Neimark-Sacker Bifurcation For A Discrete-Time Neural Network Model With Multiple Delays

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Abstract—**In this paper, we investigate a discrete-time neural network model with multiple delays. By analyzing the corresponding characteristic equation of the system, we discuss the stability and the existence of Neimark-Sacker bifurcation for the model. Applying the normal form method and the center manifold theory for discrete time system developed by Kuznetsov, we derive the explicit formula for determining the direction of Neimark-Sacker bifurcation and the stability of bifurcating periodic solution. Some numerical simulations which are in a good agreement with our theoretical analysis are carried out. Our results are new and complement previously known studies.**

Index Terms—**Neural network, stability, Neimark-Sacker bifurcation, time delay.**

I. Introduction

IT is well known that neural networks have potential applications in many fields such as optimization, image processing, signal processing, pattern recognition, associative memory, solving nonlinear algebraic equations and so on. Thus the dynamical analysis of neural networks has attracted a great attention in recent years [1-4]. In particular, discrete time neural networks governed by difference equations are more appropriate to describe the dynamics of neurons. Moreover, discrete time neural networks can also provide efficient models of continuous ones for numerical simulations. In recent years, there are many papers that deal with this aspect. For example, Yuan et al. [3-4] analyzed the Neimark-Sacker bifurcation of two classes of discrete neural networks, He and Cao [5] studied the stability and Neimark-Sacker bifurcation for a class of discrete-time neural networks, Xiao and Cao [6] considered the Neimark-Sacker bifurcation of a discrete-time tabu learning model, Huang et al. [7] discussed the Neimark-Sacker bifurcation of a discrete-time financial system, Dobrescu and Opris [8] made a theoretical discussion on the Neimark-Sacker bifurcation for discrete-delay Kaldor-Kalecki models. For more related work on the Neimark-Sacker bifurcation of discrete-time models, one can see [9-26].

On the other hand, considering the finite speeds of the switching and the transmission of signals of networks, we think that it is reasonable to introduce time delays into neural networks. Inspired by the discussion above, we consider the following discrete-time neural networks with multiple delays

$$\begin{cases} u_1(n+1) = \alpha u_1(n) + (1-\alpha)f_1(\beta u_1(n-\tau_1)) \\ \qquad\qquad + (1-\alpha)f_2(\gamma_1 u_2(n-\tau_2)), \\ u_2(n+1) = \alpha u_2(n) + (1-\alpha)f_3(\gamma_2 u_1(n-\tau_3)) \\ \qquad\qquad + (1-\alpha)f_4(\beta u_2(n-\tau_4)), \end{cases} \tag{1}$$

where $u_i(t)(i = 1,2)$ represents the ith activity of the neuron; $\alpha \in (0,1)$ denotes internal decay of neurons, $\beta \geq 0$ and $\gamma_i(i = 1,2)$ represent the gain parameters; $f_i : R \to R(i = 1,2,3,4)$ is a continuous transfer function and $f_i(0) = 0$. $\tau_i \geq 0(i = 1,2,3,4)$ is a delay.

In this paper, we make an attempt to discuss the Neimark-Sacker bifurcation of system (1). Here we shall point out that although there are many articles that investigate the Neimark-Sacker bifurcation of various neural networks, these papers are only concerned with neural networks without time delays. To the best of our knowledge, there are very few papers that investigate the Neimark-Sacker bifurcation of neural networks with delays. We believe that our results are new and complement previously known studies.

The rest of this paper is arranged as follows. In Section 2, we present some sufficient conditions for the asymptotical stability of the zero equilibrium and the existence of Neimark-Sacker bifurcation of (1). In Section 3, the direction and stability of the Neimark-Sacker bifurcation are analyzed by applying the normal form theory and the center manifold theorem. In Section 4, Some computer simulations are carried out to support the theoretical findings.

II. Stability and existence of Neimark-Sacker bifurcation

In this section, we will consider the local stability of the zero equilibrium and the existence of Neimark-Sacker bifurcation of system (1). First we make the following assumptions.

(A1) For $i = 1,2,3,4, f_i \in C^1(R)$ and $f_i(0) = 0$.

The linearization of system (1) near the zero equilibrium takes the form

$$\begin{cases} u_1(n+1) = \alpha u_1(n) + \beta(1-\alpha)f_1^{'}(0)u_1(n-\tau_1) \\ \qquad\qquad + \gamma_1(1-\alpha)f_2^{'}(0)u_2(n-\tau_2), \\ u_2(n+1) = \alpha u_2(n) + \gamma_2(1-\alpha)f_3^{'}(0)u_1(n-\tau_3) \\ \qquad\qquad + \beta(1-\alpha)f_4^{'}(0)u_2(n-\tau_4). \end{cases} \tag{2}$$

The characteristic equation of (2) is

$$\det \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} = 0, \tag{3}$$

where $K_1 = \lambda - (\alpha + \beta(1-\alpha)f_1^{'}(0)e^{-\lambda\tau_1})$, $K_2 = -\gamma_1(1-\alpha)f_2^{'}(0)e^{-\lambda\tau_2}$, $K_3 = -\gamma_2(1-\alpha)f_3^{'}(0)e^{-\lambda\tau_3}$, $K_4 = \lambda - (\alpha +$

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$\beta(1-\alpha)f_4^{'}(0)e^{-\lambda\tau_4})$. Then we have

$$\lambda^2 - 2(\alpha+\nu)\lambda + \alpha^2 + 2\alpha\nu + \mu = 0, \quad (4)$$

where

$$\nu = \frac{1}{2}[\beta(1-\alpha)f_1^{'}(0)e^{-\lambda\tau_1} + \beta(1-\alpha)f_4^{'}(0)e^{-\lambda\tau_4}],$$
$$\mu = \beta^2(1-\alpha)^2 f_1^{'}(0)f_4^{'}(0)e^{-\lambda(\tau_1+\tau_4)}$$
$$-\gamma_1\gamma_2(1-\alpha)^2 f_2^{'}(0)f_3^{'}(0)e^{-\lambda(\tau_2+\tau_3)}.$$

For $\nu \in (-1-\alpha, 1-\alpha)$, we denote

$$\Theta = \{(\nu,\mu) \in R^2 : P_1 < 0, P_2 < 0, P_3 > 0\}, \quad (5)$$

where

$$P_1 = 2(1-\alpha)\nu - (1-\alpha)^2 - \mu,$$
$$P_2 = -2(1+\alpha)\nu - (1+\alpha)^2 - \mu,$$
$$P_3 = -2\alpha\nu + 1 - \alpha^2 - \mu.$$

Theorem 2.1 *Assume that (A1) holds and $(\nu,\mu) \in \Theta$, then the zero equilibrium of system (1) is asymptotically stable.*

Proof We consider two cases.

(1) If $\nu^2 \geq \mu$. In this case, it follows from (4) that

$$\lambda_1 = \alpha + \nu + \sqrt{\nu^2 - \mu}, \lambda_2 = \alpha + \nu - \sqrt{\nu^2 - \mu}. \quad (6)$$

It is easy to see that the eigenvalues $\lambda_{1,2}$ of (4) are inside the unit circle if and only if

$$(\nu,\mu) \in \Theta_1 \cap \Theta_2, \quad (7)$$

where

$$\Theta_1 := \{(\nu,\mu) \in R^2 : \mu > 2(1-\alpha)\nu - (1-\alpha)^2,$$
$$\nu < 1-\alpha, \nu^2 \geq \mu\}, \quad (8)$$
$$\Theta_2 := \{(\nu,\mu) \in R^2 : \mu > -2(1+\alpha)\nu - (1+\alpha)^2,$$
$$\nu > -1-\alpha, \nu^2 \geq \mu\}. \quad (9)$$

(2) If $\nu^2 < \mu$. In this case, it follows from (4) that

$$\lambda_1 = \alpha + \nu + \sqrt{\mu - \nu^2}i, \lambda_2 = \alpha + \nu - \sqrt{\mu - \nu^2}i. \quad (10)$$

It is easy to see that the eigenvalues $\lambda_{1,2}$ of (4) are inside the unit circle if and only if

$$(\nu,\mu) \in \Theta_3, \quad (11)$$

where

$$\Theta_3 := \{(\nu,\mu) \in R^2 : \mu < -2\alpha\nu + 1 - \alpha^2, \nu^2 < \mu\}. \quad (12)$$

In view of two cases above, we can easily know that $\Theta = (\Theta_1 \cap \Theta_2) \cup \Theta_3$. Then we can know that λ_1 and λ_2 are inside the unit circle if $(\nu,\mu) \in \Theta$. Thus we can conclude that the zero equilibrium of system (1) is asymptotically stable. The proof of Theorem 2.1 is completed.

Next we regard μ as the bifurcation parameter to discuss the Neimark-Sacker bifurcation of zero equilibrium. For $\nu^2 < \mu$, we let

$$\lambda(\mu) = \alpha + \nu + \sqrt{\mu - \nu^2}i. \quad (13)$$

Obviously, the eigenvalues of (4) are conjugate complex pair $\lambda(\mu)$ and $\overline{\lambda(\mu)}$. Hence we have

$$|\lambda| = \sqrt{\alpha^2 + 2\alpha\nu + \mu}. \quad (14)$$

Thus we have $|\lambda| = 1$ if and only if

$$\mu = \mu_0 := -2\alpha\nu + 1 - \alpha^2. \quad (15)$$

It is easy to see that for $\nu^2 < \mu < \mu_0, |\lambda| < 1$. Noticing that $|\lambda(\mu_0)| = 1$, we can conclude that μ_0 is a critical value that destroys the stability of zero equilibrium.

Lemma 2.1. *Assume that $(A1)$ holds and $-\alpha < \nu < 1-\alpha$, then*
(i) $\left[\frac{d|\lambda(\mu)|}{d\mu}\right]_{\mu=\mu_0} > 0,$
(ii) $\lambda^k(\mu_0) \neq 1$ *for* $k = 1,2,3,4,$
where $\lambda(\mu)$ and μ_0 are defined by (13) and (15), respectively.

Proof In view of $\nu \in (-\alpha, 1-\alpha)$, we know that $\nu^2 < \mu_0$. It follows from (14) and (15) that

$$\left[\frac{d|\lambda(\mu)|}{d\mu}\right]_{\mu=\mu_0} = \frac{1}{2}\frac{1}{\sqrt{\alpha^2 + 2\alpha\nu + \mu}}\Big|_{\mu=\mu_0} = \frac{1}{2} > 0. \quad (16)$$

Then (i) holds true. Next we consider (ii). $\lambda^k(\mu_0) = 1 (k = 1,2,3,4)$ if and only if the argument $\arg\lambda(\mu_0) \in \{0, \pm\frac{\pi}{2}, \pm\frac{2\pi}{3}, \pi\}$. In view of $\nu^2 < \mu$, (15) and

$$\lambda(\mu_0) = \alpha + \nu + \sqrt{\mu_0 - \nu^2}i. \quad (17)$$

we have

$$|\lambda(\mu_0)| = 1, \mathrm{Re}\lambda(\mu_0) > 0, \mathrm{Im}\lambda(\mu_0) > 0. \quad (18)$$

Then

$$\arg\lambda(\mu_0) \in \{0, \pm\frac{\pi}{2}, \pm\frac{2\pi}{3}, \pi\}$$

does not hold. Thus we can conclude that $\lambda^k(\mu_0) \neq 1$ for $k = 1,2,3,4$. This completes the proof of Lemma 2.1.

According to the analysis above, we have the following results.

Theorem 2.2. *Let μ_0 be defined by (15). Assume that $(A1)$ holds and $\nu \in (-\alpha, 1-\alpha)$. Then we have the following results.*
(i) If $\mu > \mu_0$, then the zero equilibrium of (1) is unstable;
(ii) If $\nu^2 < \mu < \mu_0$, then the zero equilibrium of (1) is asymptotically stable;
(iii) The Neimark-Sacker bifurcation occurs near $\mu = \mu_0$, i.e., system (1) has a unique closed invariant curve bifurcation from the zero equilibrium around $\mu = \mu_0$.

III. Properties of Neimark-Sacker bifurcation

In this section, we will consider the properties of Neimark-Sacker bifurcation by applying the normal form method and the center manifold theory for discrete-time system developed by Kuznetsov [15]. In order to obtain our main results, we make the following assumption.

(A2) For $i = 1,2,3,4, f_i \in C^{(3)}(R,R), f_i(0) = f_i^{''}(0) = 0$ and $f_i^{'}(0)f_i^{'''}(0) \neq 0$.

We can rewrite system (1) as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \rightarrow$$
$$\begin{bmatrix} \alpha + \beta(1-\alpha)f_1^{'}(0)e^{-\lambda\tau_1} & \gamma_1(1-\alpha)f_2^{'}(0)e^{-\lambda\tau_2} \\ \gamma_2(1-\alpha)f_3^{'}(0)e^{-\lambda\tau_3} & \alpha + \beta(1-\alpha)f_4^{'}(0)e^{-\lambda\tau_4} \end{bmatrix}$$
$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} G_1(u,\mu) \\ G_2(u,\mu) \end{bmatrix}, \quad (19)$$

where $u=(u_1,u_2)^T\in R^2$ and

$$G_1(u,\mu)=\frac{1}{6}f_1^{'''}(0)(1-\alpha)\beta^3u_1^3 +\frac{1}{6}f_2^{'''}(0)(1-\alpha)\gamma_1^3u_2^3+O(||u||^4),$$
$$G_2(u,\mu)=\frac{1}{6}f_3^{'''}(0)(1-\alpha)\gamma_2^3u_1^3 +\frac{1}{6}f_4^{'''}(0)(1-\alpha)\gamma_2^3u_2^3+O(||u||^4).$$

Let $B(\mu)=$

$$\left[\begin{array}{cc}\alpha+\beta(1-\alpha)f_1^{'}(0)e^{-\lambda\tau_1} & \gamma_1(1-\alpha)f_2^{'}(0)e^{-\lambda\tau_2}\\ \gamma_2(1-\alpha)f_3^{'}(0)e^{-\lambda\tau_3} & \alpha+\beta(1-\alpha)f_4^{'}(0)e^{-\lambda\tau_4}\end{array}\right],$$

$$\varrho_1=\nu+\sqrt{\mu-\nu^2}i-\beta(1-\alpha)f_1^{'}(0)e^{-\lambda\tau_1},$$

$$\varrho_2=\nu+\sqrt{\mu-\nu^2}i-\beta(1-\alpha)f_4^{'}(0)e^{-\lambda\tau_4}.$$

In view of the definition of ν, we get $\bar{\varrho}_1=\varrho_2$. Let $q(\mu)\in C^2$ be an eigenvector of $B(\mu)$ corresponding to eigenvalue $\lambda(\mu)$, then we have

$$B(\mu)q(\mu)=\lambda(\mu)q(\mu). \tag{20}$$

Let $p(\mu)\in C^2$ be an eigenvector of the transposed matrix $B^T(\mu)$ corresponding to its eigenvalue, then we get

$$B^T(\mu)p(\mu)=\overline{\lambda(\mu)}p(\mu). \tag{21}$$

Then

$$q\sim\left(1,\frac{\gamma_2(1-\alpha)f_3^{'}(0)e^{-\lambda\tau_3}}{\varrho_2}\right)^T, \tag{22}$$

$$p\sim\left(1,\frac{\gamma_1(1-\alpha)f_2^{'}(0)e^{-\lambda\tau_2}}{\bar{\varrho}_2}\right)^T, \tag{23}$$

Let

$$q=\left(1,\frac{\gamma_2(1-\alpha)f_3^{'}(0)e^{-\lambda\tau_3}}{\varrho_2}\right)^T, \tag{24}$$

$$p=\frac{\bar{r}_2}{\bar{r}_2-r_2}\left(1,\frac{\gamma_1(1-\alpha)f_2^{'}(0)e^{-\lambda\tau_2}}{\bar{\varrho}_2}\right)^T. \tag{25}$$

Then we get $\langle p,q\rangle=1$, where $\langle .,.\rangle$ means the standard scalar product in $C^2:\langle p,q\rangle=\bar{p}_1q_1+\bar{p}_2q_2$. any vector $u\in R^2$ can be represented as

$$u=wq(\mu)+\overline{wd(\mu)}, \tag{26}$$

for some complex w. Clearly,

$$w=\langle p(\mu),u\rangle. \tag{27}$$

Then (19) can be transformed into the following form

$$w\rightarrow\lambda(\mu)w+g(w,\bar{w},\mu), \tag{28}$$

where $\lambda(\mu)$ can be written as $\lambda(\mu)=(1+\varphi(\mu))e^{i\theta(\mu)}$, $\varphi(\mu)$ is a smooth function with $\varphi(\mu_0=0$ and

$$g(w,\bar{w},\mu)=\sum_{k+l\geq2}\frac{1}{k!l!}g_{kl}(\mu)w^k\bar{w}^l. \tag{29}$$

By (H2), we have

$$G_1(\xi,\mu)=\frac{1}{6}f_1^{'''}(0)(1-\alpha)\beta^3\xi_1^3 +\frac{1}{6}f_2^{'''}(0)(1-\alpha)\gamma_1^3\xi_2^3+O(||\xi||^4), \tag{30}$$

$$G_2(\xi,\mu)=\frac{1}{6}f_3^{'''}(0)(1-\alpha)\gamma_2^3\xi_1^3 +\frac{1}{6}f_4^{'''}(0)(1-\alpha)\gamma_2^3\xi_2^3+O(||\xi||^4). \tag{31}$$

Then

$$B_i(\sigma,\varsigma):=\sum_{j,k=1}^{2}\frac{\partial^2G_i(\xi,\mu_0)}{\partial\xi_j\partial\xi_k}\Big|_{\xi=0}\times\sigma_j\varsigma_k=0,i=1,2, \tag{32}$$

$$C_1(\sigma,\varsigma,\iota):=\sum_{j,k,L=1}^{2}\frac{\partial^3G_1(\xi,\mu_0)}{\partial\xi_j\partial\xi_k\partial\xi_l}\Big|_{\xi=0}\sigma_j\varsigma_k\iota_l =\beta(1-\alpha)f_1^{'''}(0)\sigma_1\varsigma_1\iota_1+\gamma_1(1-\alpha)f_2^{'''}(0)\times\sigma_2\varsigma_2\iota_2, \tag{33}$$

$$C_2(\sigma,\varsigma,\iota):=\sum_{j,k,L=1}^{2}\frac{\partial^3G_2(\xi,\mu_0)}{\partial\xi_j\partial\xi_k\partial\xi_l}\Big|_{\xi=0}\sigma_j\varsigma_k\iota_l =\gamma_2(1-\alpha)f_3^{'''}(0)\sigma_1\varsigma_1\iota_1+\beta(1-\alpha)f_4^{'''}(0)\times\sigma_2\varsigma_2\iota_2. \tag{34}$$

According to (29)-(34) and the following formulae

$$g_{20}(\mu_0)=\langle p,B(q,q)\rangle,g_{11}(\mu_0)=\langle p,B(q,\bar{q})\rangle,$$
$$g_{01}(\mu_0)=\langle p,B(\bar{q},\bar{q})\rangle,g_{21}(\mu_0)=\langle p,C(q,q,\bar{q})\rangle,$$

we have

$$g_{20}(\mu_0)=g_{11}(\mu_0)=g_{02}(\mu_0)=0,$$
$$g_{21}(\mu_0)=\bar{p}_1C_1(q,q,\bar{q})+\bar{p}_2C_2(q,q,\bar{q}).$$

In view of $e^{-i\theta(\mu_0)}=\overline{\lambda(\mu_0)}$, we get

$$\begin{aligned}a(\mu_0)\;=\;&\mathrm{Re}\left[\frac{e^{-i\theta(\mu_0)}g_{21}}{2}\right]\\ &-\mathrm{Re}\left[\frac{(1-2e^{-i\theta(\mu_0)})e^{-2i\theta(\mu_0)}}{2(1-e^{-i\theta(\mu_0)})}g_{20}g_{11}\right],\\ &-\frac{1}{2}|g_{11}|^2-\frac{1}{4}|g_{02}|^2=\mathrm{Re}\left[\frac{e^{-i\theta(\mu_0)}g_{21}}{2}\right].\end{aligned}$$

Based on the analysis above, we get the following theorem.

Theorem 3.1. *Assume that $(A2)$ and $\nu\in(-\alpha,1-\alpha)$ are satisfied. Then the direction and stability of Neimark-Sacker bifurcation of system (1) can be determined by sign$(a(\mu_0))$. If sign$(a(\mu_0))>0$, then the Neimark-Sacker bifurcation of system (1) at $\mu=\mu_0$ is supercritical, moreover, the unique closed invariant curve bifurcating from the zero equilibrium for $\mu=\mu_0$ is asymptotically stable. If sign$(a(\mu_0))<0$, then the Neimark-Sacker bifurcation of system (1) at $\mu=\mu_0$ is subcritical, moreover, the unique closed invariant curve bifurcating from the zero equilibrium for $\mu=\mu_0$ is asymptotically unstable.*

IV. Computer simulations

In this section, we present some numerical results to illustrative the feasibility and effectiveness of our results obtained in the previous section.

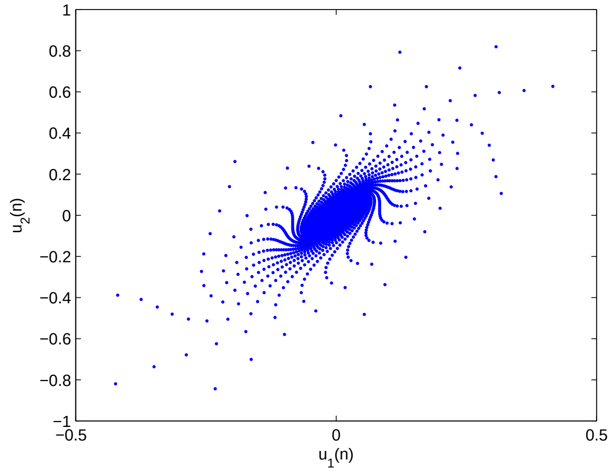

Fig. 1. Dynamical behavior of system (35) with $\gamma_1 = 0.23$ and $\gamma_2 = 0.16$.

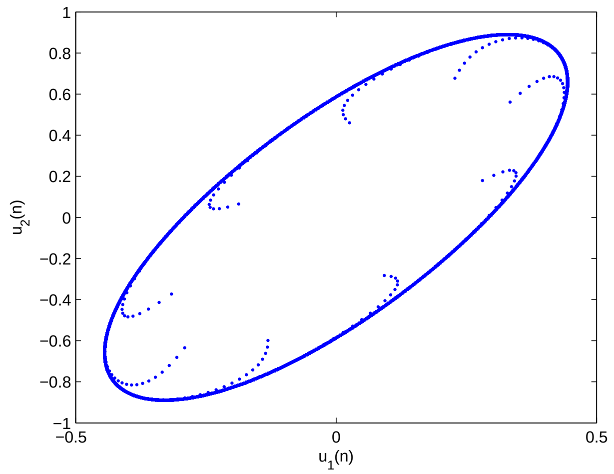

Fig. 2. Dynamical behavior of system (35) with $\gamma_1 = 0.23$ and $\gamma_2 = 0.31$.

Example 4.1. Consider the following system

$$\begin{cases} u_1(n+1) = \alpha u_1(n) + (1-\alpha) f_1(\beta u_1(n-\tau_1)) \\ \qquad\qquad\quad + (1-\alpha) f_2(\gamma_1 u_2(n-\tau_2)), \\ u_2(n+1) = \alpha u_2(n) + (1-\alpha) f_3(\gamma_2 u_1(n-\tau_3)) \\ \qquad\qquad\quad + (1-\alpha) f_4(\beta u_2(n-\tau_4)), \end{cases} \tag{35}$$

where $\alpha = 0.3, \beta = 0.35, \gamma_1 = 0.23, \gamma_2 = 0.28, f(u) = \tanh u, \tau_1 = \tau_2 = 0.24, \tau_3 = \tau_4 = 0.35$. Then $f_i(0) = 0, f_i^{'}(0) = 1 > 0, f_i^{''}(0) = 0, f_i^{'''}(0) = -2 < 0$. From (15), we get $\mu_0 = 0.83$. It is easy to check that all the conditions in Theorem 2.1 and Theorem 3.1 are fulfilled. Thus we can conclude that the zero equilibrium of (35) is asymptotically stable. When μ crosses the critical value $\mu_0 = 0.83$, the zero equilibrium of (35) is unstable and an asymptotically invariant cycle bifurcating from the zero equilibrium will appear. These results are shown in Fig. 1-2.

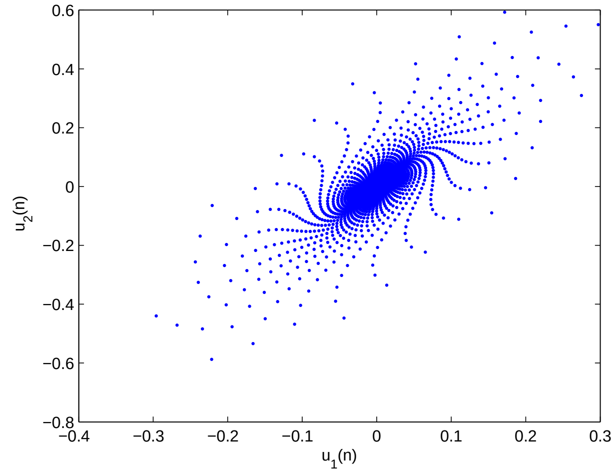

Fig. 3. Dynamical behavior of system (36) with $\gamma_1 = 0.32$ and $\gamma_2 = 0.31$.

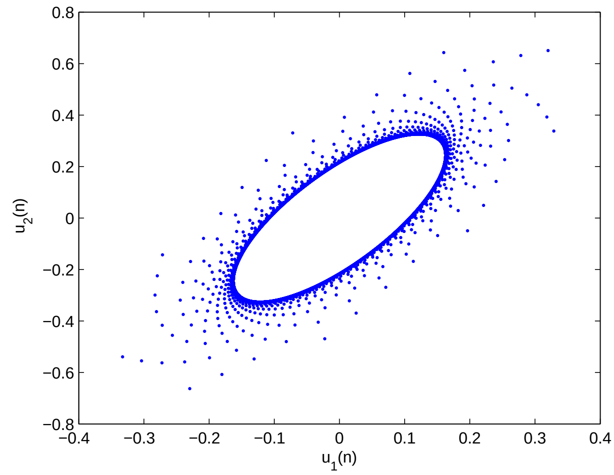

Fig. 4. Dynamical behavior of system (36) with $\gamma_1 = 0.32$ and $\gamma_2 = 0.45$.

Example 4.2. Consider the following system

$$\begin{cases} u_1(n+1) = \alpha u_1(n) + (1-\alpha) f_1(\beta u_1(n-\tau_1)) \\ \qquad\qquad\quad + (1-\alpha) f_2(\gamma_1 u_2(n-\tau_2)), \\ u_2(n+1) = \alpha u_2(n) + (1-\alpha) f_3(\gamma_2 u_1(n-\tau_3)) \\ \qquad\qquad\quad + (1-\alpha) f_4(\beta u_2(n-\tau_4)), \end{cases} \tag{36}$$

where $\alpha = 0.452, \beta = 0.463, \gamma_1 = 0.32, \gamma_2 = 0.41, f(u) = \tanh u, \tau_1 = \tau_2 = 0.31, \tau_3 = \tau_4 = 0.25$. Then $f_i(0) = 0, f_i^{'}(0) = 1 > 0, f_i^{''}(0) = 0, f_i^{'''}(0) = -2 < 0$. From (15), we get $\mu_0 = 0.76$. It is easy to check that all the conditions in Theorem 2.1 and Theorem 3.1 are fulfilled. Thus we can conclude that the zero equilibrium of (36) is asymptotically stable. When μ crosses the critical value $\mu_0 = 0.76$, the zero equilibrium of (36) is unstable and an asymptotically invariant cycle bifurcating from the zero equilibrium will appear. These results are shown in Fig. 3-4.

V. Conclusions

In the present paper, we consider a discrete-time neural network model with multiple delays. By choosing suitable bifurcation parameter, we investigate the stability and the existence of Neimark-Sacker bifurcation. The explicit formulae for determining the direction of Neimark-Sacker bifurcation and the stability of bifurcating periodic solution are given by applying the normal form method and the center manifold theory for discrete time system developed by Kuznetsov [17]. Some computer simulations are carried out to support our theoretical findings. Our results are new and complement previously known studies in [3-4]. The obtained results in this paper play an important in designing neural networks. Up to now, there are only few papers that deal with the Neimark-Sacker bifurcation for neural networks with three neurons or more neurons. We will left them for future work.

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