On the Eigenvalues Distribution of Preconditioned Block Two-by-two Matrix

Mu-Zheng Zhu[†] and Ya-E Qi[‡]

***Abstract*—The spectral properties of a class of block 2×2 matrix are studied, which arise in the numercial solutions of PDE-constrained optimization problems. Based on the Schur complement approximate approach and inexact Uzawa preconditioner, the eigenvalues distribution of preconditioned matrix is discussed by the similarity transformation. Moreover, The numerical experiments originated in PDE-constrained optimization problem are presented to show that the theoretical bound of the eigenvalues is in good agreement with its practical bound.**

***Index Terms*—block two-by-two linear systems, schur complement approach, inexact Uzawa preconditioner, eigenvalues distribution, PDE-constrained optimization problems.**

I. Introduction

In this paper, we investigate spectral properties of block 2×2 matrix in following linear systems:

$$\mathscr{A}x \equiv \begin{pmatrix} W & T \\ T & -W \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \equiv b, \tag{1}$$

where $W, T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite (SPSD) and one of them is symmetric positive definite (SPD). Without loss of generality, we assume W is SPD. The matrix $\mathscr{A}$ is nonsingular if and only if $\mathrm{null}(W)\cap \mathrm{null}(T)=\{0\}$ [10]. Thus, the linear system (1) has a unique solution when the (1, 1)-block in matrix $\mathscr{A}$ is nonsingular.

The linear system (1) can be formally regard as a special case of the saddle point problems [5]–[7]. They frequently arise from finite element discretizations of PDE-constrained optimization problems [7], [10], [23], [31], complex symmetric linear systems [2], [8], [9], [11], finite element discretizations of first-order linearization of the two-phase flow problems based on Cahn-Hilliard equation [4], [16], matrix completions problems [18], and so on [6], [12], [24], [26]. A large variety of applications and numerical solution methods of linear system (1) have been comprehensively reviewed by Benzi, Golub, and Liesen [12].

In recent years, the eigenvalue distribution of block 2×2 linear systems has been deeply studied [1], [13], [14], [22], [29]. On the one hand, the bounds for eigenvalues of $\mathscr{A}$ in (1) can be used to analyze the spectral properties of preconditioners such as symmetric indefinite preconditioners, inexact constraint preconditioners and primal-based penalty preconditioners for linear systems (1); see [13]–[15], [21], [28]. On the other hand, the estimates for eigenvalues of $\mathscr{A}$ in (1) can give theoretical basis for the CG method solving linear system (1) in a nonstandard inner product; see [20], [27], [28].

Our focus is on an important block lower triangular preconditioner, called inexact Uzawa preconditioner, which exploit the knowledge of a good approximation for the (negative) Schur complement. The aim of this paper is to investigate the spectral properties and provide the eigenvalues distribution of preconditioned matrix.

The remainder of this paper is organized as follows. In Section II, an new Schur approximation with parameter is presented and the approximate degree is studied. In Section III, an inexact Uzawa preconditioner is introduced and the eigenvalues distribution of preconditioned matrix are analyzed by the use of a similarity transformation. In Section IV, the numerical experiments are given to show that the theoretical bound of eigenvalues distribution is in good agreement with the practical bound though PDE-constrained optimization problems. Finally, in Section V we end this paper with some conclusions.

II. Schur complement approximate

In this section, we consider the approximate degree between the Schur complement $S := W + TW^{-1}T$ of the matrix $\mathscr{A}$ and its approximate

$$S(\alpha) = (W+\alpha T)W^{-1}(W+\alpha^* T),$$

where $\mathrm{Re}\,(\alpha)>0$ and $\alpha\alpha^*=1$.

As $W^{-\frac{1}{2}}TW^{-\frac{1}{2}}$ is symmetric positive semi-definite, then an orthogonal matrix $Q\in\mathbb{R}^{n\times n}$ and a diagonal matrix $\Sigma = (\sigma_{ii})_{n\times n}$, $\sigma_{ii}\ge 0$ are exist to such that

$$W^{-\frac{1}{2}}TW^{-\frac{1}{2}} = Q^T\Sigma Q,$$

so the following expression is true:

$$\begin{aligned} S(\alpha)^{-1} &= (W+\alpha^* T)^{-1}W(W+\alpha T)^{-1} \\ &= W^{-\frac{1}{2}}(I+2\mathrm{Re}\,(\alpha)Q^T\Sigma Q + Q^T\Sigma^2 Q)^{-1}W^{-\frac{1}{2}} \\ &= (QW^{-\frac{1}{2}})^{-1}(I+2\mathrm{Re}\,(\alpha)\Sigma+\Sigma^2)^{-1}(QW^{-\frac{1}{2}}). \end{aligned}$$

It is obvious that $S = S(\mathbf{i})$, where $\mathbf{i}=\sqrt{-1}$ is the imaginary unit. Thus,

$$S = W^{\frac{1}{2}}Q^T(I+\Sigma^2)QW^{-\frac{1}{2}}.$$

Then

$$S(\alpha)^{-1}S = (QW^{-\frac{1}{2}})^{-1}H(QW^{-\frac{1}{2}}), \tag{2}$$

Manuscript received January 21, 2016; This work was supported by the Research Foundation of the Higher Education Institutions of Gansu Province with code 2014A-110 and 2014B-123 and the project of the Key Laboratory of Hexi Corridor of characteristic resources utilization in Gansu Province with code XZ1402.

[†] Mu-Zheng Zhu is with the School of Mathematics and Statistics, Hexi University, Zhangye, Gansu, 734000 P. R. China; e-mail: zhumzh07@yahoo.com.

[‡] Ya-E Qi is corresponding author with the College of Chemistry and Chemical Engineering, Hexi University, Zhangye, Gansu, 734000 P. R. China.

(Advance online publication: 26 November 2016)

where $H=(1+\mathrm{Re}\,(\alpha)\Sigma+\Sigma^2)^{-1}(I+\Sigma^2)$ is a diagonal matrix and its diagonal elements

$$h_{ii}=\frac{1+\sigma_{ii}^2}{1+2\mathrm{Re}\,(\alpha)\sigma_{ii}+\sigma_{ii}^2}\le 1,$$

and the equality hold up if and only if $\mathrm{Re}\,(\alpha)=0$.

It can be found from (2) that $S(\alpha)^{-1}S$ and H have the same eigenvalues. According

$$h_{ii}=\frac{1+\sigma_{ii}^2}{1+2\mathrm{Re}\,(\alpha)\sigma_{ii}+\sigma_{ii}^2}\ge\frac{1}{1+\dfrac{2\mathrm{Re}\,(\alpha)\sigma_{ii}}{1+\sigma_{ii}^2}}$$

$$\ge\frac{1}{1+\mathrm{Re}\,(\alpha)}\quad(1\le i\le n),$$

we easily have

$$\lambda(S(\alpha)^{-1}S)\in\Big[\frac{1}{1+\mathrm{Re}\,(\alpha)},\ 1\Big]. \tag{3}$$

Similarly, an approximation $\widehat{W}$ can be found for W, and $\lambda(\widehat{W}^{-1}W)\in[\underline{\mu},\quad\bar{\mu}]$, where $\underline{\mu}>0,\ \bar{\mu}<1$.

III. Preconditioner and eigenvalues distribution

There are many “indefinite" preconditioner with block 2×2 form are presented, whose indefiniteness is tailored to compensate for the indefiniteness of systems matrix, and in this sense that the preconditioned matrix has only eigenvalues with positive real part. The eigenvalue distribution for this type preconditioners with Schur complement are also widely discussed. Theses include indefinite block diagonal preconditioners [13], [19], block triangular preconditioners [30], for inexact Uzawa preconditioners [17], and block approximate factorization preconditioners [3].

Our focus is on the inexact Uzawa preconditioner

$$\widehat{M}=\begin{pmatrix}\widehat{W}&0\\T&-S(\alpha)\end{pmatrix}=\begin{pmatrix}I_n&0\\T\widehat{W}^{-1}&I_n\end{pmatrix}\begin{pmatrix}\widehat{W}&\\&-S(\alpha)\end{pmatrix}, \tag{4}$$

which is considered in [24]. The eigenvalues distribution of the preconditioned matrix $\widehat{M}^{-1}\mathscr{A}$ are discussed in this section.

When $\widehat{W}=W$ and $S(\alpha)=S$, it is well known that the preconditioner (4) is such that the preconditioned matrix has all eigenvalues equal to 1 and minimal polynomial of degree at most 2 [12].

However, using these “ideal" preconditioners requires exact solves with W and S, which is often impractical, just the computation of S can be prohibitive [24]. Here we investigate the effect of using approximations $S(\alpha)$ instead Schur complement S. We analyze how the eigenvalue distributions are affected by providing bounds, where “bounds” for non-real eigenvalues, have to be understood as combinations of inequalities proving their clustering in a confined region of the complex plane.

Assume $Y\in\mathbb{R}^{n\times n}$ is an orthogonal matrix, $V<I$ ($V=\mathrm{diag}(v_1,v_2,\cdots,v_n)$, $v_1\le v_2\le\cdots\le v_n<1$) is a diagonal matrix such that $Y^TW^{\frac12}\widehat{W}^{-1}W^{\frac12}Y=V$. Then

$$\widehat{M}^{-1}\mathscr{A}=\begin{pmatrix}\widehat{W}&\\&-S(\alpha)\end{pmatrix}^{-1}\begin{pmatrix}I_n&0\\-T\widehat{W}^{-1}&I_n\end{pmatrix}\begin{pmatrix}W&T\\T&-W\end{pmatrix}$$
$$=\begin{pmatrix}\widehat{W}^{-1}&\\&S(\alpha)^{-1}\end{pmatrix}\begin{pmatrix}W&T\\-T(I-\widehat{W}^{-1}W)&(W+T\widehat{W}^{-1}T)\end{pmatrix}.$$

As $Y^TW^{\frac12}\widehat{W}^{-1}W^{\frac12}Y=V$, we have

$$\begin{pmatrix}Y^TW^{\frac12}&\\&S(\alpha)^{\frac12}\end{pmatrix}\begin{pmatrix}\widehat{W}^{-1}&\\&S(\alpha)^{-1}\end{pmatrix}\begin{pmatrix}W^{-\frac12}Y&\\&S(\alpha)^{\frac12}\end{pmatrix}$$
$$\cdot\begin{pmatrix}Y^TW^{-\frac12}&\\&S(\alpha)^{-\frac12}\end{pmatrix}\begin{pmatrix}W&T\\-T(I-\widehat{W}^{-1}W)&(W+T\widehat{W}^{-1}T)\end{pmatrix}$$
$$\cdot\begin{pmatrix}W^{-\frac12}Y&\\&S(\alpha)^{-\frac12}\end{pmatrix}$$
$$=\begin{pmatrix}V&\\&I\end{pmatrix}\begin{pmatrix}Y^TW^{-\frac12}&-Y^TW^{-\frac12}T\\-S(\alpha)^{-\frac12}T(I-\widehat{W}^{-1}W)&S(\alpha)^{-\frac12}(W+T\widehat{W}^{-1}T)\end{pmatrix}$$
$$\cdot\begin{pmatrix}W^{-\frac12}Y&\\&S(\alpha)^{-\frac12}\end{pmatrix}$$
$$=\begin{pmatrix}V&\\&I\end{pmatrix}\begin{pmatrix}I&G^T\\-G(I-V)&(\widehat{C}+GVG^T)\end{pmatrix}=:J,$$

where $G=S(\alpha)^{-\frac12}TW^{-\frac12}Y$, $\widehat{C}=S(\alpha)^{-\frac12}WS(\alpha)^{-\frac12}$.

It is evident that $\widehat{C}+GVG^T=S(\alpha)^{-\frac12}SS(\alpha)^{-\frac12}$, i.e.,

$$\lambda(\widehat{C}+GVG^T)\in\Big[\frac{1}{1+\mathrm{Re}\,(\alpha)},\ 1\Big].$$

According the assumption $V<I$, $V^{\frac12}$ and $(I-V)^{\frac12}$ exist. Taking a similar transformation with respect to $\mathrm{blkdiag}((I-V)^{\frac12}V^{-\frac12},I)$ on J, we have

$$\begin{pmatrix}(I-V)^{\frac12}V^{-\frac12}&\\&I\end{pmatrix}J\begin{pmatrix}V^{\frac12}(I-V)^{-\frac12}&\\&I\end{pmatrix}$$
$$=\begin{pmatrix}V&(V-V^2)^{\frac12}G^T\\-G(V-V^2)^{\frac12}&(\widehat{C}+GVG^T)\end{pmatrix} \tag{5}$$
$$\triangleq:\begin{pmatrix}A&B^T\\-B&C\end{pmatrix}=K$$

where $A=V$ and $C=\widehat{C}+GVG^T$ are all SPD, $B=G(\Sigma-\Sigma^2)^{\frac12}$.

Such matrices are nonnegative definite in $\mathbb{R}^n$. Hence, their eigenvalues have positive real part [24]. Thus, if the preconditioned matrix is similar to a matrix of the above form (5), the indefiniteness of the original matrix (1) is lost. However, we must note that this is at the expense of the loss of the symmetry, meaning that a portion of the eigenvalues will be in general complex.

According the above discuss, $\widehat{M}^{-1}\mathscr{A}$ is similar to the matrix of the form K, thus the eigenvalue analysis of the preconditioned matrix $\widehat{M}^{-1}\mathscr{A}$ can be reduced to that of the matrix of the form K. Nextly the eigenvalues distribution of the preconditioned matrix is discussed and the main results is given.

Lemma 1. *([24]) Define*

$$K=\begin{pmatrix}A&B^T\\-B&C\end{pmatrix},$$

where $A\in\mathbb{R}^{n\times n}$ is SPD, $C\in\mathbb{R}^{m\times m}$ is positive semi-definite, $m\le n$. Assume B has full rank or C is positive definite in null space of B^T. Let $S_C=C+BA^{-1}B^T$, and $S_A=A+B^TC^{-1}B$ when C is positive definite, then the real eigenvalues λ of K satisfies the following condition:

$$\min\Big(\lambda_{\min}(A),\ \lambda_{\min}(S_C)\Big)\le\lambda\le\max\Big(\lambda_{\max}(A),\ \lambda_{\max}(C)\Big).$$

The non-real eigenvalues of K satisfies the following condition:

$$\frac12\Big(\lambda_{\min}(A)+\lambda_{\min}(C)\Big)\le Re(\lambda)\le\frac12\Big(\lambda_{\max}(A)+\lambda_{\max}(C)\Big)$$

$$\Big|Im(\lambda)\Big|\le\Big(\lambda_{\max}(BB^T)\Big)^{\frac12},\quad and\quad \Big|\lambda-\xi\Big|\le\xi,$$

where

$$\xi=\begin{cases}\dfrac{\lambda_{\max}(S_A)\ \lambda_{\max}(S_C)}{\lambda_{\max}(S_A)+\lambda_{\max}(S_C)}, & C \text{ is positive definite,}\\ \lambda_{\max}(S_C), & \text{otherwise.}\end{cases}$$

Theorem 2. *The real eigenvalues λ of the inexact Uzawa preconditioned matrix $\widehat{M}^{-1}\mathscr{A}$ satisfy*

$$\min\left(v_1,\ \frac{1}{1+Re(\alpha)}\right)\le\lambda\le 1.$$

The non-real eigenvalues λ of the inexact Uzawa preconditioned matrix $\widehat{M}^{-1}\mathscr{A}$ satisfy

$$\frac{1}{2}\left(v_1+\frac{1}{1+Re(\alpha)}\right)\le Re(\lambda)\le\frac{1}{2}(v_n+1),$$

$$\left|Im(\lambda)\right|\le\frac{1}{2}\left(\frac{\sigma_n^2}{1+2Re(\alpha)\sigma_n+\sigma_n^2}\right)^{\frac{1}{2}},$$

and

$$\left|\lambda-\frac{1}{2}\right|\le\frac{1}{2}.$$

Here, v_1, v_n is the minimum and maximum diagonal element of matrix V mentioned above respectively, σ_n is the maximum diagonal element of matrix Σ mentioned above.

Proof. As A and C are all positive definite matrices, the condition in Lemma 1 is satisfied. Now we consider

$$\sigma_{\max}(B)=\left(\lambda_{\max}(BB^T)\right)^{\frac{1}{2}}.$$

As $B=G(V-V^2)^{\frac{1}{2}}$, we have

$$\begin{aligned}\lambda_{\max}(BB^T)&=\lambda_{\max}\left(G(V-V^2)G^T\right)\le\frac{1}{4}\lambda_{\max}(GG^T)\\&\le\frac{1}{4}\ \frac{\sigma_n^2}{1+2\mathrm{Re}\,(\alpha)\sigma_n+\sigma_n^2}.\end{aligned}$$

Then we have

$$\left|\mathrm{Im}\,(\lambda)\right|\le\frac{1}{2}\left(\frac{\sigma_n^2}{1+\mathrm{Re}\,(\alpha)\sigma_n+\sigma_n^2}\right)^{\frac{1}{2}}.$$

Because $A=V<I$, $W>\widehat{W}$, i.e., $\widehat{W}^{-1}>W^{-1}$. Thus the following expression is true:

$$C=S(\alpha)^{-\frac{1}{2}}\left(W+T\widehat{W}^{-1}T\right)S(\alpha)^{-\frac{1}{2}}\ge S(\alpha)^{-\frac{1}{2}}SS(\alpha)^{-\frac{1}{2}}\ge\frac{1}{1+\mathrm{Re}\,(\alpha)}I.$$

As $C=\widehat{C}+GVG^T<\widehat{C}+GG^T<I$, we have

$$\frac{1}{1+\mathrm{Re}\,(\alpha)}\le\lambda<1.$$

Next the eigenvalues distribution of S_C, S_A is discussed. To analyze the Schur complement S_C, one firstly has to obtain it explicitly. One way is to consider the Schur complement,

$$S_C=C+BA^{-1}B^T=\widehat{C}+GVG^T+G(V-V^2)V^{-1}G^T=\widehat{C}+GG^T,$$

then $\lambda(S_C)=\lambda\left(S(\alpha)^{-1}S\right)$ holds. As seen in formula (3), we have

$$\lambda(S_C)\in\left[\frac{1}{1+\mathrm{Re}\,(\alpha)},\ 1\right].$$

Because

$$\begin{aligned}S_A&=A+B^TC^{-1}B\\&=V^{\frac{1}{2}}(I+(I-V)^{\frac{1}{2}}G^T(\widehat{C}+GVG^T)^{-1}G(I-V)^{\frac{1}{2}})V^{\frac{1}{2}}\\&=V^{\frac{1}{2}}RV^{\frac{1}{2}},\end{aligned}$$

we obtain the following result by using the Sherman-Morrisson-Woodbury formula (SMW):

$$\begin{aligned}R^{-1}&=I-(I-V)^{\frac{1}{2}}G^T\cdot\left[\widehat{C}+GVG^T+G(I-V)G^T\right]^{-1}G(I-V)^{\frac{1}{2}}\\&=I-(I-V)^{\frac{1}{2}}G^T(\widehat{C}+GG^T)^{-1}G(I-V)^{\frac{1}{2}}.\end{aligned}$$

As $G^T(\widehat{C}+GG^T)^{-1}G$ and $GG^T(\widehat{C}+GG^T)^{-1}$ have the same set of nonzero eigenvalues and they are banded by

$$\max_x\frac{x^TGG^Tx}{x^T(\widehat{C}+GG^T)x}\le 1.$$

One then finds $R^{-1}\ge I-(I-V)^{\frac{1}{2}}(I-V)^{\frac{1}{2}}=V$, i.e., $H\le V^{-1}$. Further, we have $\lambda(S_A)\le 1$.

Thus,

$$\xi=\frac{\lambda_{\max}(S_A)\lambda_{\max}(S_C)}{\lambda_{\max}(S_A)+\lambda_{\max}(S_C)}=\frac{1}{2}.\qquad\square$$

Indeed, when W is trivial, more special conclusion can be reached.

Corollary 3. *If $\widehat{W}=W$ then $V=I$, the matrix*

$$K=\begin{pmatrix}I & \\ & C+GG^T\end{pmatrix}$$

has only real eigenvalues, and the distribution of eigenvalues is

$$\lambda(K)\in\left[\frac{1}{1+Re(\alpha)},\quad 1\right].$$

IV. Numerical results

In this section, it is illustrated by using numerical examples that the theoretical bound for eigenvalues of preconditioned matrix are agreement with its practical bound. All the tests are performed in MATLAB R2013a with machine precision 10^{-16}.

we consider the distributed control problem which consists of a cost functional (6) to be minimized subject to a partial differential equation (PDE) problem posed on a domain $\Omega\subset\mathbb{R}^2$ [25], [31]:

$$\min_{u,f}\frac{1}{2}\|u-u_*\|_2^2+\beta\|f\|_2^2, \tag{6}$$

$$\text{subject to}\quad -\nabla^2u=f,\quad \text{in } \Omega=[0,\ 1]^2, \tag{7}$$

$$\text{with}\quad u=u_*,\quad \text{on } \partial\Omega, \tag{8}$$

where the function

$$u_*=\begin{cases}(2x-1)^2(2y-1)^2, & (x,y)\in\left[0,\ \frac{1}{2}\right]^2,\\ 0, & \text{otherwise},\end{cases}$$

β is a regularization parameter, $\partial\Omega$ is the regions boundaries of Ω. Such problems is firstly introduced by Lions in [32].

There are two approaches to obtain the solution of the PDE-constrained optimization problems (6 – 8). The one is discretize-then-optimize and the other is optimize-then-discretize. By using the discretize-then-optimize approach and the Q_1 finite element discretize in this paper, the following linear systems can be obtained:

$$\begin{pmatrix}2\beta M & 0 & -M\\ 0 & M & K^T\\ -M & K & 0\end{pmatrix}\begin{pmatrix}f\\ u\\ \lambda\end{pmatrix}=\begin{pmatrix}0\\ \tilde{b}\\ d\end{pmatrix},$$

where $M, K\in\mathbb{R}^{n\times n}$ is the mass matrix and stiffness matrix (the discrete Laplacian) respectively. They are all SPD matrices. $d\in\mathbb{R}^n$ is the terms coming from the boundary values, $\tilde{b}\in\mathbb{R}^n$ is the discrete Galerkin projection of u_*, λ is the Laplace operator vector.

As M is a symmetric matrix, $\lambda=2\beta f$ and the following linear systems of the form

$$\mathscr{A}z=\begin{pmatrix}M & \sqrt{2\beta}K\\ \sqrt{2\beta}K & -M\end{pmatrix}\begin{pmatrix}\mu\\ \sqrt{2\beta}f\end{pmatrix}=\begin{pmatrix}\tilde{b}\\ \sqrt{2\beta}d\end{pmatrix}, \tag{9}$$

can be obtained. we note that this system of linear equations have saddle point structure.

The eigenvalues distribution of the preconditioned matrix $\widehat{M}^{-1}\mathscr{A}$ with $\widehat{W}=1.1W$ and $V=\frac{10}{11}I<I$ are listed in Tables I – III, The symbol "–" denotes that the case is not exist.

From these Tables, we can find that the practical eigenvalues just fall in the interval theoretical eigenvalues, the theoretical bound of image are good agreement with the practical bound of image. To further illustrate this, The eigenvalues distribution of the preconditioned matrix $\widehat{M}^{-1}\mathscr{A}$ is given in Fig. 1.

V. Conclusions

In this paper, our focus is directed at the spectral property of the block 2×2 linear systems, which frequently arise from saddle point problems and PDE-constrained optimization problems. Based on the Schur complement approximate approach, an inexact Uzawa preconditioner is introduced and the eigenvalues distribution of preconditioned matrix is discussed by the similarity transformation. The results of our numerical experiments utilizing test matrices from PDE-constrained optimization problems demonstrate that the theoretical bound for eigenvalues of the preconditioned matrix is good agreement with its practical bound.

Acknowledgements

The author would like to thank the anonymous referees for his/her careful reading of the manuscript and useful comments and improvements.

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TABLE I: Bound of Eigenvalues of $\widehat{M}^{-1}\mathscr{A}$ for $\widehat{W}=1.1W,\ h=2^{-4}$.

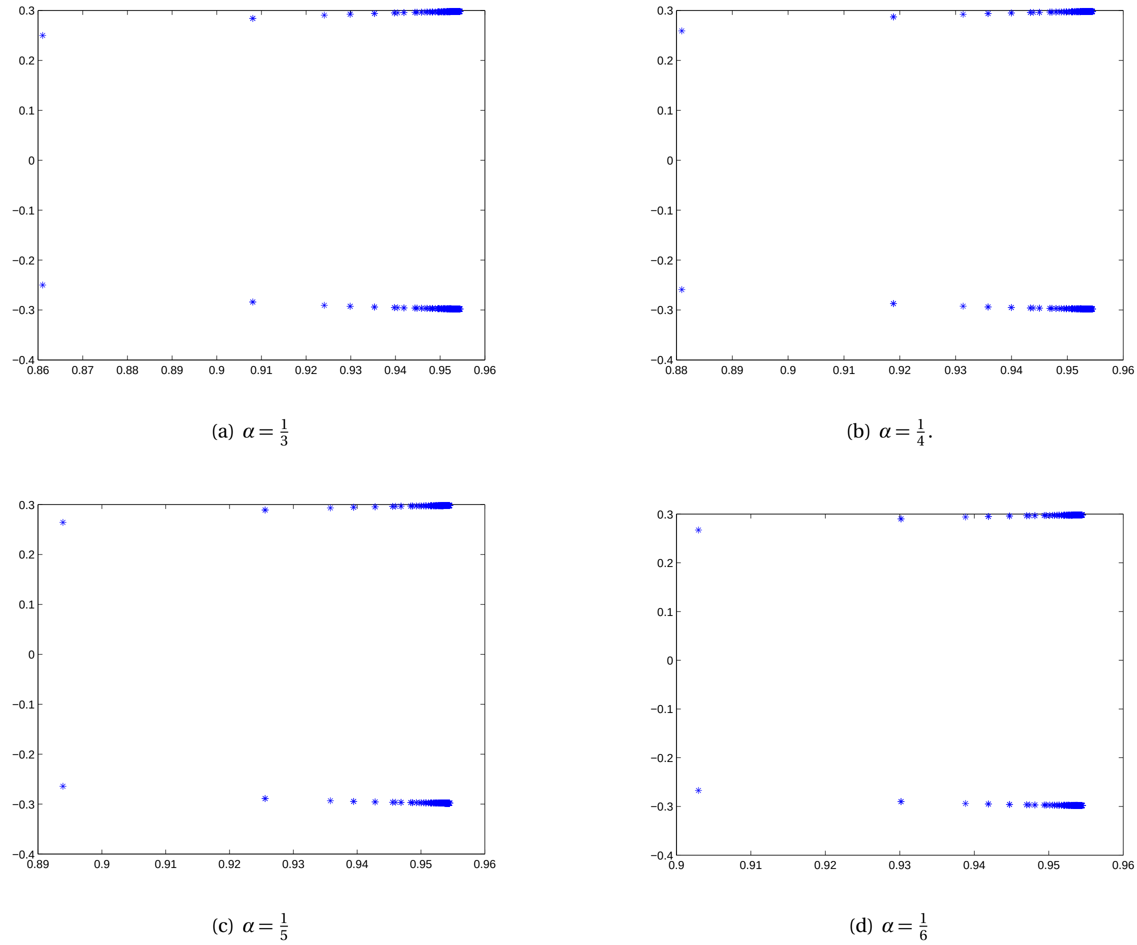
β	$Re(\alpha)$	real eigenvalues		complex eigenvalues			
		Theoretical	Practical	Theoretical real	Practical real	Theoretical image	Practical image
10^{-1}	1/3	[0.7500, 1]	–	[0.8295, 0.9545]	[0.9618, 0.9544]	[-0.3061, 0.3061]	[-0.2980 0.2980]
	1/4	[0.8000, 1]	–	[0.8545, 0.9545]	[0.9257, 0.9544]	[-0.3162, 0.3162]	[-0.2980, 0.2980]
	1/5	[0.8333, 1]	–	[0.8712, 0.9545]	[0.9312, 0.9545]	[-0.3227, 0.3227]	[-0.2981, 0.2981]
	1/6	[0.8571, 1]	–	[0.8831, 0.9545]	[0.9349, 0.9545]	[-0.3273, 0.3273]	[-0.2981, 0.2981]
10^{-2}	1/3	[0.7500, 1]	–	[0.8295, 0.9545]	[0.8612, 0.9541]	[-0.3061, 0.3061]	[-0.2980, 0.2980]
	1/4	[0.8000, 1]	–	[0.8545, 0.9545]	[0.8811, 0.9542]	[-0.3161, 0.3161]	[-0.2980, 0.2980]
	1/5	[0.8333, 1]	–	[0.8712, 0.9545]	[0.8940, 0.9543]	[-0.3227, 0.3227]	[-0.2980, 0.2980]
	1/6	[0.8571, 1]	–	[0.8831, 0.9545]	[0.9030, 0.9543]	[-0.3273, 0.3273]	[-0.2980, 0.2980]

TABLE II: Bound of Eigenvalues of $\widehat{M}^{-1}\mathscr{A}$ for $\widehat{W}=1.1W,\ h=2^{-5}$.

β	$Re(\alpha)$	real eigenvalues		complex eigenvalues			
		Theoretical	Practical	Theoretical real	Practical real	Theoretical image	Practical image
10^{-1}	1/3	[0.7500, 1]	–	[0.8295, 0.9545]	[0.9167, 0.9545]	[-0.3062, 0.3062]	[-0.2981, 0.2981]
	1/4	[0.8000, 1]	–	[0.8545, 0.9545]	[0.9256, 0.9545]	[-0.3162, 0.3162]	[-0.2981, 0.2981]
	1/5	[0.8333, 1]	–	[0.8712, 0.9545]	[0.9311, 0.9545]	[-0.3227, 0.3227]	[-0.2981, 0.2981]
	1/6	[0.8571, 1]	–	[0.8831, 0.9545]	[0.9349, 0.9545]	[-0.3273, 0.3273]	[-0.2981, 0.2981]
10^{-2}	1/3	[0.7500, 1]	–	[0.8295, 0.9545]	[0.8610, 0.9544]	[-0.3062, 0.3062]	[-0.2980, 0.2980]
	1/4	[0.8000, 1]	–	[0.8545, 0.9545]	[0.8810, 0.9545]	[-0.3162, 0.3162]	[-0.2981, 0.2981]
	1/5	[0.8333, 1]	–	[0.8712, 0.9545]	[0.8939, 0.9545]	[-0.3227, 0.3227]	[-0.2981, 0.2981]
	1/6	[0.8571, 1]	–	[0.8831, 0.9545]	[0.9030, 0.9545]	[-0.3273, 0.3273]	[-0.2981, 0.2981]

TABLE III: Bound of Eigenvalues of $\widehat{M}^{-1}\mathscr{A}$ for $\widehat{W}=1.1W,\ h=2^{-6}$.

β	$Re(\alpha)$	real eigenvalues		complex eigenvalues			
		Theoretical	Practical	Theoretical real	Practical real	Theoretical image	Practical image
10^{-1}	1/2	[0.6667, 1]	–	[0.7879, 0.9545]	[0.8998, 0.9545]	[-0.2887, 0.2887]	[-0.2981, 0.2981]
	1/3	[0.7500, 1]	–	[0.8295, 0.9545]	[0.9167, 0.9545]	[-0.3062, 0.3062]	[-0.2981, 0.2981]
	1/4	[0.8000, 1]	–	[0.8545, 0.9545]	[0.9256, 0.9545]	[-0.3162, 0.3162]	[-0.2981, 0.2981]
	1/5	[0.8333, 1]	–	[0.8712, 0.9545]	[0.9311, 0.9545]	[-0.3227, 0.3227]	[-0.2981, 0.2981]
	1/6	[0.8571, 1]	–	[0.8831, 0.9545]	[0.9349, 0.9545]	[-0.3273, 0.3273]	[-0.2981, 0.2981]
10^{-2}	1/2	[0.6667, 1]	–	[0.7879, 0.9545]	[0.8262, 0.9535]	[-0.2887, 0.2887]	[-0.2981, 0.2981]
	1/3	[0.7500, 1]	–	[0.8295, 0.9545]	[0.8610, 0.9545]	[-0.3062, 0.3062]	[-0.2981, 0.2981]
	1/4	[0.8000, 1]	–	[0.8545, 0.9545]	[0.8809, 0.9545]	[-0.3162, 0.3162]	[-0.2981, 0.2981]
	1/5	[0.8333, 1]	–	[0.8712, 0.9545]	[0.8939, 0.9545]	[-0.3227, 0.3227]	[-0.2981, 0.2981]
	1/6	[0.8571, 1]	–	[0.8831, 0.9545]	[0.9029, 0.9545]	[-0.3273, 0.3273]	[-0.2981, 0.2981]



(a) $\alpha = \frac{1}{3}$

(b) $\alpha = \frac{1}{4}$.

(c) $\alpha = \frac{1}{5}$

(d) $\alpha = \frac{1}{6}$

Fig. 1: Eigenvalues distribution of preconditioned matrix $\widehat{M}^{-1}\mathscr{A}$ $\quad(\beta = 10^{-2}, h = 2^{-5})$.

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