The Wigner-Ville Distribution in the Linear Canonical Transform Domain

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***Abstract*—The linear canonical transform (LCT) is a powerful tool for signal processing and optics. It is, therefore, worthwhile and interesting to consider the Wigner-Ville distribution (WVD) in the LCT domain. In this paper, we propose a new definition for the WVD associated with the instantaneous autocorrelation function in the LCT domain, which we name as WL, and also obtain some properties of WL. As a further generalization of WL, a new definition for the WVD associated with the offset linear canonical transform (OLCT) is given. Finally, we have achieved some applications of the new WVD in the LCT and OLCT domain to verify the derived theory.**

***Index Terms*—Offset linear canonical transform, linear canonical transform, Wigner-Ville distribution.**

I. Introduction

THE linear canonical transform (LCT) is introduced in 1970s by Collins, Moshinsky and Quesne as an integral transform with four parameters $(a,b;c,d)$ [1], [2]. It has been widely applied in several areas, including applied mathematics, optics and signal processing. With intensive research, many properties of the LCT are well studied. The Fourier transform (FT) [3], [4], fractional Fourier transform (FrFT) [5]–[8], Fresnel transform [9], Laplace transform and time scaling operations are all special cases of the LCT. The LCT is also known under different names as the Collins formula [1], the affine Fourier transform [10], the generalized Fresnel transform [11], the ABCD transform [12], etc.

The Wigner-Ville distribution (WVD) is a special type of quasi-probability distribution, which was proposed by Wigner in 1932 to study quantum corrections for classical statistical mechanics. It intended to supplant the wave function which appeared in the Schrodinger equation with a probability distribution in phase space. Later in 1948 J. Ville re-derived it as a quadratic representation of the local time-frequency energy of a signal. Among many methods, such as the iterative algorithm [13], [14], the chirp-Fourier transform method [15], the Radon-ambiguity transform [16], and the Wigner-Hough transform [17], the WVD is shown to be an important method in the linear-frequency-modulated (LFM) signal detection and parameter estimation, which is also essential in the signal processing community [18].

Based on the properties of the LCT, the FrFT, and the classical WVD, Pei and Ding [19] investigated the WVD and ambiguity function (AF) of the signal $F_{a,b,c,d}(u)$ and discuss the relations among the common fractional and canonical operators. Unlike the definition of WVD associated with the LCT in [19], Bai in [18] proposed generalized kind of WVD in the LCT domain, namely WDL, which can be thought as the affine transform of the autocorrelation function of $f(x)$.

In this paper, we propose a new definition for the WVD in the LCT domain, which we name as WL, and establish the various properties of the newly defined WL. Furthermore, we provide a new way to calculate the instantaneous frequency and the group delay. Meanwhile, a new definition for the WVD associated with the OLCT (WOL) is also proposed. This is considered as a further generalization of the WL. The applications of both the WL and WOL are also discussed.

The paper is organized as follows: Section II is Preliminary. In Section III, a new definition of WVD associated with the LCT is proposed along with its main properties. A new definition of the WVD associated with the OLCT is depicted in Section IV. The applications of the newly defined WVD in the LFM signal processing are investigated in details in Section V. and Section VI is the conclusion of this paper.

II. Preliminary

A. The Linear Canonical Transform (LCT)

The LCT of a signal $f(t)$ with parameter matrix A is defined as follows [1], [2], [18]–[29]

$$F_A(u)=\begin{cases}\int_{\Re} f(t)\sqrt{\dfrac{1}{i2\pi b}}e^{\frac{i}{2}\left(\frac{a}{b}t^2-\frac{2}{b}ut+\frac{d}{b}u^2\right)}dt, b\neq 0\\ \sqrt{d}e^{\frac{i}{2}cdu^2}f(du), \qquad b=0,\end{cases} \tag{1}$$

where the parameter matrix $A=(a,b,c,d)$, and the parameters $a,b,c,d\in\Re$ satisfying $ad-bc=1$. When the parameter $b=0$ is of no particular interest to our object. Therefore, we always assume $b\neq 0$ in this paper.

Recently, the LCT has had a great development [20]. Developing relevant theories for LCT can help to achieve more insights on its special cases and to carryover knowledge gained from one subject to others. For more detailed definitions and properties of the LCT, one can refer to [26].

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B. The Offset Linear Canonical Transform (OLCT)

The OLCT [30]–[32] is a six-parameter (a,b,c,d,u_0,ω_0) class of linear integral transform. It is a time-shifted and frequency-modulated version of the LCT. The OLCT with real parameters of $A=(a,b,c,d,u_0,\omega_0)$ of a signal $f(t)$ is defined by [30]–[32]

$$F_A(u)=\begin{cases}\int_{\Re} f(t)\sqrt{\frac{1}{i2\pi b}}e^{\frac{i}{2b}du_0^2}e^{\frac{i}{2b}\left[at^2+2t(u_0-u)-2u(du_0-b\omega_0)+du^2\right]}dt, b\neq 0\\ \sqrt{d}e^{\frac{i}{2}cd(u-u_0)^2+i\omega_0 u}f\left[d(u-u_0)\right], \qquad b=0,\end{cases} \tag{2}$$

It is easy to verify that the OLCT with parameters $(a,b,c,d,u_0,\omega_0)=(a,b,c,d,0,0)$ reduces to the LCT. And the FT, the fractional FT, the offset FT, the FRST and the frequency modulation, time scaling and time shifting are all special cases of the OLCT. As a generalization of many other linear transforms, the OLCT has found wide applications in applied mathematics, signal processing and optics [31]–[33].

C. The Classical WVD

The instantaneous autocorrelation function of a signal $f(t)$ is defined as follows [18]

$$R_f(t,\tau)=f\left(t+\frac{\tau}{2}\right)f^*\left(t-\frac{\tau}{2}\right),$$

and the WVD of a signal $f(t)$ defined as the FT of $R_f(t,\tau)$ for τ

$$W(t,\omega)=\int_{\Re} R_f(t,\tau)e^{-i\omega\tau}d\tau. \tag{3}$$

The WVD is one of the most useful time-frequency analysis tools Some properties of WVD are listed in [18], [19], [26], [27].

D. Previous Research Outputs

The WVD associated with the LCT (LCWD) given in [19] is defined as

$$WD_A(u,v)=\int_{\Re} F_A\left(u+\frac{\tau}{2}\right)F_A^*\left(u-\frac{\tau}{2}\right)e^{-iv\tau}d\tau,$$

where $F_A(u)$ is the LCT of signal $f(t)$ with parameter matrix A.

Recently, Bai et. al obtained WVD in the LCT domain, named the WDL, by substituting the orthogonal kernel $e^{-i\omega\tau}$ of the FT with the non-orthogonal kernel $\sqrt{1/i2\pi b}e^{\frac{i}{2}\left(\frac{a}{b}\tau^2-\frac{2}{b}u\tau+\frac{d}{b}u^2\right)}$ of the LCT. The main properties and applications of the WDL are also investigated in [18].

III. The New Definition and Properties of WVD

A. The Instantaneous Autocorrelation Function

We first give a new definition of the instantaneous autocorrelation function associate with LCT as the following

$$RL_f(t,\tau)=f\left(t+\frac{\tau}{2}\right)f^*\left(t-\frac{\tau}{2}\right)e^{\frac{ia}{b}\tau\left(t-\frac{\tau}{2}\right)}, \tag{4}$$

where $A=(a,b;c,d)$, $ad-bc=1.$ We obtain the classical instantaneous autocorrelation function when $a=0$, hence the new definition can be seen as the generalization of the classical instantaneous autocorrelation function.

B. The WVD in the LCT Domain

We definite the WVD in the LCT domain as

$$WL_f(t,u)=\int_{\Re} RL_f(t,\tau)e^{\frac{i}{2}\left(\frac{a}{b}\tau^2-\frac{2}{b}u\tau\right)}d\tau. \tag{5}$$

In order to make different from the existing results about the WVD, we denote it as $WL_f(t,u)$ and simplified as WL.

C. The Properties of the WL

The properties of the WL are investigated in this subsection as following.

Conjugation symmetry property

$$WL_f^*(t,u)=WL_f(t,u).$$

Time marginal property

$$\int_{\Re} WL_f(t,u)du=2\pi|b||f(t)|^2.$$

Frequency marginal property

$$\int_{\Re} WL_f(t,u)dt=2\pi|b|\left|\hat{f}(u)\right|^2.$$

Energy distribution property

$$\int_{\Re^2} WL_f(t,u)dtdu=2\pi|b|\int_{\Re}|f(t)|^2dt.$$

Moyal identical equation

$$\int_{\Re^2} WL_f(t,u)WL_f^*(t,u)dtdu=2\pi|b||\langle f,f\rangle|^2.$$

Translation property

For $f'(t)=f(t-\lambda)$, then

$$WL_{f'}(t,u)=WL_f(t-\lambda,u-a\lambda).$$

Modulation property

The WL of $f'(t)=f(t)e^{2\pi i u_0 t}$ has the form

$$WL_{f'}(t,u)=WL_f(t,u-2\pi u_0 b).$$

Multiplied signal

For $f(t)=g(t)h(t)$,

$$WL_f(t,u)=\frac{1}{2\pi|b|}\int_{\Re} WL_g(t,u-u')W_h(t,2\pi bu')du'.$$

Instantaneous frequency

For any signal $f(t)=|f(t)|e^{i\varphi(t)}$, its instantaneous frequency can be derived as following.

$$u_i(t)=\frac{\int_{\Re} uWL_f(t,u)du}{\int_{\Re} WL_f(t,u)du}.$$

Here, the denominator of above equation is the time marginal property, so the $u_i(t)$ can be rewritten as

$$u_i(t)=\frac{b}{i}\left(\varphi'(t)+\frac{at}{b}\right).$$

Group delay

For signal $\hat{f}(u)=\left|\hat{f}(u)\right|e^{i\phi(u)}$, its group delay can be derived as

$$\tau_i(u)=\frac{\int_{\Re} tWL_f(t,u)dt}{\int_{\Re} WL_f(t,u)dt}.$$

In time domain, the $\tau_i(u)$ can be rewritten as

$$\tau_i(u)=\frac{b}{4\pi|b|}\phi'(u).$$

D. The uncertainty principle of the WL

The Heisenberg's uncertainty principle [26], [28], [29] plays an important role in physics and communication. In this subsection, we obtain the uncertainty principle of the WL.

First we give two important equalities.

$$\int_{\Re} t^n WL_f(t,u)du=2\pi|b|\int_{\Re} t^n\left|f(t)\right|^2dt,$$

$$\int_{\Re} u^n WL_f(t,u)dt=2\pi|b|\int_{\Re} u^n\left|\hat{f}(u)\right|^2du.$$

Let $n=2,$ then we obtain the uncertainty principle of the WL as follows

$$2\pi|b|\int_{\Re} t^2\left|f(t)\right|^2dt+2\pi|b|\int_{\Re} u^2\left|\hat{f}(u)\right|^2du\ge\frac{\pi|b|}{2}.$$

IV. The new definition of WVD in the OLCT domain

From (4) we can definite the WVD in the OLCT domain as following

$$WOL_f(t,u)=\int_{\Re} RL_f(t,\tau)e^{\frac{i}{2b}\left[a\tau^2+2\tau(u_0-u)\right]}d\tau. \qquad (6)$$

We denote the WVD in the OLCT domain for parameter $A=(a,b,c,d,u_0,\omega_0)$ by $WOL_f(t,u)$ and simplified as the WOL of signal $f(t)$.

The relationship between WL and WOL is easy to verify, that when the parameter A reduces to $A=(a,b;c,d)$, the WOL reduces to the WL. Bellow we listed some basic properties of the WOL.

Conjugation symmetry property

$$WOL_f^*(t,u)=WOL_f(t,u).$$

Time marginal property

$$\int_{\Re} WOL_f(t,u)du=2\pi|b|\left|f(t)\right|^2.$$

Frequency marginal property

$$\int_{\Re} WOL_f(t,u)dt=2\pi|b|\left|\hat{f}(u)\right|^2.$$

Energy distribution property

$$\int_{\Re^2} WOL_f(t,u)dtdu=2\pi|b|\int_{\Re}\left|f(t)\right|^2dt.$$

Moyal identical equation

$$\int_{\Re^2} WOL_f(t,u)WOL_f^*(t,u)dtdu=2\pi|b|\left|\langle f,f\rangle\right|^2.$$

V. Applications

A. Signal Reconstruction by WL and WOL

According to (1) and (5) we obtain

$$\sqrt{\frac{1}{i2\pi b}}e^{\frac{id}{2b}u^2}WL_f(t,u)=\left[L_A h(t,\tau)\right](t,u),$$

Where

$$h(t,\tau)=\frac{1}{2\pi b}f\left(t+\frac{\tau}{2}\right)f^*\left(t-\frac{\tau}{2}\right)e^{\frac{ia}{b}\tau\left(t-\frac{\tau}{2}\right)}.$$

Hence we have

$$\begin{aligned}h(t,\tau)&=\left[L_{A^{-1}}L_A(h(t,\tau))(t,u)\right](t,\tau)\\&=\int_{\Re^2}\sqrt{\frac{1}{i2\pi b}}e^{\frac{id}{2b}u^2}WL_f(t,u)\sqrt{\frac{i}{2\pi b}}e^{-\frac{i}{2b}\left(a\tau^2-2u\tau+du^2\right)}du.\end{aligned}$$

let $t=\frac{\tau}{2},$ and we obtain that

$$\begin{aligned}&\frac{1}{2\pi b}f(\tau)f^*(0)\\&=\int_{\Re^2}\sqrt{\frac{1}{i2\pi b}}e^{\frac{id}{2b}u^2}WL_f(t,u)\sqrt{\frac{i}{2\pi b}}e^{-\frac{i}{2b}\left(a\tau^2-2u\tau+du^2\right)}du.\end{aligned}$$

This equation shows that the signal $f(t)$ can be reconstructed by the WL.

By the similar method, based on equation (2) and (6) we can obtain

$$\begin{aligned}&\sqrt{\frac{1}{i2\pi b}}e^{i\frac{d}{2b}u_0^2}e^{\frac{i}{2b}\left[-2u(du_0-b\omega_0)+du^2\right]}WOL_f(t,u)\\&=\left[O_A h(t,\tau)\right](t,u),\end{aligned}$$

Hence we have that

$$\begin{aligned}h(t,\tau)&=\left[O_{A^{-1}}(O_A h(t,\tau))(t,u)\right](t,\tau)=\\&=\int_{\Re}\sqrt{\frac{1}{i2\pi b}}e^{i\frac{d}{2b}u_0^2}e^{\frac{i}{2b}\left[-2u(du_0-b\omega_0)+du^2\right]}WOL_f(t,u)\\&\cdot\sqrt{\frac{-1}{i2\pi b}}e^{-i\frac{d}{2b}u_0^2}e^{-\frac{i}{2b}\left[a\tau^2+2\tau(u_0-u)-2u(du_0-b\omega_0)+du^2\right]}du.\end{aligned}$$

Let $t=\frac{\tau}{2}$, and we obtain

$$f(\tau)f^*(0)=\int_{\Re^2} RL_f(t,\tau)d\tau du.$$

The signal $f(t)$ can be also reconstructed by the WOL.

B. One-component LFM signal

Suppose the LFM signal modeled as $f(t)=e^{j\left(\omega_0 t+mt^2/2\right)}$; where ω_0 represent the initial frequency and m is frequency rate of signal $f(t)$, respectively. From the definition of the WL, we obtain the WL of the one-component LFM signal by

$$WL_f=2\pi\delta\left(\frac{u}{b}-\frac{at}{b}-(\omega_0+mt)\right).$$

From the definition of the WOL, WOL of $f(t)$ is

$$WOL_f = 2\pi\delta\left(\frac{u-u_0}{b} - \frac{at}{b} - (\omega_0 + mt)\right).$$

This equation shows that if we choose special parameter, the WL/WOL of $f(t)$ will produce an impulse in (t,u) plane.

C. Bi-component signal

When the processing signal is modeled as a bi-component finite-length signal as follows

$$f(t) = \begin{cases} e^{i\left(\omega_1 t + m_1 t^2/2\right)} + e^{i\left(\omega_2 t + m_2 t^2/2\right)}, & |t| < \dfrac{T}{2}, \\ 0, & |t| \geq \dfrac{T}{2}, \end{cases}$$

This signal can be expressed as summation of two one-component LFM signals $f(t) = f_1(t) + f_2(t)$, and the WL of $f(t)$ can be represented by the WL of $f_1(t)$ and $f_2(t)$ as follows

$$WL_f(t,u) = \int_{\Re} f\left(t+\frac{\tau}{2}\right) f^*\left(t-\frac{\tau}{2}\right) e^{i\left(-\frac{1}{b}u\tau + \frac{a}{b}t\tau\right)} d\tau$$
$$= WL_A^{f_1}(t,u) + WL_A^{f_2}(t,u) + WL_A^{f_1,f_2}(t,u) + WL_A^{f_2,f_1}(t,u),$$

and the WOL

$$WOL_f(t,u) = \int_{\Re} f\left(t+\frac{\tau}{2}\right) f^*\left(t-\frac{\tau}{2}\right) e^{i\left[\frac{a}{b}t\tau + \frac{\tau}{b}(u_0 - u)\right]} d\tau$$
$$= WOL_A^{f_1}(t,u) + WOL_A^{f_2}(t,u) + WOL_A^{f_1,f_2}(t,u) + WOL_A^{f_2,f_1}(t,u).$$

The auto-terms of the signal are represented by first two terms, whereas the others are the cross-terms.

VI. Conclusion

Based on the LCT and the classical WVD theory, this paper proposes a new kind of definition of WVD in the LCT domain, namely WL, and its generalization which we name as WOL also depicted in this paper. Main properties of the WL, including uncertainty principle are derived in detail. Furthermore, we provide a new way to calculate the instantaneous frequency and the group delay. Moreover, signal reconstruction by WL and WOL has been shown in this paper. In addition, the newly defined WL and WOL are applied in the LFM signal detection. The future works will be the applications of the newly defined WL in the nonstationary signal processing and develop relevant theories for the WVD and AF in the OLCT domain.

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