Smoothing Analysis of Collective Relaxation for Solving 2D Stokes Flow by Multigrid Method

Xingwen Zhu and Lixiang Zhang

Abstract—**Smoothing properties of the collective relaxations in multigrid method for solving 2D Stokes flow on the non-staggered grid are investigated by means of local Fourier analysis (LFA). For multigrid relaxation, the non-staggered discretizing scheme of Stokes flow is generally stabilized by adding an artificial pressure term. Therefore, an important problem is how to determine the zone of parameter of this term. To do that, a collective red-black Jacobi point (CRB-JACP) relaxation for the 2D Stokes flow is established. Firstly, the *h*-ellipticity for the 2D Stokes system is obtained with the parameter of the artificial pressure term. Then the Fourier representation of CRB-JACP relaxation for discretizing Stokes flow is given by the form of square matrix, whose eigenvalues are being computed. And a mathematical relation of the smoothing factor between the artificial pressure term and the *h*-ellipticity is well yielded. The results show that the numerical schemes for solving 2D Stokes flow by multigrid method on CRB-JACP have a specific convergence zone of parameter of added artificial pressure term.**

Index Terms—**Smoothing factor, local Fourier analysis, multigrid method, Stokes flow, *h*-ellipticity, collective relaxation**

I. Introduction

MULTIGRID methods are generally considered one of the fastest numerical methods which have an optimally computational complexity for solving partial differential equations (PDEs) [1]-[7], especially for 3D steady impressible Newtonian flows, governed by Navier-Stokes equations, namely,

$$\begin{cases} -\Delta \vec{u} + \vec{\nabla} p = \vec{f} & (x,y,z)\in\Omega \\ \vec{\nabla}\cdot\vec{u} = 0 & (x,y,z)\in\Omega \\ \vec{u} = \vec{g} & (x,y,z)\in\partial\Omega \end{cases} \tag{1}$$

where $\vec{u} = (u(x,y,z), v(x,y,z), w(x,y,z))$ is the velocity field, $p = p(x,y,z)$ represents the pressure, $\vec{f} = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$ is the external force field, $(x,y,z)\in\Omega\subseteq\mathbb{R}^3$, $\partial\Omega$ is the Dirichlet boundary of the computing domain.

In multigrid methods, the smoothing relaxation operator in simulation plays a key role. Several relaxation techniques have been developed for solving systems of PDEs, which are generally classified into two categories: collective and decoupled relaxations [8]. In [2], the collective relaxation is considered as a straightforward generalization of a scalar system. The early decoupled relaxation is based on a distributive Gauss-Seidel technique [9], and is currently extended to an incomplete LU factorization [10]. Recently, the relaxation techniques have been used for solving the Stokes systems [11]-[13]. Particularly, the successive over relaxation in the parallel multigrid method is applied to the linear complementarity problem [14], and the implicit Runge-Kutta relaxation for solving one-dimensional Burgers' equation is presented [15].

For multigrid methods LFA is a very useful tool. It is used to design efficient algorithms and to predict convergence factors for solving PDEs with high order accuracy [1]-[7]. Also, convergence properties of multigrid methods, with a collective point relaxation, for optimal control problems are well investigated with LFA [16]-[20]. In [21], a distributive relaxation method, which is used for solving a poroelasticity equation, is improved by using LFA. An efficiency multigrid solver on LFA for the Navier-Stokes equations is designed in [22]. The relaxations based on Hermitian and augmented Lagrangian splittings for the Oseen problem are given by LFA [23]. All-at-once multigrid approach for optimality systems with LFA are discussed in details and an analytical expression of convergence factors is given by using symbolic computation [17], [18].

Extensive analysis of the solution to Stokes equations can be found in literature due to their wide range of applications [1], [2], [6] and [8]. In particular, the multigrid performance for Stokes equations has been a topic of study during many years and a benchmark for studies of different techniques proposed in this field, as the local Fourier analysis for example in [1]-[8], [10] and [11]. In this paper, we apply this technique to study the *h*-ellipticity and smoothing factor of the considered stabilized scheme. As for the *h*-ellipticity, the similar analysis and conclusions can be found in several monographs dealing with multigrid method and LFA such as [1]-[7], especially, the well-known book by Trottenberg, Oosterlee and Schuller. The result we present is different from [1]. The Laplacian factor in determinant of discrete 2D Stokes operator is omitted, and the calculation of the *h*-ellipticity is simplified in [1]. However, we use another method. The *h*-ellipticity is directly computed by the

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determinant of discrete 2D Stokes operator. And a different expression of the parameter of the artificial pressure term is obtained. A detailed analysis on the smoothing factor of the CRB-JACP relaxation is provided by means of LFA. The numerical result has already been given in [2]. In this paper, the analytical results are presented in detail, and the relation between the parameter in artificial pressure term and smoothing factor is obtained. The results show that the relaxation method subjected by this paper is divergence with the high *h*-ellipticity, which is agreeable with the numerical results in [2].

In this paper, the analysis of the stabilized central difference discretization of 2D Stokes equations on a non-staggered grid is presented. It is well-known that a central difference scheme for Stokes problem provides an unstable discretization as opposed to MAC (Marker and Cell) scheme on staggered grids. This leads to the addition of a stabilization term, which is proportional to h^2 times the Laplacian operator, to the continuity equation. The resulting scheme is described by LFA. A quantitative analysis used to study the smoothing properties of the relaxation processes within a multigrid framework. The *h*-ellipticity of the discrete operator, taking into account the adding artificial pressure term, is studied. Also, the performance of a collective red-black Jacobi point (CRB-JACP) relaxation technique is analyzed. An analytical expression of the smoothing factor of the CRB-JACP relaxation technique is obtained and the relations between such smoothing factor, the artificial pressure term and the *h*-ellipticity are presented.

A symbolic operation process by Mathematica software is carried out to derive an explicit formulation of the smoothing factor for the multigrid method. Especially, the cylindrical algebraic decomposition (CAD) function in the Mathematica build-in command is used [24].

II. LFA and Discrete Stokes System

A. LFA and Discrete Stokes System

A crucial point for using multigrid method is to identify multigrid components, which are used to construct an efficient interplay between relaxation and coarse grid corrections. A useful tool for a proper selection of the components is LFA. In [1]-[7] and [21], LFA is applied to develop efficient multigrid methods for solving linear elliptic equations with constant coefficients. The work is based on a simplification, which is obtained by neglecting the boundary conditions, and all occurring operators are extended to an infinite grid. On an infinite grid in numerical simulation, the approximation and the corresponding error and residual are represented by linear combinations of certain exponential functions or Fourier modal functions, which are used as a unitary basis of the space of bounded infinite grid functions [1]-[7].

Throughout this paper, discretizing 2D Stokes flow on a non-staggered grid is expressed as below

$$G_h = \{\vec{x} = (x_1, x_2) := (k_1 h, k_2 h) \mid (k_1, k_2) \in \mathbb{Z}^2\} \quad (2)$$

where is h the size of the uniform mesh. On grid (2), a unitary basis of vector-valued Fourier modes [1]-[3] is given by

$$\vec{\varphi}_h(\vec{\theta}, \vec{x}) := \begin{pmatrix} exp(i\vec{\theta}\cdot\vec{x} / h) \\ exp(i\vec{\theta}\cdot\vec{x} / h) \\ exp(i\vec{\theta}\cdot\vec{x} / h) \end{pmatrix} \quad (3)$$

where $\vec{\theta} = (\theta_1, \theta_2) \in \Theta := (-\pi, \pi]^2$, $\vec{x} \in G_h$ and $i = \sqrt{-1}$. Thus, a Fourier space is defined as

$$F(\vec{\theta}) := span\{\vec{\varphi}_h(\vec{\theta}, \vec{x}) \mid \vec{\theta} \in \Theta\} \quad (4)$$

For grid (2), a 2D scalar discrete operator P_h is given by

$$P_h := [l_{\vec{n}}]_h \quad (5)$$

where $l_{\vec{n}} \in \mathbb{R}$, $\vec{n} \in J \subset \mathbb{Z}^2$, which contains $(0,0)$, the Fourier symbol of P_h is defined as

$$\tilde{P}_h(\vec{\theta}) := \sum_{\vec{n}\in J} l_{\vec{n}} exp(i\vec{\theta}\cdot\vec{n}) \quad (6)$$

Using (3), the definition of the Fourier symbol (6) is generalized to the case of a system for further use in multigrid.

Thus, a main idea of LFA is to study different multigrid relaxations by evaluating their effects on the used Fourier modes. From [2], [16] and [21], We know that if a standard coarsening in two dimensions is selected, a low frequency $\vec{\theta} = \vec{\theta}^{00} = (\theta_1, \theta_2) \in \Theta^{2h}_{low} = (\frac{-\pi}{2}, \frac{\pi}{2}]^2$ is coupled with three high frequencies $\{\vec{\theta}^{11}, \vec{\theta}^{10}, \vec{\theta}^{01}\} \in \Theta^{2h}_{high}$ in the transition state from G_h to G_{2h}, where

$$\vec{\theta}^{\vec{\alpha}} = \vec{\theta} - (\alpha_1 sign(\theta_1), \alpha_2 sign(\theta_2))\pi \ , \quad (7)$$

and $\Theta^{2h}_{high} = \Theta \setminus \Theta^{2h}_{low}$, $\vec{\alpha} \in \Lambda = \{00, 11, 10, 01\}$, in which $\vec{\alpha} = (\alpha_1, \alpha_2)$ is to be denoted by $(\alpha_1, \alpha_2) := \alpha_1\alpha_2$.

In this paper, the standard coarsening is assumed implicitly. Thus, the Fourier space (4) is needed to be subdivided into the corresponding 2*h*-harmonic subspaces

$$F_{2h}(\vec{\theta}) := span\left\{\vec{\varphi}_h(\vec{\theta}^{\vec{\alpha}}, \vec{x}) \middle| \vec{\alpha} \in \Lambda\right\} \quad (8)$$

B. Discrete of 2D Stokes equations

From (1), a 2D Stokes operator is written as

$$L = \begin{pmatrix} -\Delta & 0 & \partial_x \\ 0 & -\Delta & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix} \quad (9)$$

and its discretizing form is stated as

$$L'_h = \begin{pmatrix} -\Delta_h & 0 & \partial^h_x \\ 0 & -\Delta_h & \partial^h_y \\ \partial^h_x & \partial^h_y & 0 \end{pmatrix} \quad (10)$$

By using the standard central differencing on grid (2) with uniform mesh size h we obtain that $-\Delta_h$, ∂^h_x and ∂^h_y are second-order central difference operators with the following discrete stencils

$$-\Delta_h = \frac{1}{h^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}_h, \partial_x^h = \frac{1}{2h}\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}_h,$$

$$\partial_y^h = \frac{1}{2h}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}_h. \tag{11}$$

In fact, because the above non-staggered scheme (10) is not stable, it needs to add an artificially elliptic pressure term $-ch^2\Delta_h$ into the continuity equation to improve numerical stabilization in computation, where c is a positive real parameter, see e.g. [1] and [11]. By applying (11), the discretizing operator is changed as

$$L_h = \begin{pmatrix} -\Delta_h & 0 & \partial_x^h \\ 0 & -\Delta_h & \partial_y^h \\ \partial_x^h & \partial_y^h & -ch^2\Delta_h \end{pmatrix} \tag{12}$$

III. Measure of *h*-ellipticity for the Grid Discretization (12)

In [1], [2] and [21], the *h*-ellipticity is a necessary and sufficient condition for the existence of the point relaxation. The *h*-ellipticity measure is often used to decide whether or not a certain discretization is appropriate for the multigrid treatment. A sufficient amount of the *h*-ellipticity indicates that pointwise error of smoothing procedures can be constructed, and the computational process of the *h*-ellipticity for (12) is done. Applying (3), the measure of the *h*-ellipticity for systems of PDEs is defined by

$$E_h(M_h) = \frac{\min_{\vec{\theta}\in\Theta_{high}^{2h}} \det(\hat{M}_h(\vec{\theta}))}{\max_{\vec{\theta}\in\Theta} \det(\hat{M}_h(\vec{\theta}))} \tag{13}$$

where the complex matrix $\hat{M}_h(\vec{\theta})$ is a Fourier symbol of the discrete system operator M_h, i.e.,

$$M_h\vec{\varphi}_h(\vec{\theta},\vec{x}) = \hat{M}_h(\vec{\theta})\vec{\varphi}_h(\vec{\theta},\vec{x}) \tag{14}$$

From (5) and (6), the Fourier symbols of the discrete stencils (11) are yielded as

$$\left.\begin{aligned} \tilde{L}_{11}^h(\vec{\theta}) &= -\tilde{\Delta}_h(\vec{\theta}) = \frac{1}{h^2}(4-2\cos\theta_1-2\cos\theta_2) \\ \tilde{L}_{13}^h(\vec{\theta}) &= \tilde{\partial}_x^h(\vec{\theta}) = \frac{1}{h}i\sin\theta_1 \\ \tilde{L}_{23}^h(\vec{\theta}) &= \tilde{\partial}_y^h(\vec{\theta}) = \frac{1}{h}i\sin\theta_2 \end{aligned}\right\} \tag{15}$$

For the discrete Stokes system (10), from (3), (14) and (15), the Fourier symbol of L_h' is given below

$$\hat{L}_h'(\vec{\theta}) = \begin{pmatrix} \tilde{L}_{11}^h(\vec{\theta}) & 0 & \tilde{L}_{13}^h(\vec{\theta}) \\ 0 & \tilde{L}_{11}^h(\vec{\theta}) & \tilde{L}_{23}^h(\vec{\theta}) \\ \tilde{L}_{13}^h(\vec{\theta}) & \tilde{L}_{23}^h(\vec{\theta}) & 0 \end{pmatrix} \tag{16}$$

According to (13), it is easy to obtain $E_h(L_h') = 0$. It implies that the point relaxation scheme is not proper for the smoothing properties for (10), which means that L_h' is not *h*-ellipticity, and the checkerboard is instability [1]. To improve the stability of the discretization an artificial elliptic pressure term $-ch^2\Delta_h$ need to be introduced. The measure of the *h*-ellipticity for (12) is obtained in a following theorem.

Theorem 1 By adding the artificial elliptic pressure term $-ch^2\Delta_h$ into continuity equation of the Stokes system, the *h*-ellipticity of the non-staggered discrete system (12) is measured as

$$E_h(c) = \begin{cases} \dfrac{27c(8c-1)^2}{4} & 0 < c < \dfrac{1}{28} \\ \dfrac{27(8c-1)^2(4c+1)}{128} & \dfrac{1}{28} \le c \le \dfrac{1}{24} \\ \dfrac{4c+1}{256c} & c > \dfrac{1}{24} \end{cases} \tag{17}$$

where c is a positive real parameter.

Proof. From (3), (14) and (15), the Fourier symbol of the discrete Stokes system (12) is written as

$$\hat{L}_h(\vec{\theta}) = \begin{pmatrix} \tilde{L}_{11}^h(\vec{\theta}) & 0 & \tilde{L}_{13}^h(\vec{\theta}) \\ 0 & \tilde{L}_{11}^h(\vec{\theta}) & \tilde{L}_{23}^h(\vec{\theta}) \\ \tilde{L}_{13}^h(\vec{\theta}) & \tilde{L}_{23}^h(\vec{\theta}) & ch^2\tilde{L}_{11}^h(\vec{\theta}) \end{pmatrix} \tag{18}$$

For the sake of convenient calculation, We denote the trigonometric functions by

$$s_1 = \sin^2\frac{\theta_1}{2}, s_2 = \sin^2\frac{\theta_2}{2}. \tag{19}$$

Then, $\vec{\theta}\in\Theta$ and $\vec{\theta}\in\Theta_{high}^{2h}$ are changed to $\vec{s}\in S = [0,1]^2$, $\vec{s}\in S_{high} = [0,1]^2 - [0,\frac{1}{2}]^2$, respectively, where $\vec{s} = (s_1, s_2)$. Further, by substituting (19) into (15), the determinant of matrix (20) is expressed by

$$\begin{aligned} \det(\hat{L}_h(\vec{\theta})) &= \tilde{L}_{11}(\vec{\theta})(ch^2\tilde{L}_{11}(\vec{\theta}) - \tilde{L}_{13}(\vec{\theta})^2 - \tilde{L}_{23}(\vec{\theta})^2) \\ &= \frac{16}{h^4}(s_1+s_2)\begin{bmatrix} 4c(s_1+s_2)^2 + s_1 \\ +s_2 - (s_1^2+s_2^2) \end{bmatrix} \end{aligned} \tag{20}$$

Furthermore, by substituting (20) into (13), the *h*-ellipticity for (12) is obtained as

$$E_h(L_h) = \frac{\min_{\vec{\theta}\in\Theta_{high}^{2h}} \det(\hat{L}_h(\vec{\theta}))}{\max_{\vec{\theta}\in\Theta} \det(\hat{L}_h(\vec{\theta}))} = \frac{\min_{\vec{s}\in S_{high}} F(s_1,s_2)}{\max_{\vec{s}\in S} F(s_1,s_2)} \tag{21}$$

where

$$F(s_1,s_2)=(s_1+s_2)\begin{bmatrix}4c(s_1+s_2)^2+s_1\\+s_2-(s_1^2+s_2^2)\end{bmatrix} \quad (22)$$

Solving $\dfrac{\partial F}{\partial s_1}=0$ and $\dfrac{\partial F}{\partial s_2}=0$ by CAD function in the Mathematics [24], one of the extreme values of (22) with $(s_1,s_2)\in S_{high}$ is obtained as

$$s_1^*=\frac{-2}{24c-3},\ s_2^*=\frac{-2}{24c-3} \quad (23)$$

where $0<c\le\dfrac{1}{24}$, and other extreme values of (22) where $(s_1,s_2)\in S$ and $(s_1,s_2)\in S_{high}$ are based on the boundary, and s_1 and s_2 are not simultaneously equal to zero for $(s_1,s_2)\in S_{high}$. Thus, from (22) and (23), the maximum and minimum values of (22) with S and S_{high} are respectively given below,

$$\max_{\vec{s}\in S}F(s_1,s_2)=\begin{cases}\dfrac{16}{27}\dfrac{1}{(8c-1)^2} & 0<c\le\dfrac{1}{24}\\ 32c & c>\dfrac{1}{24}\end{cases} \quad (24)$$

$$\min_{\vec{s}\in S_{high}}F(s_1,s_2)=\begin{cases}4c & 0<c<\dfrac{1}{28}\\ \dfrac{1}{2}c+\dfrac{1}{8} & c\ge\dfrac{1}{28}\end{cases} \quad (25)$$

From (24) and (25), (17) holds for $\forall\ c>0$, $E_h(c)\ne 0$, and $0<E_h(c)<1$. The Theorem 1 holds.□

From Theorem 1, the measure of *h*-ellipticity for (12) is nonzero and independent of the mesh size h. It means that the discrete Stokes system is stabilized by the artificial pressure term $-ch^2\Delta_h$. From (17), the properties of $E_h(L_h)$ are given as

$$\left.\begin{aligned}&\inf_{c>0}E_h(L_h)=\lim_{c\to 0}E_h(L_h)=0\\ &\sup_{c>0}E_h(L_h)=E_h(\frac{1}{28})=\frac{675}{5488}\\ &\lim_{c\to+\infty}E_h(L_h)=\frac{1}{64}\end{aligned}\right\} \quad (26)$$

From (26), there exists some $c_0\in(0,\dfrac{1}{28})$ subject to:

$$\forall\ c>c_0\ ,\frac{1}{64}\le E_h(c)\le\frac{675}{5488}. \quad (27)$$

The curve of (17) is shown in Figure 1.

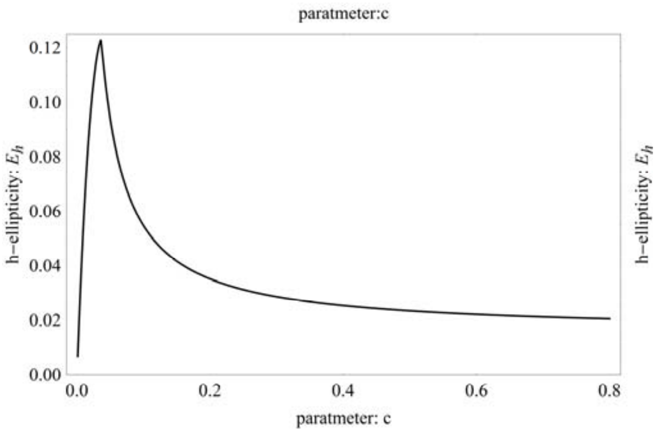

Figure 1 $E_h(c)$ curve

In [1] and [2], from (27), the smoothing factor of the point relaxation S_h is not more than 63/65, and $E_h(L_h)$ is computed omitting $\tilde{L}_{11}(\vec{\theta})$ in (22). Furthermore, an analytical solution [1] of $E_h(L_h)$ is obtained as follows.

$$E_h(c)=\begin{cases}8c(1-8c) & 0<c<\dfrac{1}{16}\\ \dfrac{1}{4} & \dfrac{1}{16}\le c\le\dfrac{1}{12}\\ \dfrac{4c+1}{64c} & \dfrac{1}{12}<c\end{cases} \quad (28)$$

IV. Error Analysis of 2D Stokes Flow (12)

From Theorem 1, (17) shows that there exists an efficient point relaxation, without loss of generality assuming that $c\in(0,\dfrac{4}{5})$. Herein the collective red-black Jacobi point (CRB-JACP) relaxation for the Stokes discrete system (12) is investigated by LFA, and the relaxation operator is denoted by S_h^{RB}. From [1], [2] and [21], the relaxation operator S_h^{RB} makes the 2*h*-harmonic subspace (8) invariant, i.e,

$$S_h^{RB}\Big|_{F_{2h}(\vec{\theta})}=:\widehat{S}_h^{RB}(\vec{\theta})\in\mathbb{C}^{12\times 12} \quad (29)$$

where $\widehat{S}_h^{RB}(\vec{\theta})$ is the Fourier representation of the relaxation operator S_h^{RB}, which is given by

$$\begin{aligned}\widehat{S}_h^{RB}(\vec{\theta})&=\widehat{S}_h^{B}(\vec{\theta})\widehat{S}_h^{R}(\vec{\theta})\\&=\frac{1}{4}\begin{pmatrix}A_{00}+I & I-A_{11} & 0 & 0\\ I-A_{00} & A_{11}+I & 0 & 0\\ 0 & 0 & A_{10}+I & I-A_{01}\\ 0 & 0 & I-A_{10} & A_{01}+I\end{pmatrix}\end{aligned}$$

$$\cdot\begin{pmatrix} A_{00}+I & A_{11}-I & 0 & 0 \\ A_{00}-I & A_{11}+I & 0 & 0 \\ 0 & 0 & A_{10}+I & A_{01}-I \\ 0 & 0 & A_{10}-I & A_{01}+I \end{pmatrix} \quad (30)$$

where I is a unit matrix, and by using (7), $A_{\vec{\alpha}}$ is expressed as

$$A_{\vec{\alpha}} = \tilde{S}_h(\vec{\theta}^{\vec{\alpha}}) = \tilde{I}_h(\vec{\theta}^{\vec{\alpha}}) - \tilde{D}_h(\vec{\theta}^{\vec{\alpha}})^{-1}\tilde{L}_h(\vec{\theta}^{\vec{\alpha}}) \quad (31)$$

This is a Fourier symbol expression of the collective Jacobi point relaxation [2] for (12), and $\tilde{I}_h(\vec{\theta}^{\vec{\alpha}})$ is 3×3 unit matrix which denotes the Fourier symbol of the unit stencil $I_h = [1]_h$, $\tilde{D}_h(\vec{\theta}^{\vec{\alpha}})$ is the Fourier symbol of the discrete stencil with the diagonal part of (12), i.e.,

$$\tilde{D}_h(\vec{\theta}^{\vec{\alpha}}) = \mathrm{diag}(\frac{4}{h^2}, \frac{4}{h^2}, 4c) \quad (32)$$

For our conveniences, two parameters are noted as

$$s_1 = \sin\frac{\theta_1}{2},\ s_2 = \sin\frac{\theta_2}{2} \quad (33)$$

with $(s_1, s_2) \in S_{low} = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]^2$ for $\Theta_{low}^{2h} = (-\frac{\pi}{2}, \frac{\pi}{2}]^2$.

In the process of Fourier smoothing analysis of CRB-JACP for (12), the standard coarsening and an ideal coarse grid correction operator [1], [2] are applied as follows,

$$Q_h^{2h}\Big|_{F_{2h}(\vec{\theta})} =: \widehat{Q}_h^{2h} = \mathrm{diag}(0,0,0,1,1,1,1,1,1,1,1,1) \in \mathbb{C}^{12\times 12} \quad (34)$$

where $\widehat{Q}_h^{2h}$ is the Fourier representation of the operator Q_h^{2h}, which annihilates the low frequency error components and makes the high frequency components unchanged. From [2] and [6], the smoothing factor of the Fourier representations for (31) and (36) is given as

$$\rho_s = \sup_{\vec{\theta}\in\Theta_{low}^{2h}} \rho(\widehat{Q}_h^{2h}\widehat{S}_h^{RB}(\vec{\theta})) \quad (35)$$

where $\rho(M)$ denotes the spectral radius of matrix M. The asymptotic smoothing process reduces the error from the high frequency components.

Applying (30) and (33), a 12×12 matrix of the discrete Stokes flow in (12) is expressed as

$$\widehat{Q}_h^{2h}\widehat{S}_h^{RB}(\vec{\theta}) = \mathrm{diag}(A_{11}, A_{22}) \quad (36)$$

where A_{11} and A_{22} are 6×6 square matrices. From (15) and (29)-(34), A_{11} and A_{22} are expressed as, respectively,

$$A_{11} = \frac{1}{4}\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix},\ A_{22} = \frac{1}{4}\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \quad (37)$$

where R_{11}, R_{22} are 3×3 zero square matrices, and

$$R_{21} = \begin{pmatrix} \begin{bmatrix} \frac{s_1^2 - s_1^4}{2c} + 2(s_1^2+s_2^2) \\ -2(s_1^2+s_2^2)^2 \end{bmatrix} & \frac{s_1\sqrt{1-s_1^2}s_2\sqrt{1-s_2^2}}{2c} & is_1h\sqrt{1-s_1^2}(1-2s_1^2-2s_2^2) \\ \frac{s_1\sqrt{1-s_1^2}s_2\sqrt{1-s_2^2}}{2c} & \begin{bmatrix} \frac{s_2^2 - s_2^4}{2c} + 2(s_1^2+s_2^2) \\ -2(s_1^2+s_2^2)^2 \end{bmatrix} & is_2h\sqrt{1-s_2^2}(1-2s_1^2-2s_2^2) \\ \frac{is_1\sqrt{1-s_1^2}(1-2s_1^2-2s_2^2)}{ch} & \frac{is_2\sqrt{1-s_2^2}(1-2s_1^2-2s_2^2)}{ch} & \begin{bmatrix} 2(s_1^2+s_2^2) - 2(s_1^2+s_2^2)^2 \\ -\frac{s_1^4 - s_1^2 + s_2^4 - s_2^2}{2c} \end{bmatrix} \end{pmatrix},$$

$$R_{22} = \begin{pmatrix} \begin{bmatrix} \frac{s_1^4 - s_1^2}{2c} - 2(s_1^2+s_2^2) \\ +2(s_1^2+s_2^2)^2 \end{bmatrix} & \frac{-s_1\sqrt{1-s_1^2}s_2\sqrt{1-s_2^2}}{2c} & -is_1h\sqrt{1-s_1^2}(1-2s_1^2-2s_2^2) \\ \frac{-s_1\sqrt{1-s_1^2}s_2\sqrt{1-s_2^2}}{2c} & \begin{bmatrix} \frac{s_2^4 - s_2^2}{2c} - 2(s_1^2+s_2^2) \\ +2(s_1^2+s_2^2)^2 \end{bmatrix} & -is_2h\sqrt{1-s_2^2}(1-2s_1^2-2s_2^2) \\ \frac{-is_1\sqrt{1-s_1^2}(1-2s_1^2-2s_2^2)}{ch} & \frac{-is_2\sqrt{1-s_2^2}(1-2s_1^2-2s_2^2)}{ch} & \begin{bmatrix} 2(s_1^2+s_2^2)^2 - 2(s_1^2+s_2^2) \\ +\frac{s_1^4 - s_1^2 + s_2^4 - s_2^2}{2c} \end{bmatrix} \end{pmatrix},$$

$$T_{11}=\begin{pmatrix} \left[\begin{array}{l} \dfrac{s_1^4-s_1^2}{2c}+2(s_1^2-s_2^2) \\ +2(s_1^2-s_2^2)^2 \end{array}\right] & \dfrac{s_1\sqrt{1-s_1^2}\,s_2\sqrt{1-s_2^2}}{2c} & is_1h\sqrt{1-s_1^2}\,(1+2s_1^2-2s_2^2) \\ \dfrac{s_1\sqrt{1-s_1^2}\,s_2\sqrt{1-s_2^2}}{2c} & \left[\begin{array}{l} -\dfrac{s_2^2-s_2^4}{2c}+2(s_1^2-s_2^2) \\ +2(s_1^2-s_2^2)^2 \end{array}\right] & -is_2h\sqrt{1-s_2^2}\,(1+2s_1^2-2s_2^2) \\ \dfrac{is_1\sqrt{1-s_1^2}\,(1+2s_1^2-2s_2^2)}{ch} & \dfrac{-is_2\sqrt{1-s_2^2}\,(1+2s_1^2-2s_2^2)}{ch} & \left[\begin{array}{l} 2(s_1^2-s_2^2)^2+2(s_1^2-s_2^2) \\ +\dfrac{s_1^4+s_2^4-(s_1^2+s_2^2)}{2c} \end{array}\right] \end{pmatrix},$$

$$T_{12}=\begin{pmatrix} \left[\begin{array}{l} \dfrac{s_1^2-s_1^4}{2c}-2(s_1^2-s_2^2)^2 \\ -2(s_1^2-s_2^2) \end{array}\right] & \dfrac{-s_1\sqrt{1-s_1^2}\,s_2\sqrt{1-s_2^2}}{2c} & -is_1h\sqrt{1-s_1^2}\,(1+2s_1^2-2s_2^2) \\ \dfrac{-s_1\sqrt{1-s_1^2}\,s_2\sqrt{1-s_2^2}}{2c} & \left[\begin{array}{l} \dfrac{s_2^2-s_2^4}{2c}-2(s_1^2-s_2^2) \\ -2(s_1^2-s_2^2)^2 \end{array}\right] & is_2h\sqrt{1-s_2^2}\,(1+2s_1^2-2s_2^2) \\ \dfrac{-is_1\sqrt{1-s_1^2}\,(1+2s_1^2-2s_2^2)}{ch} & \dfrac{is_2\sqrt{1-s_2^2}\,(1+2s_1^2-2s_2^2)}{ch} & \left[\begin{array}{l} -2(s_1^2-s_2^2)^2-2(s_1^2-s_2^2) \\ +\dfrac{(s_1^2+s_2^2)-(s_1^4+s_2^4)}{2c} \end{array}\right] \end{pmatrix},$$

$$T_{21}=\begin{pmatrix} \left[\begin{array}{l} \dfrac{s_1^2-s_1^4}{2c}+2(s_1^2-s_2^2) \\ -2(s_1^2-s_2^2)^2 \end{array}\right] & \dfrac{-s_1\sqrt{1-s_1^2}\,s_2\sqrt{1-s_2^2}}{2c} & is_1h\sqrt{1-s_1^2}\,(1-2s_1^2+2s_2^2) \\ \dfrac{-s_1\sqrt{1-s_1^2}\,s_2\sqrt{1-s_2^2}}{2c} & \left[\begin{array}{l} \dfrac{s_2^2-s_2^4}{2c}+2(s_1^2-s_2^2) \\ -2(s_1^2-s_2^2)^2 \end{array}\right] & -is_2h\sqrt{1-s_2^2}\,(1-2s_1^2+2s_2^2) \\ \dfrac{is_1\sqrt{1-s_1^2}\,(1-2s_1^2+2s_2^2)}{ch} & \dfrac{is_2\sqrt{1-s_2^2}\,(1-2s_1^2+2s_2^2)}{ch} & \left[\begin{array}{l} \dfrac{s_2^2-s_2^4+s_1^2-s_1^4}{2c} \\ +2(s_1^2-s_2^2)-2(s_1^2-s_2^2)^2 \end{array}\right] \end{pmatrix},$$

$$T_{22}=\begin{pmatrix} \left[\begin{array}{l} \dfrac{s_1^4-s_1^2}{2c}-2(s_1^2-s_2^2) \\ +2(s_1^2-s_2^2)^2 \end{array}\right] & \dfrac{s_1\sqrt{1-s_1^2}\,s_2\sqrt{1-s_2^2}}{2c} & -is_1h\sqrt{1-s_1^2}\,(1-2s_1^2+2s_2^2) \\ \dfrac{s_1\sqrt{1-s_1^2}\,s_2\sqrt{1-s_2^2}}{2c} & \left[\begin{array}{l} \dfrac{s_2^4-s_2^2}{2c}-2(s_1^2-s_2^2) \\ +2(s_1^2-s_2^2)^2 \end{array}\right] & is_2h\sqrt{1-s_2^2}\,(1-2s_1^2+2s_2^2) \\ \dfrac{-is_1\sqrt{1-s_1^2}\,(1-2s_1^2+2s_2^2)}{ch} & \dfrac{is_2\sqrt{1-s_2^2}\,(1-2s_1^2+2s_2^2)}{ch} & \left[\begin{array}{l} 2(s_1^2-s_2^2)^2-2(s_1^2-s_2^2) \\ +\dfrac{s_1^4+s_2^4-(s_1^2+s_2^2)}{2c} \end{array}\right] \end{pmatrix}.$$

Now, the smoothing factor of CRB-JACP for (12) is presented as follows.

Theorem 2 The following sentiments hold for CRB-JACP of (12) in the 2D Stokes flow in multigrid simulation.

(i). The relation of the smoothing factor and parameter c of the artificial elliptic pressure term is

$$\rho_s=\begin{cases} \dfrac{1}{8c} & \dfrac{1}{8}<c\le\dfrac{1}{4} \\ \dfrac{1+4c}{16c} & \dfrac{1}{4}<c<\dfrac{4}{5} \end{cases} \tag{38}$$

(ii). The relation between the smoothing factor and the *h*-ellipticity is

$$\rho_s = \begin{cases} 16E_h & \frac{21}{1024} < E_h \le \frac{1}{32} \\ 32E_h - \frac{1}{2} & \frac{1}{32} < E_h < \frac{3}{32} \end{cases} \tag{39}$$

Proof. From (39), the eigenvalues of (38) are obtained as

$\lambda_{1,2,3} = 0$, $\lambda_{7,8,9} = 0$, $\lambda_{10} = (s_1^2 - s_2^2)^2$,

$$\lambda_4 = \frac{1}{2}(s_1^2 + s_2^2 - 1)(s_1^2 + s_2^2),$$

$$\lambda_{5,6} = \frac{1}{8c}\left[\begin{array}{l} 4c(s_1^2 + s_2^2 - 1)(s_1^2 + s_2^2) + s_1^4 - s_1^2 \\ +s_2^4 - s_2^2 \pm 2\sqrt{\begin{array}{l} c(2s_1^2 + 2s_2^2 - 1)^2 \cdot \\ (s_1^4 - s_1^2 + s_2^4 - s_2^2) \end{array}} \end{array}\right]$$

$$\lambda_{11,12} = \frac{1}{4c}\left[\begin{array}{l} 4c(s_1^2 - s_2^2)^2 + s_1^4 - s_1^2 + s_2^4 - s_2^2 \\ \pm 4\sqrt{c(s_1^2 - s_2^2)^2(s_1^4 - s_1^2 + s_2^4 - s_2^2)} \end{array}\right]$$

For the sake of convenient discussions, we replace s_1 and s_2 with s_1^2 and s_2^2, respectively, and S_{low} is changed from $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]^2$ to $[0, \frac{1}{2}]^2$. Thus, the nonzero eigenvalues of (36) are obtained as follows,

$$\lambda_4' = \frac{1}{2}(s_1 + s_2 - 1)(s_1 + s_2),\ \lambda_{10}' = (s_1 - s_2)^2 \tag{40}$$

$$\lambda_{5,6}' = \frac{1}{8c}\left[\begin{array}{l} 4c(s_1 + s_2 - 1)(s_1 + s_2) + s_1^2 - s_1 \\ +s_2^2 - s_2 \pm 2\sqrt{\begin{array}{l} c(2s_1 + 2s_2 - 1)^2 \\ (s_1^2 - s_1 + s_2^2 - s_2) \end{array}} \end{array}\right] \tag{41}$$

$$\lambda_{11,12}' = \frac{1}{4c}\left[\begin{array}{l} 4c(s_1 - s_2)^2 + s_1^2 - s_1 + s_2^2 - s_2 \\ \pm 4\sqrt{c(s_1 - s_2)^2(s_1^2 - s_1 + s_2^2 - s_2)} \end{array}\right] \tag{42}$$

As $(s_1, s_2) \in [0, \frac{1}{2}]^2$, then by applying (40)-(42), the following values are yielded

$$\left.\begin{array}{l} \max\limits_{(s_1,s_2)\in[0,\frac{1}{2}]^2} |\lambda_4'| = |\lambda_4'|\Big|_{(0,\frac{1}{2})} = \frac{1}{8} \\ \max\limits_{(s_1,s_2)\in[0,\frac{1}{2}]^2} |\lambda_{10}'| = |\lambda_{10}'|\Big|_{(0,\frac{1}{2})} = \frac{1}{4} \end{array}\right\} \tag{43}$$

$$s_1^2 - s_1 + s_2^2 - s_2 = (s_1 - \frac{1}{2})^2 + (s_2 - \frac{1}{2})^2 - \frac{1}{2} \le 0 \tag{44}$$

From (44), (41) and (42) we conclude that we have two pairs of the conjugate complex roots, respectively. Therefore, in order to compute the maximum values of the modulus for the above eigenvalues, it only needs to compute the squares of the modulus, which means that only $|\lambda_5'|^2$ and $|\lambda_{11}'|^2$ are considered. Combining (41) and (44), the expression of $|\lambda_5'|^2$, with respect to s_1 and s_2, is given as

$$|\lambda_5'|^2 = \frac{1}{64c^2} f(s_1, s_2) \tag{45}$$

in which

$$f(s_1, s_2) = \left[\begin{array}{l} s_1^2 - s_1 + s_2^2 - s_2 \\ +4c(s_1 + s_2 - 1)^2 \end{array}\right] \cdot \left[\begin{array}{l} s_1^2 - s_1 + s_2^2 - s_2 \\ +4c(s_1 + s_2)^2 \end{array}\right] \tag{46}$$

From (46), by using CAD function in the Mathematics [24], we obtain that the extreme values of $f(s_1, s_2)$ all lie on the boundary of $S_{low} = [0, \frac{1}{2}]^2$. From the assumption of $0 < c < \frac{4}{5}$, the maximum value of (45) is obtained as

$$\begin{aligned} \max_{(s_1,s_2)\in[0,\frac{1}{2}]^2} |\lambda_6'|^2 &= \max_{(s_1,s_2)\in[0,\frac{1}{2}]^2} |\lambda_5'|^2 \\ &= \max_{(s_1,s_2)\in[0,\frac{1}{2}]^2} \frac{1}{64c^2} f(s_1, s_2)\Big|_{(\frac{1}{2},\frac{1}{2})} \\ &= \frac{1}{64c^2}(\frac{1}{4} + 2c) \end{aligned} \tag{47}$$

Combining (42) and (44), the expression of $|\lambda_{11}'|^2$, with respect to s_1 and s_2, is given as

$$|\lambda_{11}'|^2 = \frac{1}{16c^2}(g(s_1, s_2))^2 \tag{48}$$

where

$$g(s_1, s_2) = s_1 - s_1^2 + s_2 - s_2^2 + 4c(s_1 - s_2)^2 \tag{49}$$

Thus, the extreme values of (49) all lie on the boundary of $[0, \frac{1}{2}]^2$. Thus, for $0 < c \le \frac{1}{4}$, the maximum value of (48) is given as follows

$$\begin{aligned} \max_{(s_1,s_2)\in[0,\frac{1}{2}]^2} |\lambda_{12}'|^2 &= \max_{(s_1,s_2)\in[0,\frac{1}{2}]^2} |\lambda_{11}'|^2 \\ &= \max_{(s_1,s_2)\in[0,\frac{1}{2}]^2} \frac{1}{16c^2} g(s_1, s_2)^2\Big|_{(\frac{1}{2},\frac{1}{2})} \\ &= \frac{1}{64c^2} \end{aligned} \tag{50}$$

and when $c > \frac{1}{4}$, the maximum value of (50) is

$$\begin{aligned} \max_{(s_1,s_2)\in[0,\frac{1}{2}]^2} |\lambda_{12}'|^2 &= \max_{(s_1,s_2)\in[0,\frac{1}{2}]^2} |\lambda_{11}'|^2 \\ &= \max_{(s_1,s_2)\in[0,\frac{1}{2}]^2} \frac{1}{16c^2} g(s_1, s_2)^2\Big|_{(0,\frac{1}{2})} \end{aligned}$$

$$=\frac{1}{64c^2}(\frac{1}{2}+2c)^2 \tag{51}$$

Applying (43), (47), (50) and (51), the maximum value of square of the modulus for the eigenvalues of (36) is

$$\max_{\substack{k=1,2,\cdots,12\\(s_1,s_2)\in[0,\frac{1}{2}]^2}}\left|\lambda_k'\right|^2=\begin{cases}\dfrac{1}{64c^2} & 0<c\le\dfrac{1}{4}\\ \dfrac{1}{64c^2}(\dfrac{1}{2}+2c)^2 & \dfrac{1}{4}\le c<\dfrac{4}{5}\end{cases} \tag{52}$$

To make the operator S_h^{RB} be convergent, from (52), $c>\frac{1}{8}$ should hold. Therefore, (52) is rewritten as

$$\max_{\substack{k=1,2,\cdots,12\\(s_1,s_2)\in[0,\frac{1}{2}]^2}}\left|\lambda_k'\right|=\begin{cases}\dfrac{1}{8c} & \dfrac{1}{8}<c\le\dfrac{1}{4}\\ \dfrac{1+4c}{16c} & \dfrac{1}{4}\le c<\dfrac{4}{5}\end{cases} \tag{53}$$

Then (38) can be obtained by (53). Furthermore, substituting (17) into (38), we yield (39) when $\frac{1}{8}<c<\frac{4}{5}$. Thus, Theorem 2 holds.□

In [2], for $c=\frac{1}{16}$, CRB-JACP for (12) is divergent. Applying (26) and (27), the maximum value of the *h*-ellipticity for (12) is obtained at $c=\frac{1}{28}$, and from (52), the smoothing factor is $\rho=\frac{7}{2}>1$. By the same way, for $0<c\le\frac{1}{8}$ we obtain that $\rho\ge1$. So, it is concluded that the CRB-JACP relaxation of (12) is divergent with high *h*-ellipticity. From Theorem 2, the smoothing factor for the CRB-JACP relaxation of (12) is independent of the mesh size, but depends on the parameter *c* artificial elliptic pressure term.

V. CONCLUSIONS

The error smoothing process of the collective relaxation for solving 2D Stokes flow in multigrid simulation is analytically presented with details. By using (19) and (33), the Fourier symbols of (12) with the trigonometric functions for the discrete operator and relaxation are transformed to the rational functions, and the smoothing process is greatly simplified. The analytical expression of the smoothing factor for the collective relaxation is obtained successfully. The value of the smoothing factor is an upper bound for the smoothing rates and is independent of the mesh size, but it depends on the converged parameter *c* of the artificial elliptic pressure term. The coarse-grid correction operator of the multigrid components is well handled by the method suggested by this paper. The prediction of the convergence properties for the complete multigrid methods is investigated for the 2D Stokes flow.

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