have been developed for FSs, IFSs and HFSs [14-18], but there is little research on DHFSs. Consequently, it is very necessary to develop some distance and similarity measures under dual hesitant fuzzy environment. We first address this issue by putting forward the axioms for distance and similarity measures.

Definition 3 [1]. Let X be a fixed set, then a dual hesitant fuzzy set (DHFS) D on X is described as:

$$D = \{< x, h(x), g(x) >| \; x \in X\}$$

in which $h(x)$ and $g(x)$ are two sets of some values in $[0,1]$, denoting the possible membership degrees and non-membership degrees of the element $x \in X$ to the set D respectively, with the conditions:

$$0 \le \gamma, \eta \le 1, \; 0 \le \gamma^+ + \eta^+ \le 1$$

where, $\gamma \in h(x), \eta \in g(x), \gamma^+ \in h^+(x) = \bigcup_{\gamma \in h(x)} \max\{\gamma\}$, and $\eta^+ \in g^+(x) = \bigcup_{\eta \in g(x)} \max\{\eta\}$ for all $x \in X$. For convenience, the pair $d_E(x) = (h_E(x), g_E(x))$ is called a dual hesitant fuzzy element (DHFE) denoted by $d = (h, g)$.

Definition 4. Let A and B be two DHFSs on $X = \{x_1, x_2, \ldots, x_n\}$, then the distance between A and B denoted as $d(A,B)$, which satisfy the following properties:

1) $0 \le d(A,B) \le 1$;

2) $d(A,B) = 0$ *if only if* $A = B$;

3) $d(A,B) = d(B,A)$.

Definition 5. Let A and B be two DHFSs on $X = \{x_1, x_2, \ldots, x_n\}$, then the similarity measure between A and B is defined as $s(A,B)$, which satisfy the following properties:

1) $0 \le s(A,B) \le 1$;

2) $s(A,B) = 1$ *if only if* $A = B$;

3) $s(A,B) = s(B,A)$.

By analyzing Definitions 2 and 3, we can see the higher the similarity is, the smaller the distance between the two DHFEs. It is noted that $s(A,B) = 1 - d(A,B)$. Accordingly, we mainly discuss the distance measures for DHFSs in this paper, and the corresponding similarity measures can be obtained easily.

We arrange the elements in $d_E(x) = (h_E(x), g_E(x))$ in decreasing order, and let $\gamma_E^{\sigma(i)}(x)$ be the ith largest value in $h_E(x)$ and $\eta_E^{\sigma(j)}(x)$ be the jth largest value in $g_E(x)$. Let $l_h(d_E(x_i))$ be the number of values in $h_E(x_i)$ and $l_g(d_E(x_i))$ be the number of values in $g_E(x_i)$. For convenience, $l(d(x_i)) = \left(l_h(d(x_i)), l_g(d(x_i))\right)$.

In most cases, $l(d_A(x_i)) \ne l(d_B(x_i))$, i.e., $l_h(d_A(x_i)) \ne l_h(d_B(x_i))$, $l_g(d_A(x_i)) \ne l_g(d_B(x_i))$. To operate correctly, we should extend the shorter one until both of them have the same length when we compare them. Xu and Xia [17] extended the shorter one by adding different values in hesitant fuzzy environments. In fact, we can extend the shorter one by adding any value in it. The selection of this value mainly depends on the decision makers' risk preferences. Optimists anticipate desirable outcomes and may add the maximum value, while pessimists expect unfavorable outcomes and may add the minimum value. The same situation can also be found in many existing Refs [19, 20].

We develop a generalized hybrid dual hesitant weighted distance combining the generalized dual hesitant weighted distance and the generalized dual hesitant weighted Hausdorff distance as:

$$d_1(A,B) = \sum_{i=1}^{n}\left(w_i \left(\frac{1}{2l_{x_i}} \left(\sum_{j=1}^{\#h_{x_i}} \left|\gamma_A^{\sigma(j)}(x_i) - \gamma_B^{\sigma(j)}(x_i)\right|^{\lambda} + \sum_{k=1}^{\#g_{x_i}} \left|\eta_A^{\sigma(k)}(x_i) - \eta_B^{\sigma(k)}(x_i)\right|^{\lambda} \right) + \frac{\max\left\{ \max_j \left|\gamma_A^{\sigma(j)}(x_i) - \gamma_B^{\sigma(j)}(x_i)\right|^{\lambda}, \max_k \left|\eta_A^{\sigma(k)}(x_i) - \eta_B^{\sigma(k)}(x_i)\right|^{\lambda} \right\}}{2} \right) \right)^{1/\lambda};$$

$$\text{where } \lambda > 0. \tag{5}$$

$l_{x_i} = (\#h_{x_i}) + (\#g_{x_i})$, $\#h$ and $\#g$ are the numbers of the elements in h and g respectively.

$$d_{1.1}(A,B) = \sum_{i=1}^{n} w_i \left(\frac{1}{2l_{x_i}} \left(\sum_{j=1}^{\#h_{x_i}} \left|\gamma_A^{\sigma(j)}(x_i) - \gamma_B^{\sigma(j)}(x_i)\right| + \sum_{k=1}^{\#g_{x_i}} \left|\eta_A^{\sigma(k)}(x_i) - \eta_B^{\sigma(k)}(x_i)\right| \right) + \frac{\max\left\{ \max_j \left|\gamma_A^{\sigma(j)}(x_i) - \gamma_B^{\sigma(j)}(x_i)\right|, \max_k \left|\eta_A^{\sigma(k)}(x_i) - \eta_B^{\sigma(k)}(x_i)\right| \right\}}{2} \right); \tag{6}$$

$$d_{1.2}(A,B) = \sum_{i=1}^{n}\left(w_i \left(\frac{1}{2l_{x_i}} \left(\sum_{j=1}^{\#h_{x_i}} \left|\gamma_A^{\sigma(j)}(x_i) - \gamma_B^{\sigma(j)}(x_i)\right|^{2} + \sum_{k=1}^{\#g_{x_i}} \left|\eta_A^{\sigma(k)}(x_i) - \eta_B^{\sigma(k)}(x_i)\right|^{2} \right) + \frac{\max\left\{ \max_j \left|\gamma_A^{\sigma(j)}(x_i) - \gamma_B^{\sigma(j)}(x_i)\right|^{2}, \max_k \left|\eta_A^{\sigma(k)}(x_i) - \eta_B^{\sigma(k)}(x_i)\right|^{2} \right\}}{2} \right) \right)^{1/2}; \tag{7}$$

We find that the generalized dual hybrid hesitant weighted distance are one fundamental distance measure, based on which all of the other developed distance measures can be obtained under some special conditions. It is easy to see if $h = \varnothing$ or $g = \varnothing$, the distance measures for dual hesitant fuzzy sets are reduced to the ones for the HFS; If there is only one element in both h and g, the distance measures for dual hesitant fuzzy sets are reduced to the ones for the IFS.

Example. Energy is an indispensable factor for the socio-economic development of societies. Thus the correct energy policy affects economic development and environment, and so, the most appropriate energy policy selection is very important. Suppose that there are five alternatives (energy projects) $Y_i (i = 1,2,3,4,5)$ to be invested, and four attributes $G_j (j = 1,2,3,4)$ to be considered: G_1 : technological; G_2 : environmental; G_3 : socio-political; G_4: economic. The attribute weight vector is $w = (0.15, 0.3, 0.2, 0.35)^T$. Several decision makers are invited to evaluate the performance of the five alternatives. Xu and Xia [17] considered all possible evaluations for an alternative under the attributes as a HFS. Utilizing DHFSs can take much more information into account, the more values we obtain from the decision makers, the greater epistemic certainty we have. So, We use DHHS to deal with such cases: γ_{ij} indicates the degree that the alternative Y_i satisfies the attributes G_j and η_{ij} indicates the degree that the alternative Y_i does not satisfy the attributes G_j. For an alternative under an attribute, although all of the decision makers provide their evaluated values, some of these values may be repeated. However, a value repeated more times does not indicate that it has more importance than other values repeated less times. For example, the value repeated one time may be provided by a decision maker who is an expert at this area, and the value repeated twice may be provided by two decision makers who are not familiar with this area. In such cases, the value repeated one time may be more important than the one repeated twice. To get a more reasonable result, it is better that the decision makers give their evaluations anonymously. We only collect all of the possible values for an alternative under an attribute, and each value provided only means that it is a possible value, but its importance is unknown. Thus the times that the values repeated are unimportant, and it is reasonable to allow these values repeated many times appear only once. The DHFS is just a tool to deal with such cases, and all possible evaluations for an alternative under the attributes can be considered as a DHFS. The results evaluated by the decision makers are contained in a dual hesitant fuzzy decision matrix, shown in Table Ⅰ.

TABLE I

DUAL HESITANT FUZZY DECISION MATRIX.

	G_1	G_2	G_3	G_4
Y_1	{(0.5,.04,0.3); (0.4,0.2)}	{(0.9,0.8,0.7,0.1); (0.1,0)}	{(0.5,.04,0.2); (0.5,0.3,0.1)}	{(0.9,0.6,0.5,0.3); (0.1)}
Y_2	{(0.5,0.3); (0.5,0.4,0.1)}	{(0.9,0.7,0.6,0.5); (0.1)}	{(0.8,0.6,0.5,0.1); (0.2,0.1)}	{(0.7,0.4,0.3); (0.2,0)}
Y_3	{(0.7,0.6); (0.2)}	{(0.9,0.6); (0)}	{(0.7,0.5,0.3); (0.3,0.2)}	{(0.6,0.4); (0.3,0.1)}
Y_4	{(0.8,0.7,0.4,0.3); (0.2,0.1)}	{(0.7,0.4,0.2); (0.3,0.2)}	{(0.8,0.1); (0.1)}	{(0.9,0.8,0.6); (0.1,0)}
Y_5	{(0.9,0.7,0.6,0.3); (0.1)}	{(0.8,0.7,0.6,0.4); (0.2,0)}	{(0.9,0.8,0.7); (0)}	{(0.9,0.7,0.6,0.3); (0.1)}

Suppose that the ideal alternative is $A = \{\{(1)\},\{0\}\}$ seen as a special DHFS, we can calculate the distance between each alternative and the ideal alternative using our distance measures.

If we use the the generalized dual hybrid hesitant weighted distance to calculate the deviations between each alternative and the ideal alternative, then we get the rankings of these alternatives, which are listed in Tables 2, when some values of the parameter are given. We find that the rankings are different as the parameter (which can be considered as the decision makers' risk attitude) changes, consequently, the proposed distance measures can provide the decision makers more choices as the different values of the parameter are given according to the decision makers' risk attitudes.

Table Ⅱ
RESULTS OBTAINED BY THE GENERALIZED DUAL HYBRID HESITANT WEIGHTED DISTANCE.

	Y_1	Y_2	Y_3	Y_4	Y_5	Rankings
λ=1	0.3713	0.3787	0.2975	0.3205	0.2895	$Y_5 \succ Y_3 \succ Y_4 \succ Y_1 \succ Y_2$
λ=2	0.4451	0.4529	0.3523	0.4143	0.3660	$Y_3 \succ Y_5 \succ Y_4 \succ Y_1 \succ Y_2$
λ=3	0.5090	0.5053	0.3906	0.4889	0.4265	$Y_3 \succ Y_5 \succ Y_4 \succ Y_1 \succ Y_2$
λ=4	0.6558	0.5495	0.4211	0.5706	0.4758	$Y_3 \succ Y_5 \succ Y_4 \succ Y_1 \succ Y_2$
λ=5	0.5952	0.5776	0.4466	0.5892	0.5161	$Y_3 \succ Y_5 \succ Y_4 \succ Y_1 \succ Y_2$

Furthermore, Xu and Chen [22] defined several ordered weighted distance measures whose prominent characteristic is that they can alleviate (or intensify) the influence of unduly large (or small) deviations on the aggregation results by assigning them low (or high) weights. Yager [23] generalized Xu and Chen' distance measures and provided a variety of ordered weighted averaging norms, based on which he proposed several similarity measures. Merigó and Gil-Lafuente [24] introduced an ordered weighted averaging distance operator and gave its application in the selection of financial products. Xu and Xia [17] developed some ordered distance measures for HFSs. Motivated by the ordered weighted idea [17, 23], we defined a generalized dual hesitant ordered weighted distance measure:

$$d_2(A,B)=\sum_{i=1}^{n}\left(w_i\left(\frac{1}{l_{x_{\hat\sigma(i)}}}\left(\begin{array}{l}\sum_{j=1}^{\#h_{x_{\hat\sigma(i)}}}\left|\gamma_A^{\sigma(j)}(x_{\hat\sigma(i)})-\gamma_B^{\sigma(j)}(x_{\hat\sigma(i)})\right|^{\lambda}\\+\sum_{k=1}^{\#g_{x_{\hat\sigma(i)}}}\left|\eta_A^{\sigma(k)}(x_{\hat\sigma(i)})-\eta_B^{\sigma(k)}(x_{\hat\sigma(i)})\right|^{\lambda}\end{array}\right)\right)\right)^{1/\lambda};\quad (8)$$

Where $\lambda>0$, And $\hat\sigma:(1,2,3,\ldots n)\to(1,2,3,\ldots n)$ be a permutation satisfying:

$$\frac{1}{l_{x_{\hat\sigma(i)}}}\left(\begin{array}{l}\sum_{j=1}^{\#h_{x_{\hat\sigma(i)}}}\left|\gamma_A^{\sigma(j)}(x_{\hat\sigma(i)})-\gamma_B^{\sigma(j)}(x_{\hat\sigma(i)})\right|^{\lambda}\\+\sum_{k=1}^{\#g_{x_{\hat\sigma(i)}}}\left|\eta_A^{\sigma(k)}(x_{\hat\sigma(i)})-\eta_B^{\sigma(k)}(x_{\hat\sigma(i)})\right|^{\lambda}\end{array}\right)$$
$$\le\frac{1}{l_{x_{\hat\sigma(i+1)}}}\left(\begin{array}{l}\sum_{j=1}^{\#h_{x_{\hat\sigma(i+1)}}}\left|\gamma_A^{\sigma(j)}(x_{\hat\sigma(i+1)})-\gamma_B^{\sigma(j)}(x_{\hat\sigma(i+1)})\right|^{\lambda}\\+\sum_{k=1}^{\#g_{x_{\hat\sigma(i+1)}}}\left|\eta_A^{\sigma(k)}(x_{\hat\sigma(i+1)})-\eta_B^{\sigma(k)}(x_{\hat\sigma(i+1)})\right|^{\lambda}\end{array}\right),$$
$$i=1,\ 2,\ldots n-1$$

Another important issue is the determination of the weight vectors associated with the ordered weighted distance measures. Inspired by Xu and Xia [17, 18], below we give three ways to determine the weight vectors. Considering each element in A and B as a special DHFS, $d(d_A(x_{\rho(i)}),d_B(x_{\rho(i)}))$, $i=1,2,\ldots n$ as given above, and denoting $\hat\sigma,\tilde\sigma\ and\ \dot\sigma$ as ρ, we have

(1)Let

$$w_i=\frac{d(d_A(x_{\rho(i)}),d_B(x_{\rho(i)}))}{\sum_{k=1}^{n}d(d_A(x_{\rho(k)}),d_B(x_{\rho(k)}))},i=1,2,\ldots,n,\quad (9)$$

$then$

$$w_{i+1}\ge w_i\ge 0,i=1,2,\ldots n-1,and\ \sum_{i=1}^{n}w_i=1$$

(2)Let

$$w_i=\frac{e^{-d(d_A(x_{\rho(i)}),d_B(x_{\rho(i)}))}}{\sum_{k=1}^{n}e^{-d(d_A(x_{\rho(k)}),d_B(x_{\rho(k)}))}},\ i=1,2,\ldots n,\quad (10)$$

$then$

$$w_{i+1}\ge w_i\ge 0,i=1,2,\ldots n-1,and\ \sum_{i=1}^{n}w_i=1$$

(3)Let

$$\bar d(d_A,d_B)=\frac{1}{n}\sum_{k=1}^{n}d(d_A(x_{\rho(k)}),d_B(x_{\rho(k)})),$$

$$\ddot d(d(d_A(x_{\rho(i)}),d_B(x_{\rho(k)})),\bar d(d_A,d_B))=\left|\begin{array}{l}d(d_A(x_{\rho(i)}),d_B(x_{\rho(k)}))\\-\frac{1}{n}\sum_{k=1}^{n}d(d_A(x_{\rho(k)}),d_B(x_{\rho(k)}))\end{array}\right|,$$

$then\ we\ define$

$$w_i=\frac{1-\ddot d(\bar d(d_A,d_B),d(d_A(x_{\rho(i)}),d_B(x_{\rho(i)})))}{\sum_{k=1}^{n}\left(1-\ddot d(\bar d(d_A,d_B),d(d_A(x_{\rho(i)}),d_B(x_{\rho(i)})))\right)}$$

$$=\frac{1-\left|d(d_A(x_{\rho(i)}),d_B(x_{\rho(i)}))-\frac{1}{n}\sum_{k=1}^{n}d(d_A(x_{\rho(k)}),d_B(x_{\rho(k)}))\right|}{\sum_{k=1}^{n}\left(1-\left|d(d_A(x_{\rho(i)}),d_B(x_{\rho(i)}))-\frac{1}{n}\sum_{k=1}^{n}d(d_A(x_{\rho(k)}),d_B(x_{\rho(k)}))\right|\right)},$$

$$i=1,2,\ldots n\ \ from\ which\ we\ get\ \ w_i\ge 0,\sum_{i=1}^{n}w_i=1\quad (11)$$

We find that the weight vector derived from (9) is a monotonic decreasing sequence, the weight vector derived from (8) is a monotonic increasing sequence, and the weight vector derived from (11) combine the above two cases, i.e., the closer the value $d(d_A(x_{\rho(i)}),d_B(x_{\rho(i)}))$ to the mean $\frac{1}{n}\sum_{k=1}^{n}d(d_A(x_{\rho(k)}),d_B(x_{\rho(k)}))$, the larger the weight w_i. In the aforementioned example, if the attribute weight vector is unknown, then we can use the ordered weighted distance measures to calculate the distance between each alternative and the ideal alternative.

IV. APPROACHES TO MULTIPLE ATTRIBUTE GROUP DECISION MAKING WITH DUAL HESITANT FUZZY INFORMATION

Let $X=\{x_1,x_2,\ldots,x_n\}$ be a set of n alternatives, $G=\{g_1,g_2,\ldots,g_m\}$ be a set of m attributes, whose weight vector is $w=(w_1,w_2,\ldots,w_m)^T$, with $w_i\in[0,1]\ for\, i=1,2,\ldots,m, and \sum_{i=1}^{m}w_i=1$ .and let $E=\{e_1,e_2,\ldots,e_s\}$ be a set of s decision makers, whose weight vector is $\omega=(\omega_1,\omega_2,\ldots,\omega_s)^T$,with $\omega_t\in[0,1]\ for\, t=1,2,\ldots,s,$ and $\sum_{t=1}^{s}\omega_t=1$.

Let $A^{(t)}=\left(\alpha_{ij}^{(t)}\right)_{m\times n}$ be an dual hesitant fuzzy decision matrix, where

$$\alpha_{ij}^{(t)}=\{\{h_{\alpha_{ij}^{(t)}}\},\{g_{\alpha_{ij}^{(t)}}\}\}=\left\{\left\{\bigcup_{\gamma_{\alpha_{ij}^{(t)}}\in h_{\alpha_{ij}^{(t)}}}\left(\gamma_{\alpha_{ij}^{(t)}}\right)\right\},\left\{\bigcup_{\eta_{\alpha_{ij}^{(t)}}\in g_{\alpha_{ij}^{(t)}}}\left(\eta_{\alpha_{ij}^{(t)}}\right)\right\}\right\}$$

is an attribute value provided by the decision maker e_t, denoted by a DHFE, where $\left\{\bigcup_{\gamma_{\alpha_{ij}^{(t)}}\in h_{\alpha_{ij}^{(t)}}}\left(\gamma_{\alpha_{ij}^{(t)}}\right)\right\}$ indicates all of the possible degree that the alternative x_j satisfies the attribute g_i, while $\left\{\bigcup_{\eta_{\alpha_{ij}^{(t)}}\in g_{\alpha_{ij}^{(t)}}}\left(\eta_{\alpha_{ij}^{(t)}}\right)\right\}$ indicates all of the possible degree that the alternative x_j does not satisfies the attribute g_i. When all the performances of the alternatives are provided, the dual hesitant fuzzy decision matrix $A^{(t)}=\left(\alpha_{ij}^{(t)}\right)_{m\times n}$ can be constructed.

To obtain the ranking of the alternatives, we improve the method of Xu [22] and Wang [25]:

Step 1. Transform the dual hesitant fuzzy decision matrix $A^{(t)}$ into the normalized dual hesitant fuzzy decision matrix $B^{(t)}=\left(\beta^{(t)}\right)_{m\times n}$ where:

$$\beta_{ij}^{(t)}=\begin{cases}\alpha_{ij}^{(t)}, & \text{for benefit attribute } \mathrm{g}_i\\ \left(\alpha_{ij}^{(t)}\right)^c, & \text{for cost attribute } \mathrm{g}_i\end{cases},$$

$i=1,2,\ldots,m,\ j=1,2,\ldots,n.$

Step 2. Calculate the supports [25]

$$Sup\left(\beta_{ij}^{(t)},\beta_{ij}^{(p)}\right)=1-d\left(\beta_{ij}^{(t)},\beta_{ij}^{(p)}\right).$$

$t,p=1,2,\ldots s,\ i=1,2,\ldots m,\ j=1,2,\ldots n$

Step 3. Utilize the weights ω_t of the decision makers e_t $(t=1,2,\ldots,s)$ to calculate the weighted support $T\left(\left(\beta_{ij}^{(t)}\right)\right)$ of DHFE $\left(\beta_{ij}^{(t)}\right)$ by the other DHFEs $\left(\beta_{ij}^{(p)}\right)$ $(p=1,2,\ldots,s, and\ p\neq t)$

$$T\left(\beta_{ij}^{(t)}\right)=\sum_{\substack{p=1\\p\neq t}}^{s}\omega_p Sup\left(\beta_{ij}^{(t)},\beta_{ij}^{(p)}\right),$$

$t=1,2,\ldots s,\ i=1,2,\ldots m,\ j=1,2,\ldots n$

Step 4.Utilize the WGDHFPA operator [25] to obtain the hesitant fuzzy elements for the alternatives

$$\beta_{ij}=WGDHPA\left(\beta_{ij}^{(1)},\beta_{ij}^{(2)},\ldots,\beta_{ij}^{(s)}\right)$$

$$=\left(\frac{\oplus_{t=1}^{s}\omega_t\left(1+T\left(\beta_{ij}^{(t)}\right)\right)\left(\beta_{ij}^{(t)}\right)^{\lambda}}{\sum_{t=1}^{s}\omega_t\left(1+T\left(\beta_{ij}^{(t)}\right)\right)}\right)^{1/\lambda}$$

to aggregate all of the individual dual hesitant fuzzy decision matrices $B^{(t)}=\left(\beta_{ij}^{(t)}\right)_{m\times n}(t=1,2,\ldots,s)$ into the collective dual hesitant fuzzy decision matrix $B=\left(\beta_{ij}\right)_{m\times n}$.

Step 5. Calculate the score function of each attribute for the alternatives.

$s=(s(\beta_{ij}))_{n\times m}.\ i=1,2,\ldots m,\ j=1,2,\ldots n.$

Step 6. Utilize the weights w_i of attributes $g_i\,(i=1,2,\ldots,m)$ to calculate total score for all alternatives:

$$S_j=\sum_{i=1}^{m}w_i s(\beta_{ij}).\ i=1,2\ldots,m,\ j=1,2,\ldots,n.$$

Step 8. Rank the S_j $(j=1,2,\ldots,n)$ in descending order.

Step 9. Rank all of the alternatives x_j $(j=1,2,\ldots,n)$ and then select the best alternative in accordance with the collective overall preference values S_j $(j=1,2,\ldots,n)$.

As an illustrative example, consider the air defense of a naval battle group. Four missiles (alternatives) $x_i\,(i=1,2,3,4)$ remain on the candidate list. Three expert teams $e_k\,(k=1,2,3)$ to act as decision makers, whose weight vector is $w=(0.3,0.3,0.4)^T$. Four attributes are under consideration: (1) Basic capabilities, (G_1) ; (2) Operational capabilities (G_2) (3) Costs and technical effects (G_3) (4) Reliability (G_4). We choose a "perfect" missile which its four criteria are perfect. The weight vector of the attributes $G_j\,(j=1,2,3,4)$ is $w=(0.15,0.20,0.20,0.45)^T$. The experts $e_k\,(k=1,2,3)$ evaluate the missiles (alternatives) $x_i\,(i=1,2,3,4)$ with respect to the attributes $G_j\,(j=1,2,3,4)$, and construct the following three dual hesitant fuzzy decision matrices $A_k=\left(a_{ij}^{(k)}\right)_{4\times4}(k=1,2,3)$ (see Table Ⅲ). Among the considered attributes, $G_1\ and\ G_2$ are the cost attributes, $G_3\ and\ G_4$ are the benefit attributes. We transform the attribute values of cost type into the attribute values of benefit type, then $A_k=\left(a_{ij}^{(k)}\right)_{4\times4}(k=1,2,3)$ are transformed into $R_k=\left(r_{ij}^{(k)}\right)_{4\times4}(k=1,2,3)$. To obtain the ranking of the alternatives, we improve the method of Xu [22] and Wang [25] (see TableⅣ).

Table III

DUAL FUZZY DECISION MATRICE R

	G_1	G_2	G_3	G_4
A_1	{{0.5,0.4,0.3}, {0.4,0.2}}	{{0.6,0.5}, {0.3,0.2,0.1}}	{{0.6,0.4,0.3}, {0.4,0.2,0.1}}	{{0.6}, {0.4}}
A_2	{{0.8,0.7,0.6}, {0.2,0.1}}	{{0.7,0.6}, {0.3,0.2,0.1}}	{{0.7,0.6,0.5}, {0.3,0.2,0.1}}	{{0.7}, {0.2}}
A_3	{{0.9,0.8,0.7}, {0.1,0.0}}	{{0.8,0.7}, {0.2,0.1,0.0}}	{{0.8,0.7,0.6}, {0.2,0.1,0.0}}	{{0. 9}, {0.1}}
A_4	{{0.4,0.3,0.1}, {0.6,0.5}}	{{0.6,0.5}, {0.4,0.2,0.1}}	{{0.6,0.5,0.4}, {0.3,0.2,0.1}}	{{0.3}, {0.6}}

TableⅣ

SCORE VALUES OBTAINED BY ATS-WGDHFPA OPERATOR BASED ON THE GENERALIZED DUAL HESITANT WEIGHTED HAUSDORFF DISTANCE AND THE RANKING OF ALTERNATIVES.

d_1	λ=0.05	λ=0.1	λ=1	λ=10	λ=100
A_1	0.2104	0.2107	0.2157	0.2675	0.1032
A_2	0.4831	0.4833	0.4864	0.5207	0.5530
A_3	0.6528	0.6529	0.6551	0.6784	0.7081
A_4	-0.0661	-0.0658	-0.0604	-0.0115	-0.5224

We now present a figure to clearly demonstrate how the score values vary as the parameter λ increases and the aggregation arguments are kept fixed (see Fig. 1).

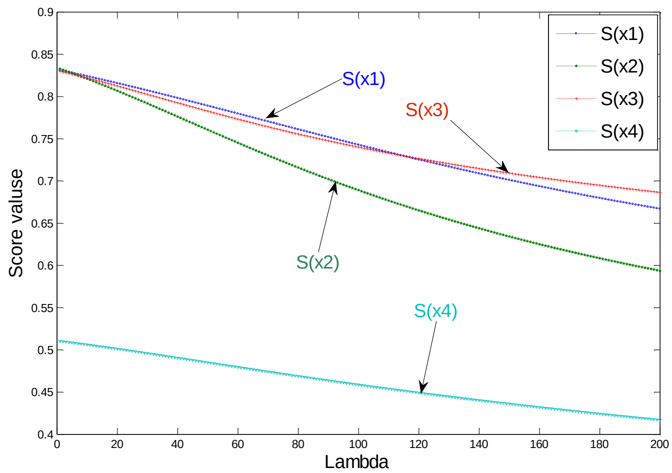

Fig. 1.Variation of the score value with respect to the parameter λ.

V. CONCLUSION

In this paper, we have given a further study about the distance measures for DHFSs. Based on ideas of the well-known Hamming distance, the Euclidean distance, the Hausdorff metric and their generalizations, we have developed a class of dual hesitant distance measures, and discussed their properties and relations as their parameters change. We have also given a variety of ordered weighted distance measures for DHFSs in which the distances are rearranged in decreasing order, and given three ways to determine the associated weighting vectors. With the relationship between distance measures and similarity measures, the corresponding similarity measures for DHFSs have been obtained. It should be pointed out that all of the above measures are based on the assumption that if the corresponding DHFEs in DHFSs do not have the same length, then the shorter one should be extended by adding the minimum value in it until both the DHFEs have the same length. In fact, we can extend the shorter DHFE by adding any value in it until it has the same length of the longer one according to the decision makers' preferences and actual situations. Finally, an approach for multi-criteria decision making has been developed based on the proposed distance measures under dual hesitant fuzzy environments.

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