Approximate Controllability of Impulsive Fractional Partial Neutral Quasilinear Functional Differential Inclusions with Infinite Delay in Hilbert Spaces

Zuomao Yan

***Abstract*—In this paper, we consider the controllability problems for a class of impulsive fractional partial neutral quasilinear functional differential inclusions with infinite delay and (α, x)-resolvent family. In particular, a set of sufficient conditions are derived for the approximate controllability of nonlinear impulsive fractional dynamical systems by assuming the associated linear system is approximately controllable. The results are established by using the concept of resolvent family, fractional calculations and fixed point techniques. Finally, an example is provided to illustrate the obtained theory.**

***Index Terms*—approximate controllability, impulsive fractional partial neutral quasilinear functional differential inclusions, (α, x)-resolvent family, infinite delay, fixed-point theorem.**

I. Introduction

THE study of impulsive differential systems is linked to their utility in simulating processes and phenomena subject to short-time perturbations during their evolution. The perturbations are performed discretely and their duration is negligible in comparison with the total duration of the processes and phenomena. For the basic theory of impulsive differential equations the reader can refer to [1], [2], [3]. The theory of impulsive partial neutral differential equations, as well as inclusions, has become an active area of investigation due to their applications in fields such as mechanics, electrical engineering, medicine biology, ecology and so on. One can refer to [1] and the references therein. Fractional order models of real systems are often more adequate than the usually used integer order models, since the description of some systems is more accurate when the fractional derivative is used. Also, fractional differential equations have recently proved to be valuable tools in modeling of many physical phenomena in various fields of science and engineering, such as physics, mechanics, chemistry, engineering, etc. For details, see [4], [5], [6], [7] and the papers [8], [9], [10]. In recent years, the existence, uniqueness and other quantitative and qualitative properties of solutions to various semilinear fractional differential systems have been extensively studied in Banach spaces; see [11], [12], [13]. Moreover, much attention has been paid to several interesting results for impulsive fractional partial differential and integrodifferential systems; see [14], [15], [16], [17], [18], [19] and the references therein.

Controllability is one of the fundamental concepts in mathematical control theory and plays an important role in control systems. The problem of the exact controllability for some fractional differential and integrodifferential systems in abstract spaces have generated considerable interest among researchers [20], [21] and so on. Especially, Debbouchea and Baleanu [22] investigated the exact controllability result of a class of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems in a Banach space by using fixed point techniques and the concept of (α, x)-resolvent family. As proved by Triggiani [23], the concept of exact controllability is very limited for many parabolic partial differential equations, the approximate controllability is more appropriate for these control systems instead of exact controllability. For semilinear functional differential and evolution control systems including delay systems in Banach spaces, there are several papers devoted to the approximate controllability; see [24], [25], [26]. The authors in [27], [28], [29] also established the approximate controllability for various kinds of nonlinear impulsive differential deterministic and stochastic systems. Moreover, by using fixed point strategy, Sakthivel et al. [30] discussed the approximate controllability of semilinear fractional differential systems without delay. The approximate controllability problem for nonlinear fractional stochastic system in Hilbert spaces has been investigated [31]. Kumar and Sukavanam [32] proved some sufficient conditions for the approximate controllability of fractional order semilinear systems with bounded delay. Sukavanam and Kumar [33] obtained the approximate controllability of a fractional order system in which the nonlinear term depends on both state and control variables. Yan [34] studied the approximate controllability of partial neutral functional differential systems of fractional order with state-dependent delay. The approximate controllability for some fractional impulsive semilinear differential systems have been studied in several papers. For example, Ge et al. [35] concerned with the Approximate controllability of semilinear evolution equations of fractional order with nonlocal and impulsive conditions. Balasubramaniam et al. [36] derived sufficient conditions for the approximate controllability of impulsive fractional integro-differential systems with nonlocal conditions in Hilbert space. Chalishajar et al. [37] discussed the approximate controllability of abstract impulsive fractional neutral evolution equations with infinite delay in Banach spaces.

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Z. Yan is with the Department of Mathematics, Hexi University, Zhangye, Gansu 734000, P.R. China, e-mail: yanzuomao@163.com.

However, many systems arising from realistic models can be described as partial fractional differential or integro-differential inclusions (see [38], [39], [40] and references therein), so it is natural to extend the concept of approximate controllability to dynamical systems represented by fractional differential or integro-differential inclusions. Yan and Jia [41] established the approximate controllability of nonlinear fractional partial neutral integrodifferential inclusions with infinite delay and impulsive effects. In this paper, we consider the approximate controllability of a class of impulsive fractional partial neutral quasilinear functional differential inclusions with infinite delay and (α, x)-resolvent family in Hilbert spaces of the form

$$\begin{aligned} \frac{d^\alpha}{dt^\alpha}D(t,x_t) \in A(t,x(t))D(t,x_t) + Bu(t) \\ +F(t,x(t),x_t), \quad (1) \\ t\in J=[0,b], t\neq t_k, k=1,\dots,m, \end{aligned}$$

$$x_0=\varphi\in\mathcal{B}, \quad (2)$$

$$\Delta x(t_k)=I_k(x_{t_k}), \quad k=1,\dots,m, \quad (3)$$

where the state $x(\cdot)$ takes values in a separable real Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, $0<\alpha\leq 1$, $A(t,\cdot)$ is a closed linear operator defined on a dense domain $D(A)$ in H into H such that $D(A)$ is independent of t. It is assumed also that $A(t,\cdot)$ generates an evolution operator in the Hilbert space H, the control function $u\in L^2(J,U)$, a Hilbert space of admissible control functions. Further, B is a bounded linear operator from U to H; the time history $x_t:(-\infty,0]\to H$, defined by $x_t(s):=x(t+s)$ belongs to an abstract phase space $\mathcal{B}$ defined axiomatically; and $F: J\times H\times\mathcal{B}\to\mathcal{P}(H)$ is a bounded closed convex-valued multi-valued map, $\mathcal{P}(H)$ is the family of all nonempty subsets of H, $G: J\times\mathcal{B}\to H, D(t,\psi)=\psi(0)-G(t,\psi), \psi\in\mathcal{B}$ $I_k:\mathcal{B}\to H(k=1,\dots,m)$, are functions subject to some additional conditions. Moreover, let $0<t_1<\cdots<t_m<b$, are prefixed points and the symbol $\Delta x(t_k)=x(t_k^+)-x(t_k^-)$, where $x(t_k^-)$ and $x(t_k^+)$ represent the right and left limits of $x(t)$ at $t=t_k$, respectively.

To the best of our knowledge, there is no work reported on the approximate controllability of impulsive fractional partial neutral quasilinear infinite delay differential inclusions in Hilbert spaces, which is expressed in the form (1)-(3). The papers [41] studied the approximate controllability of fractional impulsive integrodifferential inclusions, besides the fact that [41] applies to the approximate controllability of systems with the α-resolvent operator, the class of impulsive systems is also different from the one studied here. Further, many control systems arising from realistic models can be described as fractional impulsive partial differential inclusions with (α, x)-resolvent family. So it is natural to extend the concept of approximate controllability to dynamical systems represented by these impulsive systems. Motivated by the previously mentioned papers, we will study this interesting problem. Sufficient conditions for the approximate controllability are given by means of the nonlinear alternative of Leray-Schauder type for multivalued maps due to D. O'Regan [42] with the concept of (α, x)-resolvent family combined with approximation techniques. Especially, the known results appeared in [35], [36], [37], [41] are generalized to the fractional multi-valued settings with (α, x)-resolvent family and the case of infinite delay. Further, the operators $I_k(k=1,\dots,m)$ are continuous but without imposing completely continuous and Lipschitz condition. Therefore, the obtained results can be seen as a contribution to this emerging field.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and necessary preliminaries. Section 3 verifies the existence of mild solutions for impulsive fractional control system (1)-(3). Section 4 we establish the approximate controllability of impulsive fractional control system (1)-(3). Finally in Section 5, an example is given to illustrate our results.

II. Preliminaries

In this section, we introduce some basic definitions, notations and lemmas which are used throughout this paper.

Let $(H,\|\cdot\|)$ be a Hilbert space. $C(J,H)$ is the Hilbert space of all continuous functions from J into H with the norm $\| x \|_\infty=\sup\{\| x(t) \|: t\in J\}$ and $L(H)$ denotes the Hilbert space of bounded linear operators from H to H. A measurable function $x: J\to H$ is Bochner integrable if and only if $\| x \|$ is Lebesgue integrable. For properties of the Bochner integral see Yosida [43]. $L^1(J,H)$ denotes the Hilbert space of measurable functions $x: J\to H$ which are Bochner integrable normed by $\| x \|_{L^1}=\int_0^b \| x(t) \| dt$ for all $x\in L^1(J,H)$. Furthermore, the notation, $B_r(x,H)$ stands for the closed ball with center at x and radius $r>0$ in H.

Let $\mathcal{P}(H)$ denotes the class of all nonempty subsets of H. Let $\mathcal{P}_{bd,cl}(H)$, $\mathcal{P}_{cp,cv}(H)$, $\mathcal{P}_{bd,cl,cv}(H)$ and $\mathcal{P}_{cd}(H)$ denote respectively the family of all nonempty bounded-closed, compact-convex, bounded-closed-convex and compact-acyclic (see [44]) subsets of H. For $x\in H$ and $Y,Z\in\mathcal{P}_{bd,cl}(H)$, we denote by $D(x,Y)=\inf\{\| x-y \|: y\in Y\}$ and $\tilde{\rho}(Y,Z)=\sup_{a\in Y}D(a,Z)$, and the Hausdorff metric $H_d:\mathcal{P}_{bd,cl}(H)\times\mathcal{P}_{bd,cl}(H)\to R^+$ by $H_d(A,B)=\max\{\tilde{\rho}(A,B),\tilde{\rho}(B,A)\}$.

G is called upper semicontinuous (u.s.c.) on H if, for each $x_0\in H$, the set $G(x_0)$ is a nonempty, closed subset of H and if, for each open set S of H containing $G(x_0)$, there exists an open neighborhood S of x_0 such that $G(S)\subseteq V$. F is said to be completely continuous if $G(V)$ is relatively compact, for every bounded subset $V\subseteq H$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if F has a closed graph, i.e. $x_n\to x_*, y_n\to y_*, y_n\in G(x_n)$ imply $y_*\in G(x_*)$.

A multivalued map $G: J\to P_{bd,cl,cv}(H)$ is said to be measurable if for each $x\in H$, the function $t\mapsto D(x,G(t))$ is a measurable function on J.

Definition 1. Let $G: H\to P_{bd,cl}(H)$ be a multivalued map. Then G is called a multivalued contraction if there exists a constant $\kappa\in(0,1)$ such that for each $x,y\in H$ we have

$$H_d(G(x)-G(y))\leq\kappa \| x-y \|.$$

The constant κ is called a contraction constant of G.

Definition 2 ([6], [7]). The fractional integral of order $\mu>0$ is defined by

$$I_a^\mu f(t)=\frac{1}{\Gamma(\mu)}\int_b^t\frac{f(s)}{(t-s)^{1-\mu}}ds, \quad (4)$$

where Γ is the gamma function and $f \in L^1([a,b], R^+)$.

If $a = 0$, we can write $I_a^\mu f(t) = (g_\mu * f)(t)$, where

$$g_\mu(t) := \begin{cases} \frac{1}{\Gamma(\mu)} & t > 0, \\ 0 & t \le 0, \end{cases}$$

as usual, $*$ denotes the convolution of functions, also we have $\lim_{\mu\to 0} g_\mu(t) = \delta(t)$, which is the delta function.

Definition 3 ([6], [7]). The Riemann-Liouville fractional derivative of order $n-1 < \alpha < n$ is defined by

$${}_aD_t^\mu f(t) = \frac{1}{\Gamma(n-\mu)} \frac{d^n}{dt^n} \int_b^t (t-s)^{n-\mu-1} ds,$$

where f is an abstract continuous function on the interval $[a,b]$ and $n \in N$.

Definition 4 ([6], [7]). The Caputo fractional derivative of order $n-1 < \alpha < n$ is defined by

$${}_a^cD_t^\mu f(t) = \frac{1}{\Gamma(n-\mu)} \int_a^t (t-s)^{n-\mu-1} f^n(s) ds.$$

Definition 5 ([45]). A two parameter family of bounded linear operators $U(t,s), 0 \le s \le t \le b$, on H is called an evolution system if the following two conditions are satisfied

(i) $U(t,t) = I, U(t,\tau)U(\tau,s) = U(t,s)$ for $0 \le s,\tau \le t \le b$,

(ii) $(t,s) \to U(t,s)$ is strongly continuous for $0 \le s \le t \le b$.

Let E be the Banach space formed from $D(A)$ with the graph norm. Since $A(t)$ is a closed operator, it follows that $A(t)$ is in the set of bounded operators from E to H.

Definition 6. Let $A(t,x)$ be a closed and linear operator with domain $D(A)$ defined on a Hilbert space H and $\alpha > 0$. Let $\rho[A(t,x)]$ be the resolvent set of $A(t,x)$. We call $A(t,x)$ the generator of an (α,x)-resolvent family if there exist $\omega \ge 0$ and a strongly continuous function $R_{(\alpha,x)} : R^+ \times R^+ \to L(H)$ such that $R_{(\alpha,x)}(s,s) = I, 0 \le s \le b$, and $\{\lambda^\alpha : \mathrm{Re}(\lambda) > \omega\} \subset \rho(A)$, for $0 \le s \le t < \infty$,

$$(\lambda^\alpha I - A(s,x))^{-1} y = \int_0^\infty e^{\lambda(t-s)} R_{(\alpha,x)}(t,s) v dt, \quad \mathrm{Re}(\lambda) > \omega, (x,y) \in H^2. \quad (5)$$

In this case, $R_{(\alpha,x)}(t,s)$ is called the (α,x)-resolvent family generated by $A(t,x)$.

Remark 1.

(i) In the deleting case of s and x, (5) will be reduced to the introduced concept by [46].

(ii) We can deduce that (1)-(3) is well posed if and only if, $A(t,x)$ is the generator of (α,x)-resolvent family.

(iii) Here, $R_{(\alpha,x)}(t,s)$ can be extracted from the evolution operator of the generator $A(t,x)$.

(iv) The (α,x)-resolvent family is similar to the evolution operator for nonautonomous differential equations in a Banach space.

In this paper, we assume that the phase space $(\mathcal{B}, \|\cdot\|_\mathcal{B})$ is a seminormed linear space of functions mapping $(-\infty,0]$ into H, and satisfying the following fundamental axioms due to Hale and Kato (see e.g., in [47]).

(A) If $x : (-\infty, \theta+b] \to H$, $b > 0$, is such that $x|_{[\theta,\theta+b]} \in \mathcal{PC}([\theta,\theta+b], H)$ and $x_\theta \in \mathcal{B}$, then for every $t \in [\theta, \theta+b]$ the following conditions hold:

(i) x_t is in $\mathcal{B}$;

(ii) $\| x(t) \| \le \tilde{H} \| x_t \|_\mathcal{B}$;

(iii) $\| x_t \|_\mathcal{B} \le K(t-\theta) \sup\{\| x(s) \| : \theta \le s \le t\} + M(t-\theta) \| x_\theta \|_\mathcal{B}$, where $\tilde{H} \ge 0$ is a constant; $K, M : [0,\infty) \to [1,\infty)$, K is continuous and M is locally bounded; $\tilde{H}, K, M$ are independent of $x(\cdot)$.

(B) For the function $x(\cdot)$ in (A), the function $t \to x_t$ is continuous from $[\theta, \theta+b]$ into $\mathcal{B}$.

(C) The space $\mathcal{B}$ is complete.

Example 1. The phase space $\mathcal{PC}_r \times L^p(h,H)$. Let $1 \le p < \infty, 0 \le r < \infty$ and let $h : (-\infty,-r] \to R$ be a nonnegative measurable function which satisfies the conditions (h-5), (h-6) in the terminology of Hino et al. [48]. Briefly, this means that h is locally integrable and there is a non-negative, locally bounded function γ on $(-\infty,0]$ such that $h(\xi+\tau) \le \gamma(\xi)h(\tau)$ for all $\xi \le 0$ and $\theta \in (-\infty,-r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty,-r)$ is a set whose Lebesgue measure zero. We denote by $\mathcal{PC}_r \times L^p(h,H)$ the set consists of all classes of functions $\varphi : (-\infty,0] \to H$ such that φ is continuous on $[-r,0]$, Lebesgue-measurable, and $h \| \varphi \|^p$ is Lebesgue integrable on $(-\infty,-r)$. The seminorm is given by

$$\| \varphi \|_\mathcal{B} = \sup_{-r\le\tau\le 0} \| \varphi(\tau) \| + \left(\int_{-\infty}^{-r} h(\tau) \| \varphi \|^p \, d\tau \right)^{1/p}.$$

The space $\mathcal{B} = \mathcal{PC}_r \times L^p(h,H)$ satisfies axioms (A)-(C). Moreover, when $r = 0$ and $p = 2$, we can take $\tilde{H} = 1$, $M(t) = \gamma(-t)^{1/2}$ and $K(t) = 1 + (\int_{-t}^0 h(\tau)d\tau)^{1/2}$ for $t \ge 0$ (see [48], Theorem 1.3.8 for details).

Remark 2. Let $\varphi \in \mathcal{B}$ and $t \le 0$. The notation φ_t represents the function defined by $\varphi_t(\tau) = \varphi(t+\theta)$. Consequently, if the function $x(\cdot)$ in axiom (A) is such that $x_0 = \varphi$, then $x_t = \varphi_t$. We observe that φ_t is well-defined for $t < 0$ since the domain of φ is $(-\infty,0]$. We also note that, in general, $\varphi_t \notin \mathcal{B}$; consider, for instance, a discontinuous function in $\mathcal{PC}_r \times L^p(h,H)$ for $r > 0$.

Remark 3. In the rest of this paper M_b and K_b are the constants defined by $M_b = \sup_{t\in J} M(t)$ and $K_b = \sup_{t\in J} K(t)$.

To describe appropriately our problems we say that a function $x : [\mu,\tau] \to H$ is a normalized piecewise continuous function on $[\mu,\tau]$ if x is piecewise continuous and continuous on $[\mu,\tau]$. We denote by $\mathcal{PC}([\mu,\tau],H)$ the space formed by the normalized piecewise continuous from $[\mu,\tau]$ into H. In particular, we introduce the space $\mathcal{PC}$ formed by all functions $x : [0,b] \to H$ such that x is continuous at $t \ne t_k$, $x(t_k) = x(t_k^-)$ and $x(t_k^+)$ exists for $k = 1,2,\ldots,m$. In this paper, we always assume that $\mathcal{PC}$ is endowed with the norm $\| x \|_{\mathcal{PC}} = \sup_{t\in[0,b]} \| x(t) \|$. Then $(\mathcal{PC}, \|\cdot\|_{\mathcal{PC}})$ is a Banach space.

To simplify the notations, we put $t_0 = 0, t_{m+1} = b$ and for $x \in \mathcal{PC}$, we denote by $\hat{x}_k \in C([t_k,t_{k+1}];H), k = 0,1,\ldots,m$, the function given by

$$\hat{x}_k(t) := \begin{cases} x(t) & \text{for} \quad t \in (t_k,t_{k+1}], \\ x(t_k^+) & \text{for} \quad t = t_k. \end{cases}$$

Moreover, for $\widetilde{B} \subseteq \mathcal{PC}$ we denote by $\hat{B}_k, k = 0,1,\ldots,m$, the set $\hat{B}_k = \{\hat{x}_k : x \in \widetilde{B}\}$.

Let $x_b(x_0;u)$ be the state value of system (1)-(3) at terminal time b corresponding to the control u and the initial value $x_0 = \varphi(t) \in \mathcal{B}$. Introduce the set

$$\mathcal{B}(b, x_0) = \{x_b(x_0; u)(0) : u(\cdot) \in L^2(J, U)\},$$

which is called the reachable set of system (1)-(3) at terminal time b, its closure in H is denoted by $\overline{\mathcal{B}(b, x_0)}$.

Definition 7. The system (1)-(3) is said to be approximately controllable on the interval J if $\overline{\mathcal{B}(b, x_0)} = H$.

It is convenient at this point to define operators

$$\Gamma_0^b = \int_0^b R_{(\alpha,x)}(b, s)BB^* R^*_{(\alpha,x)}(b, s)ds,$$

$$S(a, \Gamma_0^b) = (aI + \Gamma_0^b)^{-1} \text{ for } a > 0,$$

where B^* denotes the adjoint of B and $R^*_{(\alpha,x)}(t, s)$ is the adjoint of $R_{(\alpha,x)}(t, s)$. It is straightforward that the operator Γ_0^b is a linear bounded operator.

(S1) $aS(a, \Gamma_0^b) \to 0$ as $a \to 0^+$ in the strong operator topology.

Definition 8. A function $x : (-\infty, b] \to H$ is called a mild solution of the system (1)-(3) if $x_0 = \varphi \in \mathcal{B}$ and $\Delta x(t_k) = I_k(x_{t_k}), k = 1, \ldots, m$, such that the following integral equation holds

$$\begin{aligned} x(t) &= R_{(\alpha,0)}(t, 0)[\varphi(0) - G(0, \varphi)] + G(t, x_t) \\ &\quad + \int_0^t R_{(\alpha,x)}(t, s)[Bu(s) + f(s)]ds \\ &\quad + \sum_{0<t_k<t} R_{(\alpha,x)}(t, t_k)I_k(x_{t_k}), \quad t \in J, \end{aligned}$$

where $f \in S_{F,x} = \{f \in L^1(J, H) : f(t) \in F(t, x(t), x_t)$ a.e. $t \in J\}$.

Consider the following linear fractional differential system

$$\frac{d^\alpha}{dt^\alpha} x(t) = A(t, x(t))x(t) + Bu(t), \quad t \in J = [0, b], \quad (6)$$

$$x = \varphi \in \mathcal{B}. \quad (7)$$

From [24] and [25], we have the following lemma:

Lemma 1. The assumption (S1) holds if and only if the linear fractional differential control system (6)-(7) is approximately controllable on J.

The proof of Lemma 1 can be performed along the direction of the proof of Theorem 2 in [25].

Lemma 2. A set $\widetilde{B} \subseteq \mathcal{PC}$ is relatively compact in $\mathcal{PC}$ if, and only if, the set $\hat{B}_k$ is relatively compact in $C([t_k, t_{k+1}]; H)$, for every $k = 0, 1, \ldots, m$.

Lemma 3. Let $R_{(\alpha,x)}(t, s)$ be the $R_{(\alpha,x)}$-resolvent family for the fractional problem (1)-(3). There exists a constant $K > 0$ such that

$$\begin{aligned} &\| R_{(\alpha,x)}(t, s)\omega - R_{(\alpha,y)}(t, s)\omega \| \\ &\quad \leq K \| \omega \| \int_s^t \| x(\tau) - y(\tau) \| d\tau, \end{aligned}$$

for every $x, y \in \mathcal{PC}(J, H)$ and every $\omega \in H$.

The proof is similar to the proof of Lemma 3.1 in [22], and we omit the details here.

Lemma 4 ([42] Nonlinear alternative of Leray-Schauder type for multivalued maps due to D. O'Regan). Let H be a Hilbert space with V an open,convex subset of H and $y \in H$. Suppose

(a) $\Phi : \overline{V} \to \mathcal{P}_{cd}(H)$ has closed graph, and

(b) $\Phi : \overline{V} \to \mathcal{P}_{cd}(H)$ is a condensing map with $\Phi(\overline{V})$ a subset of a bounded set in H hold. Then either

(i) Φ has a fixed point in $\overline{V}$; or

(ii) There exist $y \in \partial V$ and $\lambda \in (0, 1)$ with $y \in \lambda\Phi(y) + (1 - \lambda)\{y_0\}$.

III. Existence of solutions for impulsive fractional control system

In this section, we prove the existence of solutions for impulsive fractional control system (1)-(3). We make the following hypotheses:

(H1) The operator $A(t, x)$ generates an (α, x)-resolvent families $R_{(\alpha,x)}(t, s)$ is compact for all $t - s > 0$.

(H2) There exist constants M, σ such that $\| R_{(\alpha,x)}(t, s) \| \leq Me^{\sigma(t-s)}$ for every $s, t \in J$.

(H3) The multi-valued map $F : J \times H \times \mathcal{B} \to \mathcal{P}_{bd,cl,cv}(H)$; for each $t \in J$, the function $F(t, \cdot, \cdot) : \mathcal{B} \to \mathcal{P}_{bd,cl,cv}(H)$ is u.s.c. and for each $(x, \psi) \in H \times \mathcal{B}$, the function $F(\cdot, x, \psi)$ is measurable; for each fixed $(x, \psi) \in \mathcal{B}$, the set

$$S_{F,x,\psi} = \{f \in L^1(J, H) : f(t) \in F(t, x, \psi) \text{ for a.e } t \in J\}$$

is nonempty.

(H4) There exist continuous function $m : J \to [0, \infty)$ and a continuous nondecreasing function $\Theta : [0, \infty) \to (0, \infty)$ such that

$$\begin{aligned} \| F(t, x, \psi) \| &= \sup\{\| f \| : f \in F(t, x, \psi) \\ &\leq m(t)\Theta(\| x \| + \| \psi \|_{\mathcal{B}}), \\ &\qquad t \in J, \psi \in \mathcal{B} \end{aligned}$$

with

$$\int_1^\infty \frac{1}{s + \Theta(2s)} ds = \infty.$$

(H5) The function $G : J \times \mathcal{B} \to H$ is continuous and there exists $L > 0$ such that

$$\| G(t, \psi_1) - G(t, \psi_2) \| \leq L \| \psi_1 - \psi_2 \|_{\mathcal{B}}, \quad t \in J, \psi_1, \psi_2 \in \mathcal{B},$$

and

$$\| G(t, \psi) \| \leq L(\| \psi \|_{\mathcal{B}} + 1), \quad t \in J, \psi \in \mathcal{B}.$$

(H6) The functions $I_k : \mathcal{B} \to H$ are continuous and there exist constants c_k such that

$$\limsup_{\|\psi\|_{\mathcal{B}} \to \infty} \frac{\| I_k(\psi) \|}{\| \psi \|_{\mathcal{B}}} = c_k$$

for every $\psi \in \mathcal{B}, k = 1, \ldots, m$.

Lemma 5 ([49]). Let J be a compact interval and H be a Hilbert space. Let F be a multi-valued map satisfying (H3) and let P be a linear continuous operator from $L^1(J, H)$ to $C(J, H)$. Then, the operator

$$P \circ S_F : C(J, H) \to \mathcal{P}_{cp,cv}(H),$$

$$x \to (P \circ S_F)(x) := P(S_F, x)$$

is a closed graph in $C(J, H) \times C(J, H)$.

Theorem 1. *If the assumptions* (H1)-(H6) *are satisfied. Further, suppose that for all* $a > 0$, *then the system* (1)-(3) *has at least one mild solution on* J, *provided that* $[Kb(\| \varphi \|_{\mathcal{B}} +1)+1]MLK_b < 1$ *and*

$$K_bML + K_bM^2N_*\left[\frac{1}{a}(M_*M_1N_*)^2+1\right]\sum_{k=1}^{m} c_k < 1, \quad (8)$$

where $M_* = M\max\{1, e^{\sigma b}\}, N_* = \max\{1, e^{-\sigma b}\}, M_1 = \| B \|$.

Proof. Consider the space $\mathcal{Y} = \{x : (-\infty, b] \to H; x(0) = \varphi(0), x|_J \in \mathcal{PC}(J, H)\}$ endowed with the uniform convergence topology and define the multi-valued map $\Phi : \mathcal{Y} \to \mathcal{P}(\mathcal{Y})$ by Φx the set of $h \in \mathcal{Y}$ such that

$$h(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ R_{(\alpha,\bar{x})}(t,0)[\varphi(0) - G(0,\varphi)] + G(t,\bar{x}_t) \\ \quad + \int_0^t R_{(\alpha,\bar{x})}(t,s)Bu^a_{\bar{x}}(s)ds \\ \quad + \int_0^t R_{(\alpha,\bar{x})}(t,s)f(s)ds \\ \quad + \sum\limits_{0<t_k<t} R_{(\alpha,\bar{x})}(t,t_k)I_k(\bar{x}_{t_k}), & t \in J, \end{cases}$$

where $f \in S_{F,\bar{x}} = \{f \in L^1(J,H) : f(t) \in F(t,\bar{x},\bar{x}_t)$ a.e. $t \in J\}$, and $\bar{x} : (-\infty, 0] \to H$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on J, and

$$\begin{aligned} u^a_{\bar{x}}(s) = {} & B^*R^*_{(\alpha,\bar{x})}(b,s)S(a,\Gamma^b_0)\Big[x_b - R_{(\alpha,\bar{x})}(b,0)[\varphi(0) \\ & -G(0,\varphi)] - G(b,\bar{x}_b) - \int_0^b R_{(\alpha,\bar{x})}(b,\eta)f(\eta)d\eta \\ & -\sum_{k=1}^{m} R_{(\alpha,\bar{x})}(b,t_k)I_k(\bar{x}_{t_k})\Big], \end{aligned}$$

where $f \in S_{F,\bar{x}}$ and $\bar{x} : (-\infty, 0] \to H$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on J. In what follows, we aim to show that the operator Φ has a fixed point, which is a solution of the problem (1)-(3).

Let $\{\delta_n : n \in N\}$ be a decreasing sequence in $(0, t_1) \subset (0, b)$ such that $\lim_{n\to\infty} \delta_n = 0$. To prove the above theorem, we consider the following problem:

$$\begin{aligned} \frac{d^\alpha}{dt^\alpha}\widetilde{D}(t,x_t) \in {} & A(t,x(t))\widetilde{D}(t,x_t) + Bu(t) \\ & +F(t,x(t),x_t), \quad (9) \\ & t \in J = [0,b], t \neq t_k, k = 1,\dots,m, \end{aligned}$$

$$x_0 = \varphi \in \mathcal{B}, \quad (10)$$

$$\Delta x(t_k) = R_{(\alpha,x)}(\delta_n, 0)I_k(x_{t_k}), \quad k = 1,\dots,m, \quad (11)$$

where $\widetilde{D}(t,x_t) = \varphi(0) - R_{(\alpha,x)}(\delta_n,0)G(t,x_t)$. We shall show that the problem has at least one mild solution $x_n \in \mathcal{Y}$.

For fixed $n \in N$, set the multi-valued map $\Phi_n : \mathcal{Y} \to \mathcal{P}(\mathcal{Y})$ by $\Phi_n x$ the set of $h_n \in \mathcal{Y}$ such that

$$h_n(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ R_{(\alpha,\bar{x})}(t,0)[\varphi(0) - R_{(\alpha,\bar{x})}(\delta_n,0)G(0,\varphi)] \\ \quad +R_{(\alpha,\bar{x})}(\delta_n,0)G(t,\bar{x}_t) \\ \quad + \int_0^t R_{(\alpha,\bar{x})}(t,s)Bu^a_{n,\bar{x}}(s)ds \\ \quad + \int_0^t R_{(\alpha,\bar{x})}(t,s)f(s)ds \\ \quad + \sum\limits_{0<t_k<t} R_{(\alpha,\bar{x})}(t,t_k) \\ \quad \times R_{(\alpha,\bar{x})}(\delta_n,0)I_k(\bar{x}_{t_k}), & t \in J, \end{cases}$$

where

$$\begin{aligned} u^a_{n,\bar{x}}(s) = {} & B^*R^*_{(\alpha,\bar{x})}(b,s)S(a,\Gamma^b_0)\Big[x_b - R_{(\alpha,\bar{x})}(b,0)[\varphi(0) \\ & -R_{(\alpha,\bar{x})}(\delta_n,0)G(0,\varphi)] - R_{(\alpha,\bar{x})}(\delta_n,0)G(b,\bar{x}_b) \\ & -\int_0^b R_{(\alpha,\bar{x})}(b,\eta)f(\eta)d\eta \\ & -\sum_{k=1}^{m} R_{(\alpha,\bar{x})}(b,t_k)R_{(\alpha,\bar{x})}(\delta_n,0)I_k(\bar{x}_{t_k})\Big], \end{aligned}$$

and $f \in S_{F,\bar{x}}$. It is easy to see that the fixed point of Φ_n is a mild solution of the Cauchy problem (9)-(11).

Let $\bar{\varphi} : (-\infty, 0) \to H$ be the extension of $(-\infty, 0]$ such that $\bar{\varphi}(\theta) = \varphi(0)$ on J. We now show that Φ_n satisfies all the conditions of Lemma 4. The proof will be given in several steps.

Step 1. We shall show there exists an open set $V \subseteq \mathcal{Y}$ with $x \in \lambda\Phi_n x$ for $\lambda \in (0,1)$ and $x \notin \partial V$.

Let $\lambda \in (0,1)$ and let $x \in \lambda\Phi_n x$, then there exists an $f \in S_{F,\bar{x}}$ such that

$$\begin{aligned} x(t) = {} & \lambda R_{(\alpha,\bar{x})}(t,0)[\varphi(0) - R_{(\alpha,\bar{x})}(\delta_n,0)G(0,\varphi)] \\ & +\lambda R_{(\alpha,\bar{x})}(\delta_n,0)G(t,\bar{x}_t) \\ & +\lambda\int_0^t R_{(\alpha,\bar{x})}(t,s)BB^*R^*_{(\alpha,\bar{x})}(b,s)S(a,\Gamma^b_0) \\ & \times\Big[x_b - R_{(\alpha,\bar{x})}(b,0)[\varphi(0) - R_{(\alpha,\bar{x})}(\delta_n,0)G(0,\varphi)] \\ & -R_{(\alpha,\bar{x})}(\delta_n,0)G(b,\bar{x}_b) - \int_0^b R_{(\alpha,\bar{x})}(b,\eta)f(\eta)d\eta \\ & -\sum_{k=1}^{m} R_{(\alpha,\bar{x})}(b,t_k)R_{(\alpha,\bar{x})}(\delta_n,0)I_k(\bar{x}_{t_k})\Big](s)ds \\ & +\lambda\int_0^t R_{(\alpha,\bar{x})}(t,s)f(s)ds + \lambda\sum_{0<t_k<t} R_{(\alpha,\bar{x})}(t,t_k) \\ & \times R_{(\alpha,\bar{x})}(\delta_n,0)I_k(\bar{x}_{t_k}), \quad t \in J, \quad (12) \end{aligned}$$

for some $\lambda \in (0,1)$. However, on the other hand, from the condition (H6), we conclude that there exist positive constants $\epsilon_k (k = 1,\dots,m), \gamma_1$ such that, for all $\| \psi \|_{\mathcal{B}} > \gamma_1$,

$$\| I_k(\psi) \| \leq (c_k + \epsilon_k) \| \psi \|_{\mathcal{B}},$$

$$\begin{aligned} K_bML + K_bM^2N_*\left[\frac{1}{a}(M_*M_1N_*)^2+1\right] \\ \times\sum_{k=1}^{m}(c_k+\epsilon_k) < 1. \quad (13) \end{aligned}$$

Let

$$F_1 = \{\psi : \| \psi \|_{\mathcal{B}} \leq \gamma_1\}, \quad F_2 = \{\psi : \| \psi \|_{\mathcal{B}} > \gamma_1\},$$

$$C_1 = \max\{\| I_k(\psi) \|, x \in F_1\}.$$

Therefore,

$$\| I_k(\psi) \| \leq C_1 + (c_k + \epsilon_k) \| \psi \|_{\mathcal{B}} . \quad (14)$$

Then, by (H2), (H4), (H5) and (14), from (12) we have for $t \in J$,

$$\begin{aligned}
&\| x(t) \| \\
&\quad \leq Me^{\sigma t}[\tilde{H} \| \varphi \|_{\mathcal{B}} + Me^{\sigma\delta_n} L(\| \varphi \|_{\mathcal{B}} + 1)] \\
&\qquad + Me^{\sigma\delta_n} L(\| \bar{x}_t \|_{\mathcal{B}} + 1) + Me^{\sigma t}\frac{1}{a} Me^{\sigma b} M_1^2 \\
&\qquad \times \int_0^t e^{-2\sigma s}\bigg[\| x_b \| + Me^{\sigma b}[\| \varphi(0) \| \\
&\qquad + Me^{\sigma\delta_n} L(\| \varphi \|_{\mathcal{B}} + 1)] + Me^{\sigma\delta_n} L(\| \bar{x}_b \|_{\mathcal{B}} + 1) \\
&\qquad + Me^{\sigma b}\int_0^b e^{-\sigma\eta} m(\eta)\Theta(\| \bar{x}(\eta) \| + \| \bar{x}_\eta \|_{\mathcal{B}})d\eta \\
&\qquad + \sum_{k=1}^m Me^{\sigma(b-t_k)} Me^{\sigma\delta_n} \\
&\qquad \times [C_1 + (c_k+\epsilon_k) \| \bar{x}_{t_k} \|_{\mathcal{B}}]\bigg] ds \\
&\qquad + Me^{\sigma t}\int_0^t e^{-\sigma s} m(s)\Theta(\| \bar{x}(s) \| + \| \bar{x}_s \|_{\mathcal{B}})ds \\
&\qquad + \sum_{k=1}^m Me^{\sigma(t-t_k)} Me^{\sigma\delta_n}[C_1 + (c_k+\epsilon_k) \| \bar{x}_{t_k} \|_{\mathcal{B}}].
\end{aligned}$$

It is easy to see that

$$\| \bar{x}_t \|_{\mathcal{B}} \leq M_b \| \varphi \|_{\mathcal{B}} + K_b \| x \|_t, \quad t \in [0, b],$$

where $\| x \|_t = \sup_{0\leq s\leq t} \| x(s) \|$. If $\zeta(t) = M_b \| \varphi \|_{\mathcal{B}} + K_b \| x \|_t$, we obtain that

$$\begin{aligned}
\zeta(t) &\leq M_b \| \varphi \|_{\mathcal{B}} + K_b Me^{\sigma t}[\tilde{H} \| \varphi \|_{\mathcal{B}} \\
&\quad + Me^{\sigma\delta_n} L(\| \varphi \|_{\mathcal{B}} + 1)] + K_b Me^{\sigma\delta_n} L(\zeta(t)+1) \\
&\quad + e^{\sigma t}\widetilde{M} + K_b Me^{\sigma t}\frac{1}{a} Me^{\sigma b} M_1^2 N_*^2 Me^{\sigma b} N_* \\
&\quad \times Me^{\sigma\delta_n}\sum_{k=1}^m (c_k+\epsilon_k)\zeta(s) \\
&\quad + K_b Me^{\sigma t}\int_0^t e^{-\sigma s} m(s)\Theta(2\zeta(s))ds \\
&\quad + K_b Me^{\sigma t} N_* Me^{\sigma\delta_n}\sum_{k=1}^m (c_k+\epsilon_k)\zeta(s),
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{M} &= K_b M\frac{1}{a} Me^{\delta b} M_1^2 N_*^2\bigg[\| x_b \| + Me^{\delta b}[\| \varphi(0) \| \\
&\quad + e^{\sigma\delta_n} L(\| \varphi \|_{\mathcal{B}} + 1)] + e^{\sigma\delta_n} L(\| \bar{x}_b \|_{\mathcal{B}} + 1) \\
&\quad + Me^{\sigma b}\int_0^b e^{-\sigma\eta} m(\eta)\Theta(2\zeta(\eta))d\eta \\
&\quad + Me^{\sigma b} N_* Me^{\sigma\delta_n} mC_1\bigg],
\end{aligned}$$

$$M_* = M\max\{1, e^{\delta b}\}, N_* = \max\{1, e^{-\delta b}\}, M_1 = \| B \| .$$

Since $\lim_{n\to\infty}\delta_n = 0$, it follows that

$$\begin{aligned}
\zeta(t) &\leq M_b \| \varphi \|_{\mathcal{B}} + K_b Me^{\sigma t}[\tilde{H} \| \varphi \|_{\mathcal{B}} \\
&\quad + ML(\| \varphi \|_{\mathcal{B}} + 1)] + K_b ML(\zeta(t)+1) \\
&\quad + e^{\sigma t}\widetilde{M} + K_b Me^{\sigma t}\frac{1}{a} Me^{\sigma b} M_1^2 N_*^2 Me^{\sigma b} N_* \\
&\quad \times M\sum_{k=1}^m (c_k+\epsilon_k)\zeta(s) \\
&\quad + K_b Me^{\sigma t}\int_0^t e^{-\sigma s} m(s)\Theta(2\zeta(s))ds \\
&\quad + K_b Me^{\sigma t} N_* M\sum_{k=1}^m (c_k+\epsilon_k)\zeta(s),
\end{aligned}$$

By $\widetilde{L} = K_b ML + K_b M^2 N_*[\frac{1}{a}(M_* M_1 N_*)^2 + 1]\sum_{k=1}^m (c_k + \epsilon_k) < 1$, we obtain

$$\begin{aligned}
e^{-\sigma t}\zeta(t) &\leq \frac{1}{1-\widetilde{L}}\bigg[N_* M_b \| \varphi \|_{\mathcal{B}} + K_b M[\tilde{H} \| \varphi \|_{\mathcal{B}} \\
&\quad + ML(\| \varphi \|_{\mathcal{B}} + 1)] + N_* K_b ML + \widetilde{M} \\
&\quad + K_b M\int_0^t e^{-\sigma s} m(s)\Theta(2\zeta(s))ds\bigg].
\end{aligned}$$

Denoting by $w(t)$ the right-hand side of the above inequality, we have

$$v(t) \leq e^{\sigma t} w(t) \quad \text{for all } t \in J,$$

and

$$\begin{aligned}
w(0) &= \frac{1}{1-\widetilde{L}}\bigg[N_* M_b \| \varphi \|_{\mathcal{B}} + K_b M[\tilde{H} \| \varphi \|_{\mathcal{B}} \\
&\quad + ML(\| \varphi \|_{\mathcal{B}} + 1)] + N_* K_b ML + \widetilde{M}\bigg],
\end{aligned}$$

$$\begin{aligned}
w'(t) &= \frac{1}{1-K_b L_1} K_b Me^{-\sigma t} m(t)\Theta(v(t)) \\
&\leq \frac{1}{1-\widetilde{L}} K_b Me^{-\sigma t} m(t)\Theta(2e^{\sigma t} w(t)), \quad t \in J.
\end{aligned}$$

Then for each $t \in J$ we have

$$\begin{aligned}
&(e^{\sigma t} w(t))' \\
&\quad = \sigma e^{\sigma t} w(t) + w'(t)e^{\sigma t} \\
&\quad \leq \delta e^{\sigma t} w(t) + \frac{1}{1-\widetilde{L}} K_b Mm(t)\Theta(2e^{\sigma t} w(t)) \\
&\quad \leq \max\left\{\sigma, \frac{1}{1-\widetilde{L}} K_b Mm(t)\right\} \\
&\quad \times [e^{\sigma t} w(t) + \Theta(2e^{\sigma t} w(t))], \quad t \in J.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\int_{w(0)}^{e^{\sigma t} w(t)} \frac{d\varsigma}{\varsigma + \Theta(2\varsigma)} \\
&\quad \leq \int_0^b \max\left\{\sigma, \frac{1}{1-\widetilde{L}} K_b Mm(s)\right\} ds < \infty.
\end{aligned}$$

This inequality shows that there is a constant $\widetilde{K}$ such that $e^{\sigma t} w(t) \leq \widetilde{K}, t \in J$, and hence $\| x \|_{\mathcal{PC}} \leq e^{\sigma t} w(t) \leq \widetilde{K}$, where $\widetilde{K}$ depends only on M, σ, b and on the functions $m(\cdot)$ and $\Theta(\cdot)$. Then, there exists r^* such that $\| x \|_{\mathcal{PC}} \neq r^*$. Set

$$V = \{x \in \mathcal{Y} : \| x \|_{\mathcal{PC}} < r^*\}.$$

From the choice of V, there is no $x \in \partial V$ such that $x \in \lambda\Phi x$ for $\lambda \in (0, 1)$.

Step 2. Φ_n has a closed graph.

Let $x^{(j)} \to x^*, h_n^{(j)} \in \Phi_n x^{(j)}, x^{(j)} \in \overline{V}$ and $h_n^{(j)} \to h_n^*$. From Axiom (A), it is easy to see that $(\overline{x^{(j)}})_s \to \overline{x^*}_s$ uniformly for $s \in (-\infty, b]$ as $n \to \infty$. We prove that

$h^* \in \Phi_n \overline{x^*}$. Now $h_n^{(j)} \in \Phi_n \overline{x^{(j)}}$ means that there exists $f^{(j)} \in S_{F,\overline{x^{(j)}}}$ such that, for each $t \in J$,

$$\begin{aligned}
&h_n^{(j)}(t)\\
&= R_{(\alpha,\overline{x^{(j)}})}(t,0)[\varphi(0) - R_{(\alpha,\overline{x^{(j)}})}(\delta_n,0)G(0,\varphi)]\\
&\quad + R_{(\alpha,\overline{x^{(j)}})}(\delta_n,0)G(t,(\overline{x^{(j)}})_t)\\
&\quad + \int_0^t R_{(\alpha,\overline{x^{(j)}})}(t,s)BB^*R^*_{(\alpha,\overline{x^{(j)}})}(b,s)S(a,\Gamma_0^b)\\
&\quad \times\Big[x_b - R_{(\alpha,\overline{x^{(j)}})}(b,0)[\varphi(0)\\
&\quad - R_{(\alpha,\overline{x^{(j)}})}(\delta_n,0)G(0,\varphi)]\\
&\quad - R_{(\alpha,\overline{x^{(j)}})}(\delta_n,0)G(b,(\overline{x^{(j)}})_b)\\
&\quad - \int_0^b R_{(\alpha,\overline{x^{(j)}})}(b,\eta)f(\eta)d\eta\\
&\quad - \sum_{k=1}^m R_{(\alpha,\overline{x^{(j)}})}(b,t_k)R_{(\alpha,\overline{x^{(j)}})}(\delta_n,0)\\
&\quad \times I_k((\overline{x^{(j)}})_{t_k})\Big](s)ds + \int_0^t R_{(\alpha,\overline{x^{(j)}})}(t,s)f(s)ds\\
&\quad + \sum_{0<t_k<t} R_{(\alpha,\overline{x^{(j)}})}(t,t_k)R_{(\alpha,\overline{x^{(j)}})}(\delta_n,0)\\
&\quad \times I_k((\overline{x^{(j)}})_{t_k}), \quad t \in J.
\end{aligned}$$

We must prove that there exists $f^* \in S_{F,\overline{x^*}}$ such that, for each $t \in J$,

$$\begin{aligned}
&h_n^*(t)\\
&= R_{(\alpha,\overline{x^*})}(t,0)[\varphi(0) - R_{(\alpha,\overline{x^*})}(\delta_n,0)G(0,\varphi)]\\
&\quad + R_{(\alpha,\overline{x^*})}(\delta_n,0)G(t,(\overline{x^*})_t)\\
&\quad + \int_0^t R_{(\alpha,\overline{x^*})}(t,s)BB^*R^*_{(\alpha,\overline{x^*})}(b,s)S(a,\Gamma_0^b)\\
&\quad \times\Big[x_b - R_{(\alpha,\overline{x^*})}(b,0)[\varphi(0)\\
&\quad - R_{(\alpha,\overline{x^*})}(\delta_n,0)G(0,\varphi)]\\
&\quad - R_{(\alpha,\overline{x^*})}(\delta_n,0)G(b,(\overline{x^*})_b)\\
&\quad - \int_0^b R_{(\alpha,\overline{x^*})}(b,\eta)f(\eta)d\eta\\
&\quad - \sum_{k=1}^m R_{(\alpha,\overline{x^*})}(b,t_k)R_{(\alpha,\overline{x^*})}(\delta_n,0)I_k((\overline{x^*})_{t_k})\Big](s)ds\\
&\quad + \int_0^t R_{(\alpha,\overline{x^*})}(t,s)f(s)ds\\
&\quad + \sum_{0<t_k<t} R_{(\alpha,\overline{x^*})}(t,t_k)R_{(\alpha,\overline{x^*})}(\delta_n,0)\\
&\quad \times I_k((\overline{x^*})_{t_k}), \quad t \in J.
\end{aligned}$$

Now, for every $t \in J$, we have

$$\begin{aligned}
&\Bigg\|\Bigg(h_n^{(j)}(t) - R_{(\alpha,\overline{x^{(j)}})}(t,0)[\varphi(0) - R_{(\alpha,\overline{x^{(j)}})}(\delta_n,0)g(0,\varphi)]\\
&\quad - R_{(\alpha,\overline{x^{(j)}})}(\delta_n,0)g(t,(\overline{x^{(j)}})_t)\\
&\quad - \int_0^t R_{(\alpha,\overline{x^{(j)}})}(t,s)BB^*R^*_{(\alpha,\overline{x^{(j)}})}(b,s)S(a,\Gamma_0^b)\\
&\quad \times\Big[x_b - R_{(\alpha,\overline{x^{(j)}})}(b,0)[\varphi(0) - R_{(\alpha,\overline{x^{(j)}})}(\delta_n,0)G(0,\varphi)]\\
&\quad - R_{(\alpha,\overline{x^{(j)}})}(\delta_n,0)G(b,(\overline{x^{(j)}})_b)\\
&\quad - \sum_{k=1}^m R_{(\alpha,\overline{x^{(j)}})}(b,t_k)R_{(\alpha,\overline{x^{(j)}})}(\delta_n,0)I_k((\overline{x^{(j)}})_{t_k})\Big](s)ds\\
&\quad - \sum_{0<t_k<t} R_{(\alpha,\overline{x^{(j)}})}(t,t_k)R_{(\alpha,\overline{x^{(j)}})}(\delta_n,0)I_k((\overline{x^{(j)}})_{t_k})\Bigg)\\
&\quad - \Bigg(h_n^*(t) - R_{(\alpha,\overline{x^*})}(t,0)[\varphi(0) - R_{(\alpha,\overline{x^*})}(\delta_n,0)G(0,\varphi)]\\
&\quad - R_{(\alpha,\overline{x^*})}(\delta_n,0)G(t,(\overline{x^*})_t)\\
&\quad - \int_0^t R_{(\alpha,\overline{x^*})}(t,s)BB^*R^*_{(\alpha,\overline{x^*})}(b,s)S(a,\Gamma_0^b)\\
&\quad \times\Big[x_b - R_{(\alpha,\overline{x^*})}(b,0)[\varphi(0)\\
&\quad - R_{(\alpha,\overline{x^*})}(\delta_n,0)G(0,\varphi)] - R_{(\alpha,\overline{x^*})}(\delta_n,0)G(b,(\overline{x^*})_b)\\
&\quad - \sum_{k=1}^m R_{(\alpha,\overline{x^*})}(b,t_k)R_{(\alpha,\overline{x^*})}(\delta_n,0)I_k((\overline{x^*})_{t_k})\Big](s)ds\\
&\quad - \sum_{0<t_k<t} R_{(\alpha,\overline{x^*})}(t,t_k)R_{(\alpha,\overline{x^*})}(\delta_n,0)I_k((\overline{x^*})_{t_k})\Bigg)\Bigg\|_{\mathcal{PC}}\\
&\to 0 \quad \text{as} \quad j \to \infty.
\end{aligned}$$

Consider the linear continuous operator $\Psi : L^1(J,H) \to C(J,H)$,

$$\begin{aligned}
&\Psi(f)(t)\\
&= \int_0^t R_{(\alpha,\bar{x})}(t,s)\Big[f(s) + BB^*R^*_{(\alpha,\bar{x})}(b,s)S(a,\Gamma_0^b)\\
&\quad \times\Big(\int_0^b R_{(\alpha,\bar{x})}(b,\eta)f(\eta)d\eta\Big)(s)\Big]ds.
\end{aligned}$$

From Lemma 5, It follows that $\Psi \circ S_F$ is a closed graph operator. Also, from the definition of Ψ, we have that, for every $t \in J$,

$$\begin{aligned}
&h_n^{(j)}(t) - R_{(\alpha,\overline{x^{(j)}})}(t,0)[\varphi(0) - R_{(\alpha,\overline{x^{(j)}})}(\delta_n,0)G(0,\varphi)]\\
&\quad - R_{(\alpha,\overline{x^{(j)}})}(\delta_n,0)G(t,(\overline{x^{(j)}})_t)\\
&\quad - \int_0^t R_{(\alpha,\overline{x^{(j)}})}(t,s)BB^*R^*_{(\alpha,\overline{x^{(j)}})}(b,s)S(a,\Gamma_0^b)\\
&\quad \times\Big[x_b - R_{(\alpha,\overline{x^{(j)}})}(b,0)[\varphi(0) - R_{(\alpha,\overline{x^{(j)}})}(\delta_n,0)G(0,\varphi)]\\
&\quad - R_{(\alpha,\overline{x^{(j)}})}(\delta_n,0)G(b,(\overline{x^{(j)}})_b)\\
&\quad - \sum_{k=1}^m R_{(\alpha,\overline{x^{(j)}})}(b,t_k)R_{(\alpha,\overline{x^{(j)}})}(\delta_n,0)I_k((\overline{x^{(j)}})_{t_k})\Big](s)ds\\
&\quad - \sum_{0<t_k<t} R_{(\alpha,\overline{x^{(j)}})}(t,t_k)R_{(\alpha,\overline{x^{(j)}})}(\delta_n,0)I_k((\overline{x^{(j)}})_{t_k})\\
&\quad \in \Psi(S_{F,\overline{x^{(j)}}}).
\end{aligned}$$

Since $\overline{x^{(j)}} \to \overline{x^*}$, for some $f^* \in S_{F,\overline{x^*}}$ it follows that, for every $t \in J$, we have

$$\begin{aligned}
&h_n^*(t) - R_{(\alpha,\overline{x^*})}(t,0)[\varphi(0) - R_{(\alpha,\overline{x^*})}(\delta_n,0)G(0,\varphi)]\\
&\quad - R_{(\alpha,\overline{x^*})}(\delta_n,0)G(t,(\overline{x^*})_t)\\
&\quad - \int_0^t R_{(\alpha,\overline{x^*})}(t,s)BB^*R^*_{(\alpha,\overline{x^*})}(b,s)S(a,\Gamma_0^b)\\
&\quad \times\Big[x_b - R_{(\alpha,\overline{x^*})}(b,0)[\varphi(0)
\end{aligned}$$

$$
\begin{aligned}
&-R_{(\alpha,\overline{x^*})}(\delta_n,0)G(0,\varphi)] - R_{(\alpha,\overline{x^*})}(\delta_n,0)G(b,(\overline{x^*})_b)\\
&-\int_0^b R_{(\alpha,\overline{x^*})}(b,\eta)f(\eta)d\eta\\
&-\sum_{k=1}^{m} R_{(\alpha,\overline{x^*})}(b,t_k)R_{(\alpha,\overline{x^*})}(\delta_n,0)I_k((\overline{x^*})_{t_k})\Big](s)ds\\
&-\int_0^t R_{(\alpha,\overline{x^*})}(t,s)f(s)ds\\
&-\sum_{0<t_k<t} R_{(\alpha,\overline{x^*})}(t,t_k)R_{(\alpha,\overline{x^*})}(\delta_n,0)I_k((\overline{x^*})_{t_k})\\
&=\int_0^t R_{(\alpha,\overline{x^*})}(t,s)\Big[f(s)+BB^*R^*_{(\alpha,\overline{x^*})}(b,s)S(a,\Gamma_0^b)\\
&\times\Big(\int_0^b R_{(\alpha,\overline{x^*})}(b,\eta)f(\eta)d\eta\Big)(s)\Big]ds.
\end{aligned}
$$

Therefore, Φ_n has a closed graph.

Step 3. We show that the operator Φ_n condensing.

For this purpose, we decompose Φ_n as $\Lambda_n + \Gamma_n$, where the map $\Lambda_n : \overline{V} \to \mathcal{Y}$ be defined by $\Lambda_n x$, the set $\rho_n \in \mathcal{Y}$ such that

$$
\rho_n(t) = \begin{cases} 0, & t \in (-\infty, 0],\\ -R_{(\alpha,\bar{x})}(t,0)R_{(\alpha,\bar{x})}(\delta_n,0)G(0,\varphi) & \\ \quad +R_{(\alpha,\bar{x})}(\delta_n,0)G(t,\bar{x}_t) & t\in J,\end{cases}
$$

and the map $\Gamma_n : \overline{V} \to \mathcal{Y}$ be defined by $\Gamma_n x$, the set $\vartheta_n \in \mathcal{Y}$ such that

$$
\vartheta_n(t) = \begin{cases} 0, \quad t\in(-\infty,0],\\ R_{(\alpha,\bar{x})}(t,0)\varphi(0) + \int_0^t R_{(\alpha,\bar{x})}(t,s)u^a_{n,\bar{x}}(s)ds\\ \quad +\int_0^t R_{(\alpha,\bar{x})}(t,s)f(s)ds\\ \quad +\sum\limits_{0<t_k<t} R_{(\alpha,\bar{x})}(t,t_k)R_{(\alpha,\bar{x})}(\delta_n,0)\\ \quad \times I_k(\overline{x}_{t_k}) \quad t\in J.\end{cases}
$$

We first show that Λ_n is a contraction while Γ_n is a completely continuous operator.

Claim 1. Λ_n is a contraction on $\overline{V}$.

Let $t \in J$ and $x^*, x^{**} \in \mathcal{Y}$. If $x \in \overline{V}$, it follows that

$$
\| \bar{x}_s \|_{\mathcal{B}} \le M_b \| \varphi \|_{\mathcal{B}} + K_b r^* := r'.
$$

From (H5) and Lemma 3, we have

$$
\begin{aligned}
&\| (\Lambda_n x^*)(t) - (\Lambda_n x^{**})(t) \|\\
&\le \| [R_{(\alpha,\overline{x^*})}(t,0)R_{(\alpha,\overline{x^*})}(\delta_n,0) - R_{(\alpha,\overline{x^{**}})}(t,0)\\
&\quad \times R_{(\alpha,\overline{x^{**}})}(\delta_n,0)]G(0,\varphi)\|\\
&\quad + \| R_{(\alpha,\overline{x^*})}(\delta_n,0)G(t,\overline{x^*}_t) - R_{(\alpha,\overline{x^{**}})}(\delta_n,0)\\
&\quad \times G(t,\overline{x^{**}}_t)\|\\
&\le \| [R_{(\alpha,\overline{x^*})}(t,0) - R_{(\alpha,\overline{x^{**}})}(t,0)]\\
&\quad \times R_{(\alpha,\overline{x^*})}(\delta_n,0)G(0,\varphi)\|\\
&\quad + \| [R_{(\alpha,\overline{x^*})}(\delta_n,0) - R_{(\alpha,\overline{x^{**}})}(\delta_n,0)]\\
&\quad \times R_{(\alpha,\overline{x^{**}})}(t,0)G(0,\varphi)\|\\
&\quad + \| [R_{(\alpha,\overline{x^*})}(\delta_n,0) - R_{(\alpha,\overline{x^{**}})}(\delta_n,0)]\\
&\quad \times G(t,\overline{x^*}_t)\|\\
&\quad + \| R_{(\alpha,\overline{x^{**}})}(\delta_n,0)[G(t,\overline{x^*}_t) - G(t,\overline{x^{**}}_t)\|\\
&\le KbMe^{\sigma\delta_n}L(\|\varphi\|_{\mathcal{B}}+1)\|\overline{x^*}_t - \overline{x^{**}}_t\|_{\mathcal{B}}\\
&\quad +K\delta_n Me^{\sigma t}L(\|\varphi\|_{\mathcal{B}}+1)\|\overline{x^*}_t - \overline{x^{**}}_t\|_{\mathcal{B}}\\
&\quad +K\delta_n L(\|x_t\|_{\mathcal{B}}+1)\|\overline{x^*}_t - \overline{x^{**}}_t\|_{\mathcal{B}}\\
&\quad +Me^{\sigma\delta_n}L\|\overline{x^*}_t - \overline{x^{**}}_t\|_{\mathcal{B}}\\
&\le [KbMe^{\sigma\delta_n}L(\|\varphi\|_{\mathcal{B}}+1) + K\delta_n Me^{\sigma t}L(\|\varphi\|_{\mathcal{B}}+1)\\
&\quad +K\delta_n L(r'+1) + Me^{\sigma\delta_n}L]K_b\\
&\quad \times \sup_{s\in[0,b]}\|\overline{x^*}(s) - \overline{x^{**}}(s)\|\\
&= [KbMe^{\sigma\delta_n}L(\|\varphi\|_{\mathcal{B}}+1) + K\delta_n Me^{\sigma t}L(\|\varphi\|_{\mathcal{B}}+1)\\
&\quad +K\delta_n L(r'+1) + Me^{\sigma\delta_n}L]K_b\\
&\quad \times \sup_{s\in[0,b]}\|x^*(s) - x^{**}(s)\| \quad (\text{since } \bar{x} = x \text{ on } J)\\
&= [KbMe^{\sigma\delta_n}L(\|\varphi\|_{\mathcal{B}}+1) + K\delta_n Me^{\sigma t}L(\|\varphi\|_{\mathcal{B}}+1)\\
&\quad +K\delta_n L(r'+1) + Me^{\sigma\delta_n}L]K_b\\
&\quad \times \|x^* - x^{**}\|_{\mathcal{PC}}.
\end{aligned}
$$

Since $\lim_{n\to\infty}\sigma_n = 0$, it follows that

$$
\begin{aligned}
&\| (\Lambda_n x^*)(t) - (\Lambda_n x^{**})(t) \|\\
&\quad \le [Kb(\|\varphi\|_{\mathcal{B}}+1)+1]MLK_b \|x^* - x^{**}\|_{\mathcal{PC}}.
\end{aligned}
$$

Taking supremum over t,

$$
\|\Lambda_n x^* - \Lambda_n x^{**}\|_{\mathcal{PC}} \le L_0 \|x^* - x^{**}\|_{\mathcal{PC}},
$$

where $L_0 = [Kb(\|\varphi\|_{\mathcal{B}}+1)+1]MLK_b < 1$. Thus Λ_n is a contraction on $\overline{V}$.

Claim 2. Γ_n is convex for each $x \in \overline{V}$.

In fact, if $\vartheta_n^1, \vartheta_n^2$ belong to $\Gamma_n x$, then there exist $f_1, f_2 \in S_{F,\bar{x}}$ such that

$$
\begin{aligned}
&\vartheta_n^i(t)\\
&= R_{(\alpha,\bar{x})}(t,0)\varphi(0) + \int_0^t R_{(\alpha,\bar{x})}(t,s)BB^*\\
&\quad \times R^*_{(\alpha,\bar{x})}(b,s)S(a,\Gamma_0^b)\Big[x_b - R_{(\alpha,\bar{x})}(b,0)[\varphi(0)\\
&\quad -R_{(\alpha,\bar{x})}(\delta_n,0)G(0,\varphi)] - R_{(\alpha,\bar{x})}(\delta_n,0)G(b,\bar{x}_b)\\
&\quad -\int_0^b R_{(\alpha,\bar{x})}(b,\eta)f_i(\eta)d\eta\\
&\quad -\sum_{k=1}^m R_{(\alpha,\bar{x})}(b,t_k)R_{(\alpha,\bar{x})}(\delta_n,0)I_k(\overline{x}_{t_k})\Big](s)ds\\
&\quad +\int_0^t R_{(\alpha,\bar{x})}(t,s)f_i(s)ds\\
&\quad +\sum_{0<t_k<t} R_{(\alpha,\bar{x})}(t,t_k)R_{(\alpha,\bar{x})}(\delta_n,0)I_k(\overline{x}_{t_k}),\\
&\quad t\in J, \quad i=1,2.
\end{aligned}
$$

Let $0 \le \lambda \le 1$. For each $t \in J$ we have

$$
\begin{aligned}
&(\lambda\vartheta_n^1 + (1-\lambda)\vartheta_n^2)(t)\\
&= R_{(\alpha,\bar{x})}(t,0)\varphi(0) + \int_0^t R_{(\alpha,\bar{x})}(t,s)BB^*R^*_{(\alpha,\bar{x})}(b,s)\\
&\quad \times S(a,\Gamma_0^b)\Big[x_b - R_{(\alpha,\bar{x})}(b,0)[\varphi(0)\\
&\quad -R_{(\alpha,\bar{x})}(\delta_n,0)G(0,\varphi)] - R_{(\alpha,\bar{x})}(\delta_n,0)G(b,\bar{x}_b)\\
&\quad -\int_0^b R_{(\alpha,\bar{x})}(b,\eta)[\lambda f_1(\eta) + (1-\lambda)f_2(\eta)]d\eta\\
&\quad -\sum_{k=1}^m R_{(\alpha,\bar{x})}(b,t_k)R_{(\alpha,\bar{x})}(\delta_n,0)I_k(\overline{x}_{t_k})\Big](s)ds\\
&\quad +\int_0^t R_{(\alpha,\bar{x})}(t,s)[\lambda f_1(s) + (1-\lambda)f_2(s)]f_i(s)ds\\
&\quad +\sum_{0<t_k<t} R_{(\alpha,\bar{x})}(t,t_k)R_{(\alpha,\bar{x})}(\delta_n,0)I_k(\overline{x}_{t_k}).
\end{aligned}
$$

Since $S_{F,\bar{x}}$ is convex (because F has convex values) we have $(\lambda\vartheta_n^1 + (1-\lambda)\vartheta_n^2) \in \Gamma_n x$.

Claim 3. $\Gamma_n(\overline{V})$ is completely continuous.

To this end, we consider the decomposition Γ_n by $\Gamma_n = \Gamma_n^1 + \Gamma_n^2$, where the map $\Gamma_n^1 : \overline{V} \to \mathcal{P}(\mathcal{Y})$ be defined by $\Gamma_n^1 x$, the set $\tilde{\gamma}_n^1 \in \mathcal{Y}$ such that

$$\begin{aligned}\tilde{\gamma}_n^1(t) &\\ &= R_{(\alpha,\bar{x})}(t,0)\varphi(0) + \int_0^t R_{(\alpha,\bar{x})}(t,s)Bu_{n,\bar{x}}^a(s)ds \\ &\quad + \int_0^t R_{(\alpha,\bar{x})}(t,s)f(s)ds,\end{aligned}$$

and the map $\Gamma_n^2 : \overline{V} \to \mathcal{P}(\mathcal{V})$ be defined by $\Gamma_n^2 x$, the set $\tilde{\gamma}_n^2 \in \mathcal{Y}$ such that

$$\tilde{\gamma}_n^2(t) = \sum_{0<t_k<t} R_{(\alpha,\bar{x})}(t,t_k)R_{(\alpha,\bar{x})}(\delta_n,0)I_k(\overline{x}_{t_k})$$

(1) $\Gamma_n^1(\overline{V})$ is completely continuous.

We begin by showing $\Gamma_n^1(\overline{V})$ is equicontinuous. Let $0 < \tau_1 < \tau_2 \le b$. For each $x \in \overline{V}$, we have

$$\begin{aligned}&\| \tilde{\gamma}_n^1(\tau_2) - \tilde{\gamma}_n^1(\tau_1) \| \\ &\quad \le \| [R_{(\alpha,\bar{x})}(\tau_2,0) - R_{(\alpha,\bar{x})}(\tau_1,0)]\varphi(0) \| \\ &\qquad + \left\| \int_0^{t_1-\varepsilon} [R_{(\alpha,\bar{x})}(\tau_2,s) - R_{(\alpha,\bar{x})}(\tau_1,s)] \right. \\ &\qquad \left. \times Bu_{n,\bar{x}}^a(s)ds \right\| \\ &\qquad + \left\| \int_{\tau_1-\varepsilon}^{\tau_1} [R_{(\alpha,\bar{x})}(\tau_2,s) - R_{(\alpha,\bar{x})}(\tau_1,s)] \right. \\ &\qquad \left. \times Bu_{n,\bar{x}}^a(s)ds \right\| \\ &\qquad + \left\| \int_{\tau_1}^{\tau_2} R_{(\alpha,\bar{x})}(\tau_2,s)Bu_{n,\bar{x}}^a(s)ds \right\| \\ &\qquad + \left\| \int_0^{\tau_1-\varepsilon} [R_{(\alpha,\bar{x})}(\tau_2,s) - R_{(\alpha,\bar{x})}(\tau_1,s)]f(s)ds \right\| \\ &\qquad + \left\| \int_{\tau_1-\varepsilon}^{\tau_1} [R_{(\alpha,\bar{x})}(\tau_2,s) - R_{(\alpha,\bar{x})}(\tau_1,s)]f(s)ds \right\| \\ &\qquad + \left\| \int_{\tau_1}^{\tau_2} R_{(\alpha,\bar{x})}(\tau_2,s)f(s)ds \right\| \\ &\quad \le \| [R_{(\alpha,\bar{x})}(\tau_2,0) - R_{(\alpha,\bar{x})}(\tau_1,0)]\varphi(0) \| \\ &\qquad + \int_0^{\tau_1-\varepsilon} \| R_{(\alpha,\bar{x})}(\tau_2,s) - R_{(\alpha,\bar{x})}(\tau_1,s) \| \\ &\qquad \times M_1M_2ds + 2M_* \int_{\tau_1-\varepsilon}^{\tau_1} e^{-\sigma s}M_1M_2ds \\ &\qquad + Me^{\sigma\tau_2}\int_{\tau_1}^{\tau_2} e^{-\sigma s}M_1M_2ds + \Theta(r^*+r') \\ &\qquad \times \int_0^{\tau_1-\varepsilon} \| R_{(\alpha,\bar{x})}(\tau_2,s) - R_{(\alpha,\bar{x})}(\tau_1,s) \| m(s)ds \\ &\qquad + 2M_*\Theta(r^*+r')\int_{\tau_1-\varepsilon}^{\tau_1} e^{-\sigma s}m(s)ds \\ &\qquad + Me^{\sigma t_2}\Theta(r')\int_{\tau_1}^{\tau_2} e^{-\sigma s}m(s)ds,\end{aligned}$$

where

$$\begin{aligned}&\| u_{n,\bar{x}}^a(s) \| \\ &\quad = M_*\frac{1}{a}M_1N_*\Big[\| x_b \| + M_*[\tilde{H} \| \varphi \|_{\mathcal{B}} \\ &\qquad + L(\| \varphi \|_{\mathcal{B}} + 1)] + L(r^*+1) \\ &\qquad + Me^{\sigma b}\Theta(r^*+r')\int_0^b e^{-\sigma s}m(s)ds\Big] := M_2.\end{aligned}$$

From the above inequalities, we see that the right-hand side of $\| \tilde{\gamma}_n^1(\tau_2) - \tilde{\gamma}_n^1(\tau_1) \|$ tends to zero independent of $x \in \overline{V}$ as $\tau_2 - \tau_1 \to 0$ with ε sufficiently small, since $R_{(\alpha,\bar{x})}(t,s)$ is a strongly continuous operator and the compactness of $R_{(\alpha,\bar{x})}(t,s)$ for $t-s>0$ implies the continuity in the uniform operator topology. Thus the set $\{\Gamma_n^1 x : x \in \overline{V}\}$ is equicontinuous. The equicontinuities for the other cases $\tau_1 < \tau_2 \le 0$ or $\tau_1 \le 0 \le \tau_2 \le b$ are very simple.

Now we prove that $\Gamma_n^1(\overline{V})(t) = \{\tilde{\gamma}_n^1(t) : \tilde{\gamma}_n^1(t) \in \Gamma_n^1(\overline{V})\}$ is relatively compact for every $t \in [0,b]$.

Let $0 < t \le s \le b$ be fixed and let ε be a real number satisfying $0 < \varepsilon < t$. For $x \in \overline{V}$, we define

$$\begin{aligned}\tilde{\gamma}_n^{1,\varepsilon}(t) &\\ &= R_{(\alpha,\bar{x})}(t,0)\varphi(0) + \int_0^{t-\varepsilon} R_{(\alpha,\bar{x})}(t,s)Bu_{n,\bar{x}}^a(s)ds \\ &\quad + \int_0^{t-\varepsilon} R_{(\alpha,\bar{x})}(t,s)f(s)ds,\end{aligned}$$

where $f \in S_{F,\bar{x}}$. Using the compactness of $R_{(\alpha,\bar{x})}(t,s)$ for $t-s>0$, we deduce that the set $U_\varepsilon(t) = \{\tilde{\gamma}_n^{1,\varepsilon}(t) : x \in \overline{V}\}$ is relatively compact in H for every $\varepsilon, 0 < \varepsilon < t$. Moreover, for every $x \in \overline{V}$ we have

$$\begin{aligned}&\| \tilde{\gamma}_n^1(t) - \tilde{\gamma}_n^{1,\varepsilon}(t) \| \\ &\quad \le \left\| \int_{t-\varepsilon}^t R_{(\alpha,\bar{x})}(t,s)Bu_{n,\bar{x}}^a(s)ds \right\| \\ &\qquad + \left\| \int_{t-\varepsilon}^t R_{(\alpha,\bar{x})}(t,s)f(s)ds \right\| \\ &\quad \le 2M_*\int_{t-\varepsilon}^t e^{-\sigma s}M_1M_2ds \\ &\qquad + 2M_*\Theta(r^*+r')\int_{t-\varepsilon}^t e^{-\sigma s}m(s)ds.\end{aligned}$$

The right hand side of the above inequality tends to zero as $\varepsilon \to 0$. Since there are relatively compact sets arbitrarily close to the set $U(t) = \{\tilde{\gamma}_n^1(t) : x \in \overline{V}\}$. Hence the set $U(t)$ is relatively compact in H. By Arzelá-Ascoli theorem, we conclude that $\Gamma_n^1 : \overline{V} \to \mathcal{P}(\mathcal{Y})$ is completely continuous.

(2) $\Gamma_n^2(\overline{V})$ is completely continuous.

We begin by showing $\Gamma_n^2(\overline{V})$ is equicontinuous. For each $x \in \overline{V}, t \in (0,b)$ be fixed, $t \in [t_i, t_{i+1}]$, $\tilde{\gamma}_n^2(t) \in \Gamma_n^2(x)$, such that

$$\tilde{\gamma}_n^2(t) = \sum_{0<t_k<t} R_{(\alpha,\bar{x})}(t,t_k)R_{(\alpha,\bar{x})}(\delta_n,0)I_k(\overline{x}_{t_k}). \quad (15)$$

Next, for $\tau_1 \le s < t \le \tau_2$, $\varepsilon > 0$, we have, using the semigroup property,

$$\begin{aligned}&\| [\widehat{\tilde{\gamma}_n^2}]_i(\tau_2) - [\widehat{\tilde{\gamma}_n^2}]_i(\tau_1) \| \\ &\le \left\| \sum_{0<t_k<t} [R_{(\alpha,\bar{x})}(\tau_2, t_k) - R_{(\alpha,\bar{x})}(\tau_1, t_k)] \right. \\ &\quad \left. \times R_{(\alpha,\bar{x})}(\delta_n, 0) I_k(\overline{x}_{t_k}) \right\| \\ &\le m \sum_{k=1}^{m} \| [R_{(\alpha,\bar{x})}(\tau_2, t_k) - R_{(\alpha,\bar{x})}(\tau_1, t_k)] \\ &\quad \times R_{(\alpha,\bar{x})}(\delta_n, 0) I_k(\overline{x}_{t_k}) \| .\end{aligned}$$

As $\tau_2 - \tau_1 \to 0$, the right-hand side of the above inequality tends to zero independently of x due to the set $\{R_{(\alpha,\bar{x})}(\delta_n, 0)I_k(\overline{x}_{t_k}) : x \in \overline{V}\}$ is relatively compact in H and the strong continuity of $R_{(\alpha,\bar{x})}(\cdot,\cdot)$. So $[\widehat{\tilde{\gamma}_n^2}]_i, i = 1, 2, \ldots, m$, are equicontinuous.

Next, we show that $\Gamma_n^2(\overline{V})(t) = \{\tilde{\gamma}_n^2(t) : \tilde{\gamma}_n^2(t) \in \Gamma_2(\overline{V})\}$ is relatively compact for every $t \in [0, b]$.

For $t \in [0, b]$ and $x \in \overline{V}$, we have that

$$\begin{aligned}&[\widehat{\tilde{\gamma}_n^2}]_i(t) \\ &= \sum_{0<t_k<t} R_{(\alpha,\bar{x})}(t, t_k) R_{(\alpha,\bar{x})}(\delta_n, 0) I_k(\overline{x}_{t_k}) \\ &\in \sum_{k=1}^{m} R_{(\alpha,\bar{x})}(t, t_k) R_{(\alpha,\bar{x})}(\delta_n, 0) I_k(\overline{V}(0, H)).\end{aligned}$$

From (14), we obtain that $I_k(\bar{x}_{t_k})$ bounded in H. By the compactness of $(R_{(\alpha,\bar{x})}(t, s))_{t-s>0}$, we see $\{R_{(\alpha,\bar{x})}(\delta_n, 0)I_k(\overline{x}_{t_k}) : x \in \overline{V}, k = 1, 2, \ldots, m\}$ are relatively compact in H. Also, it follows that $[\widehat{\tilde{\gamma}_n^2}]_k(t)$ is relatively compact in H, for all $t \in [t_k, t_{k+1}]$, $k = 1, \ldots, m$. By Lemma 2, we infer that $\Gamma_n^2(\overline{V})$ is relatively compact. Now an application of the Arzelá-Ascoli theorem justifies the relatively compactness of $\Gamma_n^2(\overline{V})$. Therefore, $\Gamma_n^2(\overline{V})$ is completely continuous, and hence $\Phi_n^2(\overline{V})$ is completely continuous.

As a consequence of Steps 3, we conclude that $\Phi_n = \Lambda_n + \Gamma_n$ is a condensing map. All of the conditions of Lemma 4 are satisfied, we deduce that $\Lambda_n + \Gamma_n$ has a fixed point $x \in \mathcal{Y}$, which is a mild solution of the problem (9)-(11). Then, we have

$$\begin{aligned}&x_n(t) \\ &= R_{(\alpha,\bar{x}_n)}(t, 0)[\varphi(0) - R_{(\alpha,\bar{x}_n)}(\delta_n, 0)G(0, \varphi)] \\ &\quad + R_{(\alpha,\bar{x}_n)}(\delta_n, 0)G(t, \bar{x}_{n,t}) \\ &\quad + \int_0^t R_{(\alpha,\bar{x}_n)}(t, s) B u^a_{n,\bar{x}_n}(s) ds \\ &\quad + \int_0^t R_{(\alpha,\bar{x}_n)}(t, s) f_n(s) ds \\ &\quad + \sum_{0<t_k<t} R_{(\alpha,\bar{x}_n)}(t, t_k) R_{(\alpha,\bar{x})}(\delta_n, 0) \\ &\quad \times I_k(\bar{x}_{n,t_k}), \quad t \in [0, b], \end{aligned} \tag{16}$$

where

$$\begin{aligned}&u^a_{n,\bar{x}_n}(s) \\ &= B^* R^*_{(\alpha,\bar{x}_n)}(b, s) S(a, \Gamma_0^b) \Big[x_b - R_{(\alpha,\bar{x}_n)}(b, 0)[\varphi(0) \\ &\quad - R_{(\alpha,\bar{x}_n)}(\delta_n, 0)G(0, \varphi)] \\ &\quad - R_{(\alpha,\bar{x}_n)}(\delta_n, 0)G(b, \bar{x}_{n,b}) \\ &\quad - \int_0^b R_{(\alpha,\bar{x}_n)}(b, \eta) f_n(\eta) d\eta \\ &\quad - \sum_{k=1}^{m} R_{(\alpha,\bar{x}_n)}(b, t_k) R_{(\alpha,\bar{x}_n)}(\delta_n, 0) I_k(\bar{x}_{n,t_k}) \Big],\end{aligned}$$

and $f_n \in S_{F,\bar{x}_n}$.

Next we will show that the set $\{x_n : n \in N\}$ is relatively compact in $\mathcal{Y}$. We consider the decomposition $x_n = x_n^1 + x_n^2$ where

$$\begin{aligned}&x_n^1(t) \\ &= R_{(\alpha,\bar{x}_n)}(t, 0)[\varphi(0) - R_{(\alpha,\bar{x}_n)}(\delta_n, 0)G(0, \varphi)] \\ &\quad + R_{(\alpha,\bar{x}_n)}(\delta_n, 0)G(t, \bar{x}_{n,t}) \\ &\quad + \int_0^t R_{(\alpha,\bar{x}_n)}(t, s) B u^a_{n,\bar{x}_n}(s) ds \\ &\quad + \int_0^t R_{(\alpha,\bar{x}_n)}(t, s) f_n(s) ds \end{aligned} \tag{17}$$

for some $f_n \in S_{F,\bar{x}_n}$, and

$$\begin{aligned}&x_n^2(t) \\ &= \sum_{0<t_k<t} R_{(\alpha,\bar{x}_n)}(t, t_k) R_{(\alpha,\bar{x}_n)}(\delta_n, 0) I_k(\bar{x}_{n,t_k}).\end{aligned} \tag{18}$$

Step 4. $\{x_n^1(t) : n \in N\}$ is relatively compact in $\mathcal{Y}$.

Claim 1. $\{x_n^1 : n \in N\}$ is equicontinuous on J.

For $\varepsilon > 0, x_n \in \overline{V}$, there exists a constant $0 < \eta < \varepsilon$ such that for all $t \in (0, b]$ and $\xi \in (0, \eta)$ with $t + \xi \le b$, we have,

$$\begin{aligned}&\| x_n^1(t + \xi) - x_n^1(t) \| \\ &\le \| [R_{(\alpha,\bar{x}_n)}(t + \xi, 0) - R_{(\alpha,\bar{x}_n)}(t, 0)] \\ &\quad \times R_{(\alpha,\bar{x}_n)}(\delta_n, 0)G(0, \varphi) \| \\ &\quad + \| R_{(\alpha,\bar{x}_n)}(\delta_n, 0)[G(t + \xi, \bar{x}_{n,t+\xi}) - G(t, \bar{x}_{n,t})] \| \\ &\quad + \left\| \int_t^{t+\xi} R_{(\alpha,\bar{x}_n)}(t + \xi, s) B u^a_{n,\bar{x}_n}(s) ds \right\| \\ &\quad + \left\| \int_0^t [R_{(\alpha,\bar{x}_n)}(t + \xi, s) - R_{(\alpha,\bar{x}_n)}(t, s)] \right. \\ &\quad \left. \times B u^a_{n,\bar{x}_n}(s) ds \right\| \\ &\quad + \left\| \int_t^{t+\xi} R_{(\alpha,\bar{x}_n)}(t + \xi, s) f_n(s) ds \right\| \\ &\quad + \left\| \int_0^t [R_{(\alpha,\bar{x}_n)}(t + \xi, s) - R_{(\alpha,\bar{x}_n)}(t, s)] f_n(s) ds \right\| \\ &\le \| [R_{(\alpha,\bar{x}_n)}(t + \xi, 0) - R_{(\alpha,\bar{x}_n)}(t, 0)] \\ &\quad \times R_{(\alpha,\bar{x}_n)}(\delta_n, 0)G(0, \varphi) \| \\ &\quad + M e^{\sigma\delta_n} L[\xi + \| \bar{x}_{n,t+\xi} - \bar{x}_{n,t} \|_{\mathcal{B}}] \\ &\quad + M_* \int_t^{t+\xi} e^{-\sigma s} M_1 M_2 ds \\ &\quad + \int_0^t \| R_{(\alpha,\bar{x}_n)}(t + \xi, s) - R_{(\alpha,\bar{x}_n)}(t, s) \| M_1 M_2 ds \\ &\quad + M_* \Theta(r^* + r') \int_t^{t+\xi} e^{-\sigma s} m(s) ds + \Theta(r^* + r') \\ &\quad \times \int_0^t \| R_{(\alpha,\bar{x}_n)}(t + \xi, s) - R_{(\alpha,\bar{x}_n)}(t, s) \| m(s) ds.\end{aligned}$$

Using the compact operator property, we can choose $\xi \in (0,t)$ such that

$$\| [R_{(\alpha,\bar{x}_n)}(t+\xi,0) - R_{(\alpha,\bar{x}_n)}(t,0)] \times R_{(\alpha,\bar{x}_n)}(\delta_n,0)G(0,\varphi) \| < \frac{\varepsilon}{4}, \quad (19)$$

and

$$Me^{\delta\sigma_n}L[\xi + \| \bar{x}_{n,t+\xi} - \bar{x}_{n,t} \|_{\mathcal{B}}] < \frac{\varepsilon}{4}, \quad (20)$$

$$M_* \int_t^{t+\xi} e^{-\delta s}[\Theta(r^*+r')m(s) + M_1M_2]ds < \frac{\varepsilon}{4}, \quad (21)$$

$$\int_0^t \| R_{(\alpha,\bar{x}_n)}(t+\xi,s) - R_{(\alpha,\bar{x}_n)}(t,s) \| \times[\Theta(r^*+r')m(s) + M_1M_2]ds < \frac{\varepsilon}{4}. \quad (22)$$

By (19)-(22) one has

$$\| x_n^1(t+\xi) - x_n^1(t) \| < \varepsilon.$$

Therefore, $\{x_n^1(t) : n \in N\}$ is equicontinuous for $t \in (0,b]$. Clearly $\{x_n^1(0) : n \in N\}$ is equicontinuous.

Claim 2. $\{x_n^1(t) : n \in N\}$ is relatively compact in H.

Let $t \in (0,b]$, $\varepsilon > 0, x_n \in \overline{V}$, there exists $\xi \in (0,t)$ such that

$$\begin{aligned}
&\| x_n^1(t) - x_n^\xi(t) \| \\
&\leq \int_{t-\xi}^t \| R_{(\alpha,\bar{x}_n)}(t,s)Bu^a_{n,\bar{x}_n}(s) \| ds \\
&\quad + \int_{t-\xi}^t \| R_{(\alpha,\bar{x}_n)}(t,s)f_n(s) \| ds \\
&\leq M_* \int_{t-\xi}^t e^{-\sigma s} M_1M_2 ds \\
&\quad + M_*\Theta(r^*+r') \int_{t-\xi}^t e^{-\sigma s} m(s)ds < \varepsilon,
\end{aligned}$$

where

$$\begin{aligned}
&x_n^\xi(t) \\
&= R_{(\alpha,\bar{x}_n)}(t,0)[\varphi(0) - R_{(\alpha,\bar{x}_n)}(\delta_n,0)G(0,\varphi)] \\
&\quad + R_{(\alpha,\bar{x}_n)}(\delta_n,0)G(t,\bar{x}_{n,t}) \\
&\quad + \int_0^{t-\xi} R_{(\alpha,\bar{x}_n)}(t,s)Bu^a_{n,\bar{x}_n}(s)ds \\
&\quad + \int_0^{t-\xi} R_{(\alpha,\bar{x}_n)}(t,s)f_n(s)ds
\end{aligned}$$

for some $f_n \in S_{F,\bar{x}_n}$. From (H5), we obtain that $G(t,\bar{x}_{n,t})$ are bounded in H. By the compactness of $R_{(\alpha,\bar{x})}(t,s)$ for $t,s > 0$, we see that the set $\{x_n^\xi(t) : n \in N\}$ is relatively compact in H. Combining the above inequality, one has $\{x_n^1(t) : n \in N\}$ is relatively compact in H.

Step 5. $\{x_n^2(t) : n \in N\}$ is relatively compact in $\mathcal{Y}$.

Claim 1. $\{x_n^2 : n \in N\}$ is equicontinuous on J.

For any $\varepsilon > 0$ and $0 < t < b$. Since $R_{(\alpha,\bar{x}_n)}(\delta_n,0)$ is a compact operator, we find that the set $W = \{R_{(\alpha,\bar{x}_n)}(\delta_n,0)I_k(\overline{x}_{n,t_k}) : x_n \in \overline{V}\}$ is relatively compact in H. From the strong continuity of $(R_{(\alpha,\bar{x}_n)}(t,s))_{t\geq s}$, for $\varepsilon > 0$, we can choose $0 < \eta < b - t$ such that

$$\| [R_{(\alpha,\bar{x}_n)}(t+\xi,s) - R_{(\alpha,\bar{x}_n)}(t,s)]\nu \| < \frac{\varepsilon}{m}, \quad \nu \in W,$$

when $|\xi| < \eta$. For each $x_n \in \overline{V}, t \in (0,b)$ be fixed, $t \in [t_i, t_{i+1}]$, such that

$$\begin{aligned}
&\| [\widehat{x_n^2}]_i(t+\xi) - [\widehat{x_n^2}]_i(t) \| \\
&\leq \left\| \sum_{0<t_k<t} [R_{(\alpha,\bar{x}_n)}(t,\xi - t_k) - R_{(\alpha,\bar{x}_n)}(t,t_k)] \right. \\
&\quad \left. \times R_{(\alpha,\bar{x}_n)}(\delta_n,0)I_k(\overline{x}_{n,t_k}) \right\| \\
&\leq \sum_{k=1}^m \| [R_{(\alpha,\bar{x}_n)}(t,\xi - t_k) - R_{(\alpha,\bar{x}_n)}(t,t_k)] \\
&\quad \times R_{(\alpha,\bar{x}_n)}(\delta_n,0)I_k(\overline{x}_{n,t_k}) \| < \varepsilon.
\end{aligned}$$

As $\xi \to 0$ and ε sufficiently small, the right-hand side of the above inequality tends to zero independently of x_n, so $[\widehat{x_n^2}]_i, i = 1,2,\ldots,m$, are equicontinuous.

Claim 2. $\{x_n^2(t) : n \in N\}$ is relatively compact in H.

Let $t \in (0,b]$, $\varepsilon > 0, x_n \in \overline{V}$. Using the similar arguments as that in Step 3, we have

$$\begin{aligned}
&[\widehat{x_n^2}]_i(t) \\
&= \sum_{0<t_k<t} R_{(\alpha,\bar{x}_n)}(t,t_k)R_{(\alpha,\bar{x}_n)}(\delta_n,0)I_k(\overline{x}_{t_k}) \\
&\in \sum_{k=1}^m R_{(\alpha,\bar{x}_n)}(t,t_k)R_{(\alpha,\bar{x}_n)}(\delta_n,0)I_k(\overline{V}(0,H))
\end{aligned}$$

for all $n \in N$. One has $[\widehat{x_n^2}]_i(t), i = 1,2,\ldots,m$, is relatively compact for every $t \in [t_i, t_{i+1}]$, and $\{x_n^2(t) : n \in N\}$ is relatively compact in H.

Thus, we obtain that the set $\{x_n : n \in N\}$ is relatively compact in $\mathcal{Y}$. We may suppose that

$$x_n \to x_* \in \mathcal{Y} \quad \text{as } n \to \infty.$$

Obviously, $x_* \in \mathcal{Y}$, taking limits in (16) one has

$$\begin{aligned}
&x_*(t) \\
&= R_{(\alpha,\bar{x}_*)}(t,0)[\varphi(0) - G(0,\varphi)] + G(t,\bar{x}_{*,t}) \\
&\quad + \int_0^t R_{(\alpha,\bar{x}_*)}(t,s)Bu^a_{\bar{x}_*}(s)ds \\
&\quad + \int_0^t R_{(\alpha,\bar{x}_*)}(t,s)f_*(s)ds \\
&\quad + \sum_{0<t_k<t} R_{(\alpha,\bar{x}_*)}(t,t_k)I_k(\overline{x}_{*,t_k}), \quad t \in J \quad (23)
\end{aligned}$$

where

$$\begin{aligned}
&u^a_{\bar{x}_*}(s) \\
&= B^*R^*_{(\alpha,\bar{x}_*)}(b,s)S(a,\Gamma_0^b)\Big[x_b - R_{(\alpha,\bar{x}_*)}(b,0)[\varphi(0) \\
&\quad - G(0,\varphi)] - G(b,\bar{x}_{*,b}) \\
&\quad - \int_0^b R_{(\alpha,\bar{x}_*)}(b,\eta)f_*(\eta)d\eta \\
&\quad - \sum_{k=1}^m R_{(\alpha,\bar{x}_*)}(b,t_k)I_k(\overline{x}_{*,t_k})\Big],
\end{aligned}$$

and some $f_* \in S_{F,\bar{x}_*}$. which implies that x_* is a mild solution of the problem (1)-(3) and the proof of Theorem 1 is complete.

IV. Approximate controllability of fractional control systems

In this section, we present our main result on approximate controllability of system (1)-(3). To do this, we also need the following assumptions:

(B1) The function $G : J \times \mathcal{B} \to H$ is continuous, and there exists a constant $\widetilde{C}_1 > 0$ such that

$$\| G(t, \psi) \| \leq \widetilde{C}_1$$

for $0 \leq t \leq b, \psi \in \mathcal{B}$.

(B2) There exists a constant $\widetilde{C}_2 > 0$ such that

$$\| F(t, x, \psi) \| \leq \widetilde{C}_2, \quad (t, \psi) \in J \times \mathcal{B},$$

where

$$\| F(t, x, \psi) \| = \sup\{\| f \| : f \in F(t, x, \psi)\}.$$

(B3) The functions $I_k : \mathcal{B} \to H$ are continuous and there exist constants c_k such that

$$\| I_k(\psi) \| \leq c_k$$

for every $\psi \in \mathcal{B}, k = 1, \ldots, m$.

Theorem 2. *Assume that assumptions of Theorem* 1 *hold and, in addition, hypotheses* (B1)-(B3) *are satisfied and the linear system corresponding to system* (1)-(3) *is approximately controllable on J. Then the system* (1)-(3) *is approximately controllable on J.*

Proof. Let $x^a(\cdot)$ be a fixed point of Φ in $\mathcal{Y}$. By Theorem 1, any fixed point of Φ is a mild solution of the system (1)-(3). This means that there is $x^a \in \Phi(x^a)$, that is, there is $f \in S_{F,\bar{x}^a}$ such that

$$\begin{aligned} x^a(t) &= R_{(\alpha,\bar{x}^a)}(t, 0)[\varphi(0) - G(0, \varphi)] \\ &\quad + R_{(\alpha,\bar{x}^a)}(\delta_n, 0)G(t, \bar{x}^a_t) \\ &\quad + \int_0^t R_{(\alpha,\bar{x}^a)}(t, s)Bu^a_{\bar{x}}(s)ds \\ &\quad + \int_0^t R_{(\alpha,\bar{x}^a)}(t, s)f(s)ds \\ &\quad + \sum_{0<t_k<t} R_{(\alpha,\bar{x}^a)}(t, t_k)I_k(\overline{x}^a_{t_k}), \quad t \in J, \end{aligned}$$

where

$$\begin{aligned} u^a_{\bar{x}}(s) &= B^* R^*_{(\alpha,\bar{x}^a)}(b, s)S(a, \Gamma^b_0)\Big[x_b - R_{(\alpha,\bar{x}^a)}(b, 0)[\varphi(0) \\ &\quad - G(0, \varphi)] - G(b, \bar{x}^a_b) \\ &\quad - \int_0^b R_{(\alpha,\bar{x}^a)}(b, \eta)f(\eta)d\eta \\ &\quad - \sum_{k=1}^m R_{(\alpha,\bar{x}^a)}(b, t_k)I_k(\overline{x}^a_{t_k})\Big], \end{aligned}$$

and satisfies

$$\begin{aligned} x^a(b) &= x_b + aS(a, \Gamma^b_0)\Big\{x_b - R_{(\alpha,\bar{x}^a)}(b, 0)[\varphi(0) \\ &\quad - G(0, \varphi)] - G(b, \bar{x}^a_b) \\ &\quad - \int_0^b R_{(\alpha,\bar{x}^a)}(t, s)f(s)ds \\ &\quad - \sum_{k=1}^m R_{(\alpha,\bar{x}^a)}(b, t_k)I_k(\overline{x}^a_{t_k})\Big\}. \end{aligned}$$

By the conditions (B1) and (B3), we see that $G(b, \bar{x}^a_b)$ and $\sum_{k=1}^m R_{(\alpha,\bar{x}^a)}(b, t_k)I_k(\overline{x}^a_{t_k})$ are relatively compact in H, so there exist x_G and $x_\Sigma \in H$ such that (by passing to a subsequence)

$$G(b, \bar{x}^a_b) \to x_G, \quad \sum_{k=1}^m R_{(\alpha,\bar{x}^a)}(b, t_k)I_k(\overline{x}^a_{t_k}) \to x_\Sigma$$

in $\| \cdot \|$, respectively, as $a \to 0^+$. On the other hand, by the conditions (B2), we get

$$\int_0^b \| f(s) \|^2 \, ds \leq \widetilde{C}^2_2 b.$$

Consequently, the sequences $\{f(s)\}$ is bounded in $L^2(J, H)$. Thus there are subsequences, still denoted by $\{f(s)\}$ that converge weakly to say $f^{**}(s)$ in $L^2(J, H)$. The operator

$$l(t) \to \int_0^t R_{(\alpha,\bar{x}^a)}(t, s)l(s)ds$$

is also compact on $L^2(J, H)$, so one has that

$$\int_0^b R_{(\alpha,\bar{x}^a)}(b, s)[f(s) - f^{**}(s)]ds \to 0 \quad \text{as } a \to 0.$$

Define

$$\begin{aligned} p(x^a(\cdot)) &= x_b - R_{(\alpha,\bar{x}^a)}(b, 0)[\varphi(0) - G(0, \varphi)] \\ &\quad - G(b, \bar{x}^a_b) - \int_0^b R_{(\alpha,\bar{x}^a)}(b, s)f(s)ds \\ &\quad - \sum_{k=1}^m R_{(\alpha,\bar{x}^a)}(b, t_k)I_k(\overline{x}^a_{t_k}), \end{aligned}$$

$$\begin{aligned} q &= x_b - R_{(\alpha,\bar{x}^a)}(b, 0)[\varphi(0) - G(0, \varphi)] \\ &\quad - x_G - \int_0^b R_{(\alpha,\bar{x}^a)}(b, s)f^{**}(s)ds - x_\Sigma. \end{aligned}$$

It follows that

$$\begin{aligned} &\| p(x^a) - q \| \\ &\leq \| G(b, \bar{x}^a_b) - x_G \| \\ &\quad + \left\| \sum_{k=1}^m R_{(\alpha,\bar{x}^a)}(b, t_k)I_k(\overline{x}^a_{t_k}) - x_\Sigma \right\| \\ &\quad + \left\| \int_0^b R_{(\alpha,\bar{x}^a)}(b, s)[f(s) - f^{**}(s)]ds \right\| \\ &\quad \to 0 \quad \text{as } a \to 0^+. \end{aligned} \tag{24}$$

Then from (24) and Lemma 1, we obtain

$$\begin{aligned}&\| x^a(b) - x_b \| \\ &\quad =\| aR(a,\Gamma_0^b)p(x^a) \| \\ &\quad \leq \| aR(a,\Gamma_0^b)q \| + \| aR(a,\Gamma_0^b)[p(x^a)-q] \| \\ &\quad \leq \| aR(a,\Gamma_0^b)q \| + \| p(x^a)-q \| \\ &\quad \to 0 \quad \text{as } a \to 0^+.\end{aligned}$$

This proves the approximate controllability of system (1)-(3). The proof is completed.

V. Example

Consider the following impulsive fractional partial neutral quasilinear infinite delay differential inclusions of the form

$$\begin{aligned}&\frac{\partial^\alpha}{\partial t^\alpha}\left[z(t,x) - \int_{-\infty}^t a_1(t)a_2(s-t)z(t,x)\right] \\ &\quad \in a(x,t,z(t,x))\frac{\partial^2}{\partial x^2}\left[z(t,x)\right. \\ &\quad\quad \left. - \int_{-\infty}^t a_1(t)a_2(s-t)z(t,x)\right] + \tilde{\mu}(t,x) \\ &\quad\quad + \varpi(t,z(t,x)) \\ &\quad\quad + \int_{-\infty}^t a_3(t,s-t,x,z(s,x))ds, \qquad (25) \\ &\quad\quad 0 \leq t \leq b, 0 \leq x \leq \pi,\end{aligned}$$

$$z(t,0) = z(t,\pi) = 0,\ 0 \leq t \leq b, \qquad (26)$$

$$z(\tau,x) = \varphi(\tau,x),\quad \tau \leq 0, 0 \leq x \leq \pi, \qquad (27)$$

$$\triangle z(t_k,x) = \int_{-\infty}^{t_k} \eta_k(s-t_k)z(s,x)ds, \qquad (28)$$
$$k = 1,2,\ldots,m,$$

where $0 < \alpha \leq 1, 0 < t_1 < \cdots < t_m < b$, and φ is continuous. Let $H = L^2([0,\pi])$ with the norm $\| \cdot \|$ and define the operator $A(t,\cdot) : H \to H$ by $(A(t,\cdot)\omega)(x) = a(x,t,\cdot)\omega''$ with the domain

$$D(A(t,\omega)) := \{\omega \in H : \omega'' \in H, \omega(0) = \omega(\pi) = 0\}$$

is dense in the Banach space H and independent of t. Then

$$A\omega = \sum_{n=1}^\infty n^2 \langle \omega, \omega_n\rangle \omega_n,\quad \omega \in D(A),$$

where $(\cdot,\cdot)$ is the inner product in $L^2[0,\pi]$ and $\omega_n = Z_n \circ \omega$ is the orthogonal set of eigenvectors in $A(t,\omega)$, where $Z_n(t,s) = \sqrt{\frac{2}{\pi}}\sin n(t-s)^n, 0 < \alpha \leq 1, 0 \leq s \leq t \leq b, n = 1,2,\ldots$. Assume that

(a) The operator $[A(t,\cdot) + \lambda^\alpha I]^{-1}$ exists in $L(H)$ for any λ with $\mathrm{Re}\lambda \leq 0$ and M_α

$$\| [A(t,\cdot) + \lambda^\alpha I]^{-1} \| \leq \frac{M_\alpha}{|\lambda|+1},\ t \in [0,b].$$

(b) There exist constants $\kappa \in (0,1]$ and M_α such that

$$\begin{aligned}&\| [A(t_1,\cdot) - A(t_2,\cdot)]A^{-1}(s,\cdot) \| \\ &\quad \leq M_\alpha |t_1 - t_2|^\kappa,\ t_1,t_2 \in [0,b].\end{aligned}$$

Then, the operator $A(s,\cdot), s \in [0,b]$ generates an evolution operator $\exp(-t^\alpha A(s,\cdot)), t > 0$, which is compact, analytic and self-adjoint and there exists a constant M_α such that

$$\begin{aligned}&\| A^\kappa(s,\cdot)\exp(-t^\alpha A(s,\cdot)) \| \\ &\quad \leq \frac{M_\alpha}{t^\alpha},\ t > 0, s \in [0,b],\end{aligned}$$

where $n = 0,1$. In particular, we conclude that the evolution operator in fact is an (α,u)-resolvent family has the form:

$$R(\alpha,v)(t,s)\omega = \sum_{n=1}^\infty \exp[-n^2(t-s)^\alpha](\omega,\omega_n)\omega_n,\ \omega \in H.$$

In the next applications, the phase space $\mathcal{B} = \mathcal{PC}_0 \times L^2(h,H)$ is the space introduced in Example 1.

Additionally, we will assume that

(i) The functions $a_i : R \to R, i = 1,2$, are continuous functions with

$$L_G = \| a_1 \|_\infty \left(\int_{-\infty}^0 \frac{(a_2(s))^2}{h(s)}\right)^{\frac{1}{2}} < \infty.$$

(ii) The functions $\varpi : R^2 \to R$ is continuous and that there exists a continuous function $l_1 : R \to R$ such that

$$|\varpi(t,y)| \leq l_1(t)|y|,\quad (t,y) \in R^2.$$

(iii) The function $a_3 : R^4 \to R$ is continuous and there exist continuous functions $b_1, b_2 : R \to R$ such that

$$|a_3(t,s,x,y)| \leq b_1(t)b_2(s)|y|,\quad (t,s,x,y) \in R^4$$

with $L_F = (\int_{-\infty}^0 \frac{(b_2(s))^2}{h(s)}ds)^{\frac{1}{2}} < \infty$.

(iv) The functions $\eta_k : R \to R, k = 1,2,\ldots,m$, are continuous, and $L_k = (\int_{-\infty}^0 \frac{(\eta_k(s))^2}{h(s)}ds)^{\frac{1}{2}} < \infty$ for every $k = 1,2,\ldots,m$,

(v) The function $\tilde{\mu} : [0,b] \times [0,\pi] \to [0,\pi]$ is continuous.

For the phase space $\mathcal{B}$, we have identified $\varphi(\theta)(x) = \varphi(\theta,x) \in \mathcal{B}$, defining the maps $G : [0,b] \times \mathcal{B} \to H, F : [0,b] \times H \times \mathcal{B} \to \mathcal{P}(H)$ by

$$G(t,\varphi)(x) = \int_{-\infty}^0 a_1(t)a_2(s)\varphi(s,x)ds,$$

$$D(t,\varphi)(x) = \varphi(0)x + G(t,\varphi)(x),$$

$$\begin{aligned}&F(t,y,\varphi)(x) \\ &\quad = \varpi(t,y(t,x)) + \int_{-\infty}^0 a_3(t,s,x,\varphi(s,x))ds,\end{aligned}$$

$$I_k(\varphi)(x) = \int_{-\infty}^0 \eta_k(s)\varphi(s,x)ds.$$

and

$$Bu(t)(x) = \tilde{\mu}(t,x).$$

Using these definitions, we can represent the system (25)-(28) in the abstract form (1)-(3). Moreover, F, G, I_k are bounded linear operators on $\mathcal{B}$ with $\| F \| \leq L_F, \| G \| \leq L_G, \| I_k \| \leq L_k, k = 1,2,\ldots,m$, where $L_F = \max_{0\leq t\leq b}\{Hl_1(t) + l_F b_1(t)\}$. Thus, the assumptions (H1)-(H6) and (B1)-(B3) all hold, the associated linear system of (25)-(28) is not exactly controllable but it is approximately controllable. Hence by Theorems 1, 2, the system (25)-(28) is approximately controllable on $[0,b]$.

REFERENCES

[1] M. Benchohra, J. Henderson and S.K. Ntouyas, *Impulsive Differential Equations and Inclusions*, Hindawi Publishing Corporation, New York, 2006.

[2] V. Lakshmikanthan, D.D. Bainov and P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[3] W.M. Haddad, V. Chellabonia, S.G. Nersesov and G. Sergey, *Impulsive and Hybrid Dynamical Systems: Stability, Dissipativity, and Control*, Princeton University Press, Princeton, NJ, 2006.

[4] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, in: North-Holland Mathematics Studies, vol. 204, Elsevier, Amsterdam, 2006.

[5] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[6] I. Podlubny, *Fractional Differential Equations, Mathematics in Sciences and Engineering*, 198, Academic Press, San Diego, 1999.

[7] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[8] W.G. Glockle and T.F. Nonnemacher, "A fractional calculus approach of self-similar protein dynamics," *Biophysical Journal,* vol. 68, no. 1, pp. 46-53, 1995.

[9] F. Metzler, W. Schick, H.G. Kilian and T.F. Nonnemacher, "Relaxation in filled polymers: A fractional calculus approach," *Journal of Chemical Physics,* vol. 103, no. 16, pp. 7180-7186, 1995.

[10] V. Lakshmikantham, "Theory of fractional differential equations," *Nonlinear Analysis: Theory Methods and Applications,* vol. 60, no. 10, pp. 3337-3343, 2008.

[11] M. M. El-Borai, "Some probability densities and fundamental solutions of fractional evolution equations," *Chaos Solitons and Fractals,* vol. 14, no. 3, pp. 433-440, 2002.

[12] G.M. Mophou, "Existence and uniqueness of mild solutions to impulsive fractional differential equations," *Nonlinear Analysis: Theory Methods and Applications,* vol. 72, no. 3-4, pp. 1604-1615, 2010.

[13] C. Lizama and G.M. N'Guérékata, "Mild solutions for abstract fractional differential equations, " *Applicable Analysis,* vol. 92, no. 8, pp. 1731-1754, 2013.

[14] X.-B. Shu, Y. Lai and Y. Chen, "The existence of mild solutions for impulsive fractional partial differential equations," *Nonlinear Analysis: Theory Methods and Applications,* vol. 74, no. 5, pp. 2003-2011, 2011.

[15] A. Chauhan and J. Dabas, "Existence of mild solutions for impulsive fractional order semilinear evolution equations with nonlocal conditions," *Electronic Journal of Differential Equations,* vol. 2011, no. 107, pp. 1-10, 2011.

[16] K. Balachandran, S. Kiruthika and J.J. Trujillo, "On fractional impulsive equations of Sobolev type with nonlocal condition in Banach spaces," *Computers and Mathematics with Applications,* vol. 62, no. 3, pp. 1157-1165, 2011.

[17] A. Chauhan and J. Dabas, "Local and global existence of mild solution to an impulsive fractional functional integro-differential equation with nonlocal condition," *Communications in Nonlinear Science and Numerical Simulation,* vol. 19, no. 4, pp. 821-829, 2014.

[18] S. Suganya, M. M. Arjunan and J.J. Trujillo, "Existence results for an impulsive fractional integro-differential equation with state-dependent delay," *Applied Mathematics and Computation,* vol. 266, pp. 54-69, 2015. .

[19] Z. Yan and F. Lu, "Existence of a new class of impulsive Riemann-Liouville fractional partial neutral functional differential equations with infinite delay," *IAENG International Journal of Applied Mathematics,* vol. 45, no. 4, pp. 300-312, 2015.

[20] Z. Tai and X. Wang, "Controllability of fractional-order impulsive neutral functional infinite delay integrodifferential systems in Banach spaces," *Applied Mathematics Letters,* vol. 22, no. 11, pp. 1760-1765, 2009.

[21] Z. Yan, "Controllability of fractional-order partial neutral functional integrodifferential inclusions with infinite delay," *Journal of the Franklin Institute,* vol. 348, no. 8, pp. 2156-2173, 2011.

[22] A. Debbouchea and D. Baleanu, "Controllability of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems," *Computers and Mathematics with Applications,* vol. 62, no. 3, pp. 1442-1450, 2011.

[23] R. Triggiani, "A note on the lack of exact controllability for mild solutions in Banach spaces," *SIAM Journal on Control and Optimization,* vol.15, no. 3, pp. 407-411, 1977.

[24] N.I. Mahmudov, "Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces," *SIAM Journal on Control and Optimization,* vol. 42, no. 5, pp. 1604-1622, 2003.

[25] N.I. Mahmudov, "Approximate controllability of evolution systems with nonlocal conditions," *Nonlinear Analysis: Theory, Methods and Applications,* vol. 68, no. 3, pp. 536-546, 2008.

[26] L.W. Wang, "Approximate controllability for integrodifferential equations with multiple delays," *Journal of Optimization Theory and Applications,* vol. 143, no. 1, pp. 185-206, 2009.

[27] R. Sakthivel, N. I. Mahmudov and J. H. Kim, "Approximate Controllability of nonlinear impulsive differential systems," *Reports on Mathematical Physics,* vol. 60, no. 1, pp. 85-96, 2007.

[28] R. Sakthivel, E.R. Anandhi and S.G. Lee, "Approximate Controllability of impulsive differential inclusions with nonlocal conditions," *Dynamic Systems and Applications,* vol. 18, no. 3, pp. 637-654, 2009.

[29] R. Subalakshmi and K. Balachandran, "Approximate controllability of nonlinear stochastic impulsive integrodifferential systems in Hilbert spaces," *Chaos Solitons and Fractals,* vol. 42, no. 4, pp. 2035-2046, 2009.

[30] R. Sakthivel, Y. Ren and N.I. Mahmudov, "On the approximate controllability of semilinear fractional differential systems," *Computers and Mathematics with Applications,* vol. 62, no. 3, pp. 1451-1459, 2011.

[31] R. Sakthivel, S. Suganya and S.M. Anthoni, "Approximate controllability of fractional stochastic evolution equations," *Computers and Mathematics with Applications,* vol. 63, no. 3, pp. 660-668, 2012.

[32] S. Kumar and N. Sukavanam, "Approximate controllability of fractional order semilinear systems with bounded delay," *Journal of Differential Equations,* vol. 252, no. 11, pp. 6163-6174, 2012.

[33] N. Sukavanam and S. Kumar, "Approximate controllability of fractional order semilinear delay systems," *Journal of Optimization Theory and Applications,* vol. 151, no. 11, pp. 373-384, 2011.

[34] Z. Yan, "Approximate controllability of partial neutral functional differential systems of fractional order with state-dependent delay," *International Journal of Control,* vol. 85, no. 8, pp. 1051-1062, 2012.

[35] F.-D. Ge, H.-C. Zhou, C.-H. Kou, "Approximate controllability of semilinear evolution equations of fractional order with nonlocal and impulsive conditions via an approximating technique," *Applied Mathematics and Computation,* vol. 275, pp. 107-120, 2016.

[36] P. Balasubramaniam, V. Vembarasan and T. Senthilkumar, "Approximate controllability of impulsive fractional integro-differential systems with nonlocal conditions in Hilbert space," *Numerical Functional Analysis and Optimization,* vol. 35, no. 2, pp. 177- 197, 2014.

[37] D.N. Chalishajar, K. Malar and K. Karthikeyan, "Approximate controllability of abstract impulsive fractional neutral evolution equations with infinite delay in Banach spaces," *Electronic Journal of Differential Equations,* vol. 2013, no. 275, pp. 1-21, 2013.

[38] M. Benchohra, J. Henderson, S.K. Ntouyas and A. Ouahab, "Existence results for fractional functional differential inclusions with infinite delay and application to control theory," *Fractional Calculus and Applied Analysis,* vol. 11, no. 1, pp. 35-56, 2008.

[39] Z. Yan and X. Jia, "Impulsive problems for fractional partial neutral functional integro-differential inclusions with infinite delay and analytic resolvent operators," *Mediterranean Journal of Mathematics,* vol. 11, no. 2, pp. 393-428, 2014.

[40] Z. Yan, "On a nonlocal problem for fractional integrodifferential inclusions in Banach spaces," *Annales Polonici Mathematici,* vol. 101, no. 1, pp. 87-103, 2011.

[41] Z. Yan, X. Jia, "Approximate controllability of fractional impulsive partial neutral integrodifferential inclusions with infinite delay in Hilbert spaces," *Advances in Difference Equations,* vol. 2015, no. 1, pp. 1-31, 2015.

[42] D. O'Regan, "Nonlinear alternatives for multivalued maps with applications to operator inclusions in abstract spaces," *Proceedings of the American Mathematical Society,* vol. 127, no. 12, pp. 3557-3564, 1999.

[43] K. Yosida, *Functional Analysis*, 6th edn. Springer-Verlag, Berlin, 1980.

[44] P.M. Fitzpatrick and W.V. Petryshyn, "Fixed point theorems for multivalued noncompact acyclic mappings," *Pacific Journal of Mathematics,* vol. 54, no. 2, pp. 17-23, 1974.

[45] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, 1983.

[46] D. Araya and C. Lizama, "Almost automorphic mild solutions to fractional differential equations," *Nonlinear Analysis: Theory Methods and Applications,* vol. 69, no. 11, pp. 3692-3705, 2008.

[47] J.K. Hale and J. Kato, "Phase spaces for retarded equations with infinite delay," *Funkcialaj Ekvacioj,* vol. 21, no. 1, pp.11-41, 1978.

[48] Y. Hino, S. Murakami and T. Naito, *Functional-Differential Equations with Infinite Delay*, in: Lecture Notes in Mathematics, vol. 1473, Springer-Verlag, Berlin, 1991.

[49] A. Lasota and Z. Opial, "An application of the Kakutani-Ky-Fan theorem in the theory of ordinary differential equations," *Bulletin De L Academie Polonaise Des Sciences-Serie Des Sciences Mathematiques Astronomiques Et Physiques,* vol. 13, no. 11, pp. 781-786, 1965.