Improved Homogeneous Balance Method for Multi-Soliton Solutions of Gardner Equation with Time-Dependent Coefficients

Sheng Zhang, Zhaoyu Wang

Abstract—**Homogeneous balance method (HBM) plays an important role in solving nonlinear partial differential equations (PDEs). In this paper we improve two key steps of the HBM to construct multi-soliton solutions of a time-dependent-coefficient Gardner equation from nonlinear lattice, plasma physics and ocean dynamics. It is shown that the HBM cannot construct multi-soliton solutions of the Gardner equation but the improved HBM is valid. The improved HBM can also be used to construct multi-soliton solutions of some other nonlinear PDEs in mathematical physics.**

Index Terms—**Homogeneous balance method, multi-soliton solution, Gardner equation with time-dependent coefficients.**

I. Introduction

In nonlinear science, many physical phenomena can be described by nonlinear PDEs. Researchers often investigate exact solutions of such nonlinear PDEs to gain more insight into these physical phenomena for further applications. In the past several decades, more and more exact solutions of nonlinear PDEs have been obtained, such as those in [1], [2], [3], [4], [5], [6]. As a direct method, the HBM [7] proposed by Wang plays an important role in solving nonlinear PDEs [8], [9], [10], [11], [12], [13], [14]. With the development of soliton theory, finding multi-soliton solutions has gradually become one of the most important and significant tasks and attached much attention [15], [16], [17], [18], [19], [20]. When the inhomogeneities of media and nonuniformities of boundaries are taken into account, the variable-coefficient PDEs could describe more realistic physical phenomena than their constant-coefficient counterparts [21]. Therefore, how to generalize the existing methods to construct exact solutions of nonlinear PDEs with variable coefficients is worthy of exploring. For such motivation, the present paper will improve the HBM [7] to construct multi-soliton solutions of the following time-dependent-coefficient Gardner equation from nonlinear lattice, plasma physics and ocean dynamics [22]:

$$u_t + a(t)uu_x + b(t)u^2u_x + c(t)u_{xxx} + d(t)u_x + f(t)u = 0, \tag{1}$$

where $u(x,t)$ is the amplitude of the relevant wave model, e.g., for the internal waves in a stratified ocean, x is the horizontal coordinate and t is the time, while the time-dependent coefficients $a(t)$, $b(t)$, $c(t)$, $d(t)$ and $f(t)$ are all analytic functions and relevant to the background density and shear flow stratification.

The rest of the paper is organized as follows. In Section 2, we improve two key steps of the HBM [7] for constructing multi-soliton solution of Eq. (1). As a result, one-soliton solution, two-soliton solution and three-soliton solution of Eq. (1) are obtained, from which a formula of n-soliton solution of Eq. (1) is summarized. It is shown that that the obtained multi-soliton solutions can not be constructed by using the original HBM. In Section 3, we outline the steps of the improved HBM and then conclude this paper.

II. Multi-soliton solutions

According to the HBM [7], we suppose that Eq. (1) has a solution in the form:

$$u = i\alpha(t)\frac{\partial}{\partial x}g(w(x,t)) + \beta(t) = i\alpha(t)g'(w)w_x + \beta(t), \tag{2}$$

where i is the imaginary unit, $g(w(x,t))$, $w(x,t)$, $\alpha(t)$ and $\beta(t)$ are undetermined functions. Substituting Eq. (2) into Eq. (1) yields

$$\begin{aligned} &u_t + a(t)uu_x + b(t)u^2u_x + c(t)u_{xxx} + d(t)u_x + f(t)u \\ &\quad = [i\alpha(t)c(t)g^{(4)} - i\alpha^3(t)b(t)g'^2g'']w_x^4 + \cdots, \end{aligned} \tag{3}$$

where the unwritten part in Eq. (3) is a polynomial of various partial derivatives of $w(x,t)$, the degree of which is lower than 4. We further set the coefficient of w_x^4 to zero, an ordinary differential equation (ODE) for $g(w)$ is obtained as following:

$$\alpha(t)c(t)g^{(4)} - \alpha^3(t)b(t)g'^2g'' = 0, \tag{4}$$

which has a solution in the form:

$$g(w) = \ln w, \tag{5}$$

under the condition that

$$c(t) = \frac{1}{6}\alpha^2(t)b(t). \tag{6}$$

From Eq. (5) we have

$$g'^2g'' = \frac{1}{6}g^{(4)},\ g'g'' = -\frac{1}{2}g^{(3)},\ g'^3 = \frac{1}{2}g^{(3)},\ g'^2 = -g''. \tag{7}$$

Substituting Eqs. (6) and (7) into Eq. (3) and collecting the coefficients of $g^{(4)}$, $g^{(3)}$, g'' and g' yields

$$u_t + a(t)uu_x + b(t)u^2u_x + c(t)u_{xxx} + d(t)u_x + f(t)u$$

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$$= [\frac{1}{2}\alpha^2(t)a(t)w_x^3 + \alpha^2(t)\beta(t)b(t)w_x^3 + \frac{1}{2}i\alpha^3(t)b(t)w_x^2w_{xx}]g''' \\ +[i\alpha(t)w_xw_t + i\alpha(t)d(t)w_x^2 + i\alpha(t)\beta(t)a(t)w_x^2 \\ +i\alpha(t)\beta^2(t)b(t)w_x^2 + \alpha^2(t)a(t)w_xw_{xx} \\ +2\alpha^2(t)\beta(t)b(t)w_xw_{xx} + \frac{1}{2}i\alpha^3(t)b(t)w_{xx}^2 \\ +\frac{2}{3}i\alpha^3(t)b(t)w_xw_{xxx}]g'' + [i\alpha(t)f(t)w_x + i\alpha'(t)w_x \\ +i\alpha(t)w_{xt} + i\alpha(t)d(t)w_{xx} + i\alpha(t)a(t)\beta(t)w_{xx} \\ +i\alpha(t)b(t)\beta^2(t)w_{xx} + \frac{1}{6}i\alpha^3(t)b(t)w_{xxxx}]g' \\ +\beta'(t) + f(t)\beta(t) = 0. \tag{8}$$

Setting the coefficients of g''', g'' and g' to zeros, we get a system of nonlinear PDEs for $w(x,t)$ as follows:

$$\frac{1}{2}\alpha^2(t)a(t)w_x^3 + \alpha^2(t)\beta(t)b(t)w_x^3 + \frac{1}{2}i\alpha^3(t)b(t)w_x^2w_{xx} = 0, \tag{9}$$

$$i\alpha(t)w_xw_t + i\alpha(t)d(t)w_x^2 + i\alpha(t)\beta(t)a(t)w_x^2 \\ +i\alpha(t)\beta^2(t)b(t)w_x^2 + \alpha^2(t)a(t)w_xw_{xx} \\ +2\alpha^2(t)\beta(t)b(t)w_xw_{xx} + \frac{1}{2}i\alpha^3(t)b(t)w_{xx}^2 \\ +\frac{2}{3}i\alpha^3(t)b(t)w_xw_{xxx} = 0, \tag{10}$$

$$i\alpha(t)f(t)w_x + i\alpha'(t)w_x + i\alpha(t)w_{xt} + i\alpha(t)d(t)w_{xx} \\ +i\alpha(t)a(t)\beta(t)w_{xx} + i\alpha(t)b(t)\beta^2(t)w_{xx} \\ +\frac{1}{6}i\alpha^3(t)b(t)w_{xxxx} = 0, \tag{11}$$

$$\beta'(t) + f(t)\beta(t) = 0. \tag{12}$$

We further suppose that

$$w = 1 + \theta e^{kx+h(t)+i\frac{\pi}{2}}, \tag{13}$$

where k, θ and $h(t)$ are undetermined constants and function, respectively. Substituting Eq. (13) into the system of nonlinear PDEs (9)–(12), we then get

$$\alpha(t) = Ae^{-\int f(t)dt}, \ \beta(t) = Be^{-\int f(t)dt}, \tag{14}$$

$$h(t) = k\int [(A^2 + ikB - \frac{1}{6}k^2B^2)b(t)e^{-2\int f(t)dt} - d(t)]dt, \tag{15}$$

$$a(t) = -(2A + ikB)b(t)e^{-\int f(t)dt}, \tag{16}$$

and hence obtain a kink one-soliton solution of Eq. (1):

$$u = iAe^{-\int f(t)dt} \\ \cdot \frac{k\theta e^{kx+k\int [(A^2+ikB-\frac{1}{6}k^2B^2)b(t)e^{-2\int f(t)dt}-d(t)]dt+i\frac{\pi}{2}}}{1+\theta e^{kx+k\int [(A^2+ikB-\frac{1}{6}k^2B^2)b(t)e^{-2\int f(t)dt}-d(t)]dt+i\frac{\pi}{2}}} \\ +Be^{-\int f(t)dt}, \tag{17}$$

under the constraints of Eqs. (6) and (16).

Inspired by the multi-soliton solutions [23] of the modified Korteweg–de Vries (KdV) equation $u_t + 6u^2u_x + u_{xxx} = 0$, we suppose that

$$w = \frac{1 + e^{\xi_1 - i\frac{\pi}{2}}}{1 + e^{\xi_1 + i\frac{\pi}{2}}}, \ \xi_1 = k_1x + h_1(t), \tag{18}$$

can be employed to construct one-soliton solution of Eq. (1). Then Eq. (9) is reduced to:

$$[ia(t) + 2i\beta(t)b(t) + k\alpha(t)b(t)]e^{\xi_1} + ia(t) \\ +2i\beta(t)b(t) - k\alpha(t)b(t) = 0, \tag{19}$$

which shows $k\alpha(t)b(t) = 0$. Obviously, when $k\alpha(t) = 0$, solution (2) is a trivial one. Therefore $b(t) = 0$ is needed. Then we can see from Eq. (19) that $a(t) = 0$. In this case, Eq. (1) is a linear PDE. This is not the starting point of this paper. So following the idea of HBM [7] we can not construct one-soliton solution from Eqs. (9)–(12) and (18). However, if we substitute Eqs. (2), (5), (6) and (18) into Eq. (3) and cancel the common denominator $(e^{2\xi_1} + 1)^3$, then Eq. (3) is converted into a polynomial of e^{ξ_1}. Collecting and setting each coefficient of the same powers of e^{ξ_1} to zeros, we get a set of nonlinear ODEs:

$$\beta'(t) + f(t)\beta(t) = 0, \tag{20}$$

$$-2k_1^2\alpha(t)d(t) + 2k\alpha(t)f(t) - 2k_1^2\alpha(t)\beta(t)a(t) \\ -2k_1^2\alpha(t)\beta^2(t)b(t) - \frac{1}{3}k_1^4\alpha^3(t)b(t) \\ -2k_1\alpha(t)h_1'(t) + 2k_1\alpha'(t) = 0, \tag{21}$$

$$3\beta(t)f(t) - 4k_1^3\alpha^2(t)a(t) - 8k_1^3\alpha^2(t)\beta(t)b(t) + 3\beta'(t) = 0, \tag{22}$$

$$4k_1\alpha(t)f(t) + 4k_1\alpha'(t) = 0, \tag{23}$$

$$3\beta(t)f(t) + 4k_1^3\alpha^2(t)a(t) + 8k_1^3\alpha^2(t)\beta(t)b(t) + 3\beta'(t) = 0, \tag{24}$$

$$2k_1^2\alpha(t)d(t) + 2k_1\alpha(t)f(t) + 2k_1^2\alpha(t)\beta(t)a(t) \\ +2k_1^2\alpha(t)\beta^2(t)b(t) + \frac{1}{3}k_1^4\alpha^3(t)b(t) \\ +2k_1\alpha(t)h_1'(t) + 2k_1\alpha'(t) = 0. \tag{25}$$

Solving the system of nonlinear ODEs (20)–(25) yields

$$\alpha(t) = Ae^{-\int f(t)dt}, \ \beta(t) = Be^{-\int f(t)dt}, \tag{26}$$

$$h_1(t) = k_1\int [(A^2 - \frac{1}{6}k_1^2B^2)b(t)e^{-2\int f(t)dt} - d(t)]dt, \tag{27}$$

$$a(t) = -2Ab(t)e^{-\int f(t)dt}. \tag{28}$$

We, therefore, obtain the following one-soliton solution of Eq. (1):

$$u = iAe^{-\int f(t)dt}\left(\ln\frac{1 + e^{\xi_1 - i\frac{\pi}{2}}}{1 + e^{\xi_1 + i\frac{\pi}{2}}}\right)_x + Be^{-\int f(t)dt}, \tag{29}$$

under the constraints of Eqs. (6) and (28).

In Fig. 1, a spatial structure of the one-soliton solution (29) is shown by selecting $A = 1$, $B = 1$, $k_1 = 1$, $f(t) = t$, $b(t) = 1 - t$, $d(t) = \sin t$. The dynamical evolutions and profiles of the one-soliton solution (29) are shown in Figs. 2 and 3, respectively. We can see that the one-soliton determined by Eq. (29) propagates along the x-axis in the negative direction at first and then in the positive direction. In the process of propagation, the amplitude of left-travelling soliton increases while the one of right-travelling soliton decreases and the maximal amplitude happens between the locations of $x = -1$ and $x = 0$ at the time $t = 0$.

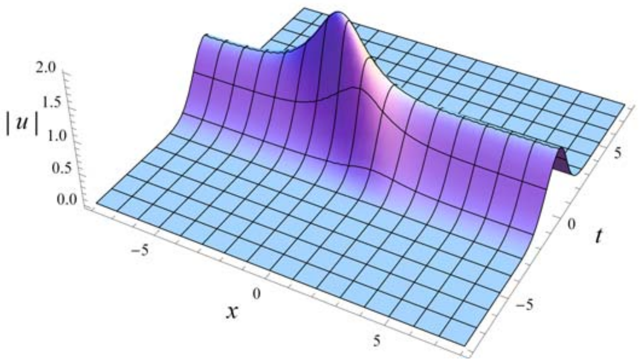


Fig. 1. Spatial structure of one-soliton solution (29).

For the two-soliton solution, we suppose that

$$w=\frac{1+\mathrm{e}^{\xi_1-i\frac{\pi}{2}}+\mathrm{e}^{\xi_2-i\frac{\pi}{2}}+\mathrm{e}^{\xi_1+\xi_2-i\pi+\theta_{12}}}{1+\mathrm{e}^{\xi_1+i\frac{\pi}{2}}+\mathrm{e}^{\xi_2+i\frac{\pi}{2}}+\mathrm{e}^{\xi_1+\xi_2+i\pi+\theta_{12}}},$$
$$\xi_j=k_jx+h_j(t),\ (j=1,2), \tag{30}$$

where k_1, k_2, θ_{12} and $h_1(t)$, $h_2(t)$ are undetermined constants and functions, respectively. Substituting Eq. (30) into Eq. (9), then equating the coefficients of the same powers of $\mathrm{e}^{3\xi_1+\xi_2}$ and $\mathrm{e}^{4\xi_1+\xi_2}$ to zeros, we have

$$-k_1^3\alpha^2(t)a(t)-2k_1^3\alpha^2(t)\beta(t)b(t)-ik_1^4\alpha^3(t)b(t)=0, \tag{31}$$
$$-ik_1^3\alpha^2(t)a(t)-2ik_1^3\alpha^2(t)\beta(t)b(t)-k_1^4\alpha^3(t)b(t)=0, \tag{32}$$

which show $k_1\alpha(t)b(t)=0$. Since $k_1\alpha(t)=0$, solution (2) to construct is a trivial one, we have to set $b(t)=0$. Then it is easy to see from Eq. (9) that $a(t)=0$. In this case, Eq. (1) becomes linear again. So from Eqs. (9)-(12) and (30) we can not construct two-soliton solution by the idea of HBM [7]. In another way, we substitute Eqs. (2), (5), (6) and (30) into Eq. (3) and cancel the common denominator $[\theta_{12}^2\mathrm{e}^{2\xi_1+2\xi_2}+2(1-\theta_{12})\mathrm{e}^{\xi_1+\xi_2}+\mathrm{e}^{2\xi_1}+\mathrm{e}^{2\xi_2}+1]^3$, then Eq. (3) is converted into a polynomial of e^{ξ_1} and e^{ξ_2}. Collecting and setting each coefficient of the same powers of $\mathrm{e}^{p\xi_1+q\xi_2}(p,q=0,1,2,\cdots,6)$ to zeros yields a set of nonlinear ODEs. From the set of nonlinear ODEs, we have

$$\alpha(t)=A\mathrm{e}^{-\int f(t)\mathrm{d}t},\ \beta(t)=B\mathrm{e}^{-\int f(t)\mathrm{d}t}, \tag{33}$$

$$h_j(t)=k_j\int[(A^2-\frac{1}{6}k_j^2B^2)b(t)\mathrm{e}^{-2\int f(t)\mathrm{d}t}-d(t)]\mathrm{d}t,$$
$$(j=1,2), \tag{34}$$

$$a(t)=-2Ab(t)\mathrm{e}^{-\int f(t)\mathrm{d}t},\ \mathrm{e}^{\theta_{12}}=\frac{(k_1-k_2)^2}{(k_1+k_2)^2}. \tag{35}$$

Thus, we obtain the following two-soliton solution of Eq. (1):

$$u=iA\mathrm{e}^{-\int f(t)\mathrm{d}t}$$
$$\cdot\left(\ln\frac{1+\mathrm{e}^{\xi_1-i\frac{\pi}{2}}+\mathrm{e}^{\xi_2-i\frac{\pi}{2}}+\mathrm{e}^{\xi_1+\xi_2-i\pi+\theta_{12}}}{1+\mathrm{e}^{\xi_1+i\frac{\pi}{2}}+\mathrm{e}^{\xi_2+i\frac{\pi}{2}}+\mathrm{e}^{\xi_1+\xi_2+i\pi+\theta_{12}}}\right)_x$$
$$+B\mathrm{e}^{-\int f(t)\mathrm{d}t}, \tag{36}$$

under the constraints of Eqs. (6) and (35).

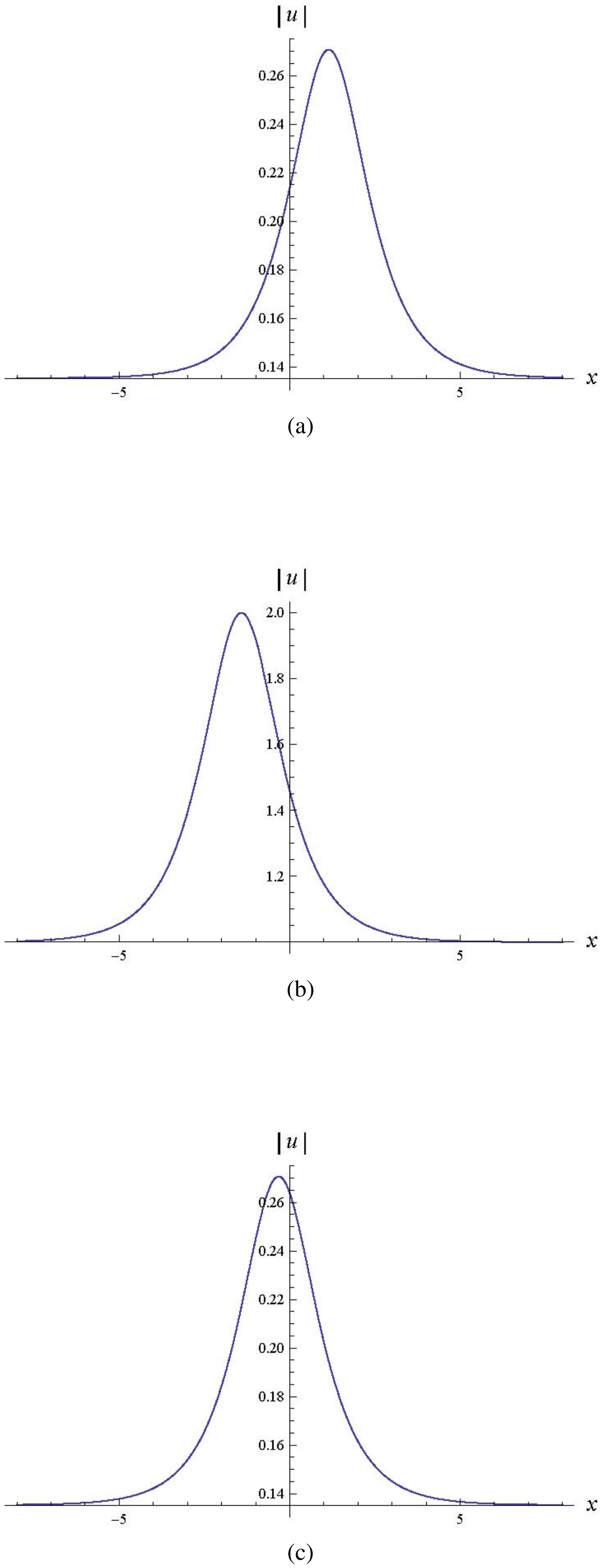
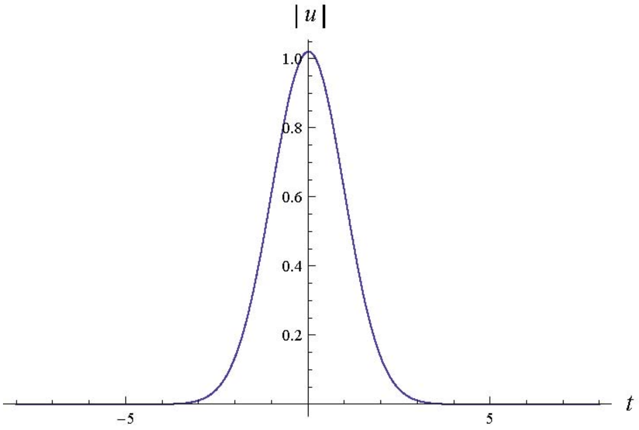
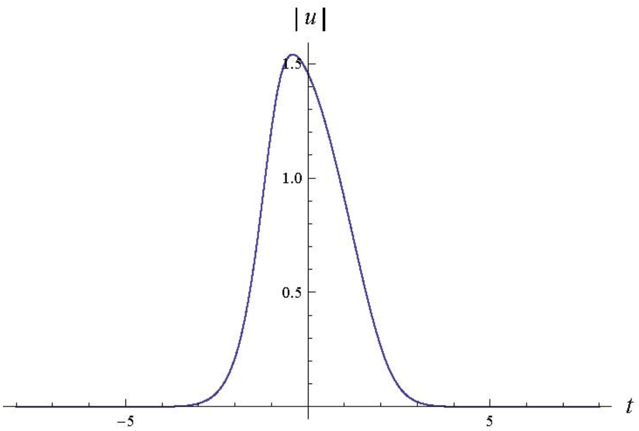


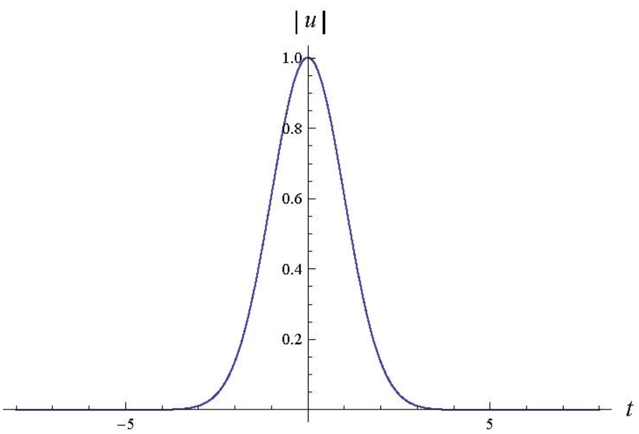
Fig. 2. Evolutions of one-soliton solution (29) at different times (a) $t=-2$, (b) $t=0$ and (c) $t=2$.



(a)



(b)



(c)

Fig. 3. Profiles of one-soliton solution (29) at different locations (a) $x = -6$, (b) $x = 0$ and (c) $x = 6$.

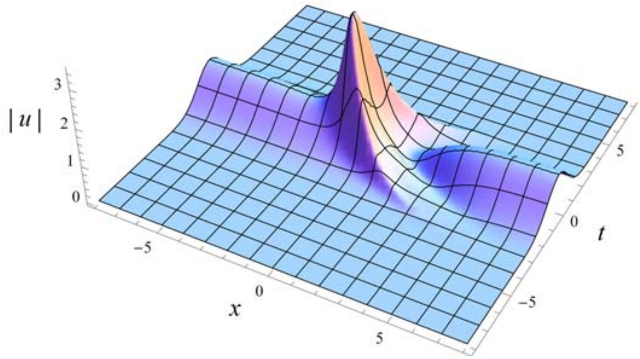


Fig. 4. Spatial structure of two-soliton solution (36).

In Fig. 4, a spatial structure of the two-soliton solution (36) is shown by selecting $A = 1$, $B = 1$, $k_1 = -1$, $k_2 = 2.1$, $f(t) = t$, $b(t) = 1 - t$, $d(t) = t$. The dynamical evolutions and profiles of the two-soliton solution (36) are shown in Figs. 5 and 6, respectively. It can be seen that the amplitude and the propagation direction of two-soliton determined by Eq. (36) have similar evolution laws as those in the one-soliton shown in Figs. 1–3.

To construct the three-soliton solution, we suppose that

$$\begin{aligned} w = {} & [1 + \mathrm{e}^{\xi_1 - i\frac{\pi}{2}} + \mathrm{e}^{\xi_2 - i\frac{\pi}{2}} + \mathrm{e}^{\xi_3 - i\frac{\pi}{2}} + \mathrm{e}^{\xi_1+\xi_2 - i\pi + \theta_{12}} \\ & + \mathrm{e}^{\xi_1+\xi_3 - i\pi + \theta_{13}} + \mathrm{e}^{\xi_2+\xi_3 - i\pi + \theta_{23}} \\ & + \mathrm{e}^{\xi_1+\xi_2+\xi_3 - i\frac{3\pi}{2} + \theta_{12}+\theta_{13}+\theta_{23}}] \\ & /[1 + \mathrm{e}^{\xi_1 + i\frac{\pi}{2}} + \mathrm{e}^{\xi_2 + i\frac{\pi}{2}} + \mathrm{e}^{\xi_3 + i\frac{\pi}{2}} + \mathrm{e}^{\xi_1+\xi_2 + i\pi + \theta_{12}} \\ & + \mathrm{e}^{\xi_1+\xi_3 + i\pi + \theta_{13}} + \mathrm{e}^{\xi_2+\xi_3 + i\pi + \theta_{23}} \\ & + \mathrm{e}^{\xi_1+\xi_2+\xi_3 + i\frac{3\pi}{2} + \theta_{12}+\theta_{13}+\theta_{23}}], \end{aligned} \tag{37}$$

$$\xi_j = k_j x + h_j(t), \ (j = 1, 2, 3), \tag{38}$$

where k_1, k_2, k_3, θ_{12}, θ_{13}, θ_{23} and $h_1(t)$, $h_2(t)$, $h_3(t)$ are undetermined constants and functions, respectively. A direct computation shows that following the idea of HBM [7] we can not construct three-soliton solution from Eqs. (9)–(12) and (37). In the similar manipulation, we substitute Eqs. (2), (5), (6) and (37) into Eq. (3) and equate each coefficient of the same powers of the exponential functions to zeros, then a set of nonlinear ODEs are derived. Solving the set of nonlinear ODEs, we have

$$\alpha(t) = A\mathrm{e}^{-\int f(t)\mathrm{d}t}, \ \beta(t) = B\mathrm{e}^{-\int f(t)\mathrm{d}t}, \tag{39}$$

$$\begin{aligned} h_j(t) = k_j \int [(A^2 - \frac{1}{6}k_j^2 B^2) b(t) \mathrm{e}^{-2\int f(t)\mathrm{d}t} - d(t)]\mathrm{d}t, \\ (j = 1, 2, 3), \end{aligned} \tag{40}$$

$$\begin{aligned} a(t) = -2Ab(t)\mathrm{e}^{-\int f(t)\mathrm{d}t}, \ \mathrm{e}^{\theta_{jl}} = \frac{(k_j - k_l)^2}{(k_j + k_l)^2}, \\ (1 \le j < l \le 3). \end{aligned} \tag{41}$$

With the help of Eqs. (2), (3), (5), (16) and (37)–(41), we obtain the following three-soliton solution of Eq. (1):

$$u = iA\mathrm{e}^{-\int f(t)\mathrm{d}t}$$

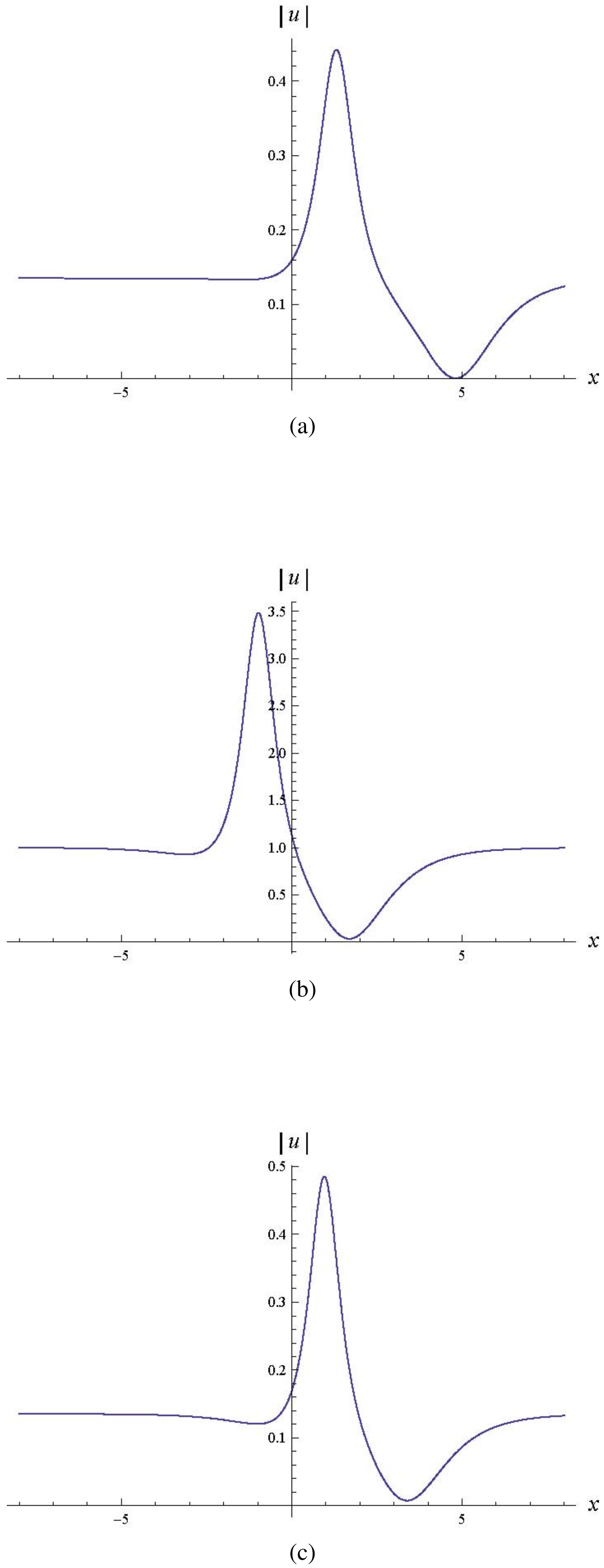


Fig. 5. Evolutions of two-soliton solution (36) at different times (a) $t = -2$, (b) $t = 0$ and (c) $t = 2$.

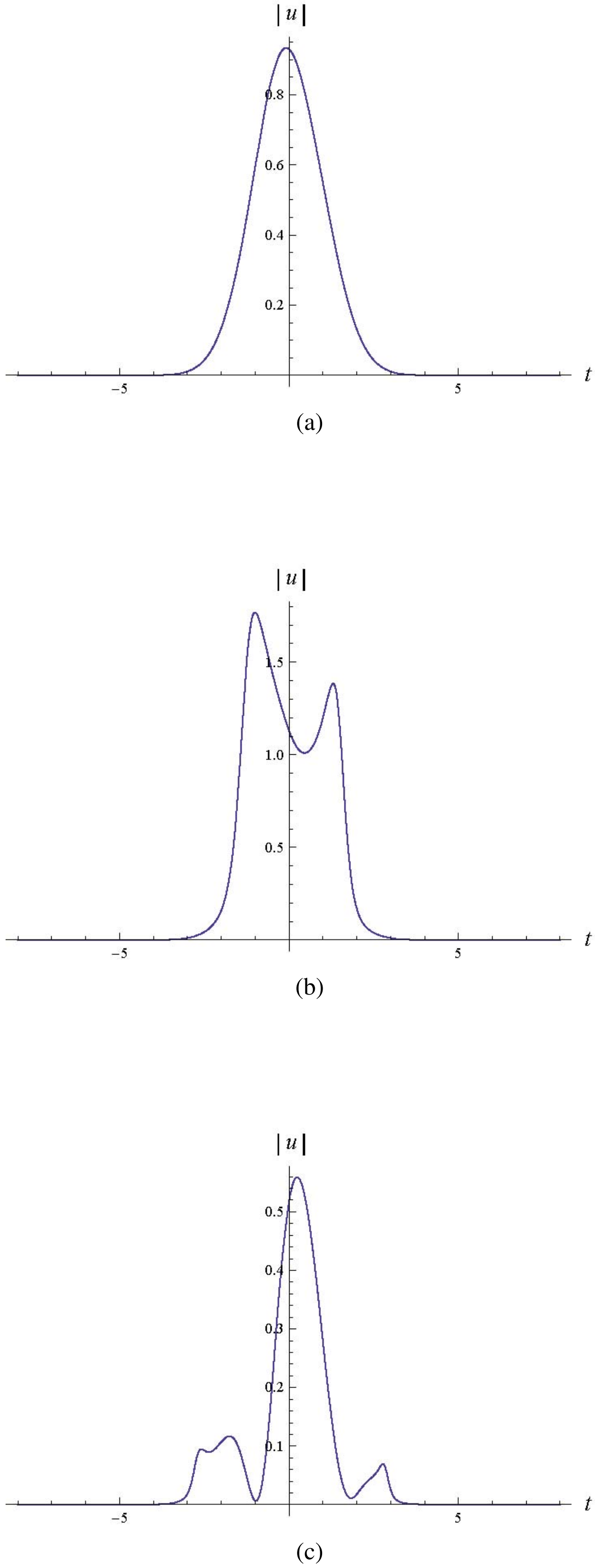


Fig. 6. Profiles of two-soliton solution (36) at different locations (a) $x = -3$, (b) $x = 0$ and (c) $x = 3$.

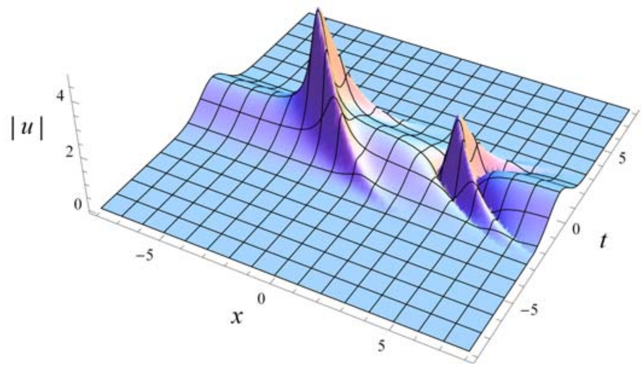


Fig. 7. Spatial structure of three-soliton solution (42).

$$\cdot\left(\ln\frac{\sum\limits_{\mu=0,1}\mathrm{e}^{\sum_{j=1}^{3}\mu_j(\xi_j-i\frac{\pi}{2})+\sum_{1\le j<l}\mu_j\mu_l\theta_{jl}}}{\sum\limits_{\mu=0,1}\mathrm{e}^{\sum_{j=1}^{3}\mu_j(\xi_j+i\frac{\pi}{2})+\sum_{1\le j<l}\mu_j\mu_l\theta_{jl}}}\right)_x$$

$$+B\mathrm{e}^{-\int f(t)\mathrm{d}t}, \tag{42}$$

under the constraints of Eqs. (6) and (41). Here the summation $\Sigma_{\mu=0,1}$ refers to all possible combination of each $\mu_i = 0,\ 1$ for $i = 1, 2, 3$.

In Fig. 7, a spatial structure of the three-soliton solution (42) is shown by selecting $A = 1$, $B = 1$, $k_1 = -1$, $k_2 = 1.5$, $k_3 = 0.1$, $f(t) = t$, $b(t) = 1 - t$, $d(t) = t$. The dynamical evolutions and profiles of the three-soliton solution (42) are shown in Figs. 8 and 9, respectively. We can see that the amplitude and the propagation direction of three-soliton determined by Eq. (42) have similar evolution laws as those in the one-soliton and two-soliton shown respectively in Figs. 1–3 and Figs. 4–6.

Generally, if take

$$w=\frac{\sum\limits_{\mu=0,1}\mathrm{e}^{\sum_{j=1}^{n}\mu_j(\xi_j-i\frac{\pi}{2})+\sum_{1\le j<l}\mu_j\mu_l\theta_{jl}}}{\sum\limits_{\mu=0,1}\mathrm{e}^{\sum_{j=1}^{n}\mu_j(\xi_j+i\frac{\pi}{2})+\sum_{1\le j<l}\mu_j\mu_l\theta_{jl}}},$$

$$\xi_j = k_j x + h_j(t), \tag{43}$$

$$h_j(t) = k_j\int[(A^2-\frac{1}{6}k_j^2B^2)b(t)\mathrm{e}^{-2\int f(t)\mathrm{d}t}-d(t)]\mathrm{d}t,$$

$$(j = 1, 2, \cdots, n), \tag{44}$$

$$\mathrm{e}^{\theta_{jl}} = \frac{(k_j-k_l)^2}{(k_j+k_l)^2},\ (1\le j<l\le n), \tag{45}$$

and substitute Eqs. (43)–(45) into Eq. (8), then equate each coefficient of the same powers of the exponential functions to zeros, we can obtain n-soliton solution of Eq. (1):

$$u = iA\mathrm{e}^{-\int f(t)\mathrm{d}t}$$

$$\cdot\left(\ln\frac{\sum\limits_{\mu=0,1}\mathrm{e}^{\sum_{j=1}^{n}\mu_j(\xi_j-i\frac{\pi}{2})+\sum_{1\le j<l}\mu_j\mu_l\theta_{jl}}}{\sum\limits_{\mu=0,1}\mathrm{e}^{\sum_{j=1}^{n}\mu_j(\xi_j+i\frac{\pi}{2})+\sum_{1\le j<l}\mu_j\mu_l\theta_{jl}}}\right)_x$$

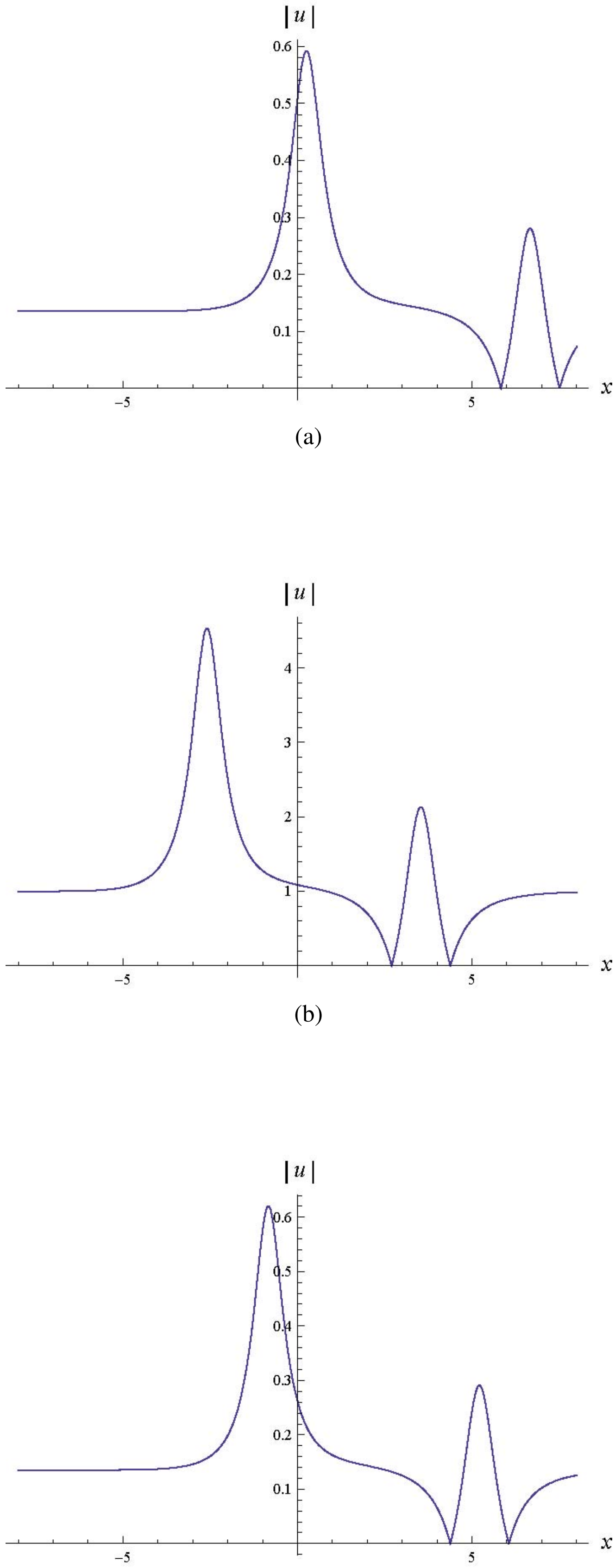


Fig. 8. Evolutions of three-soliton solution (42) at different times (a) $t = -2$, (b) $t = 0$ and (c) $t = 2$.

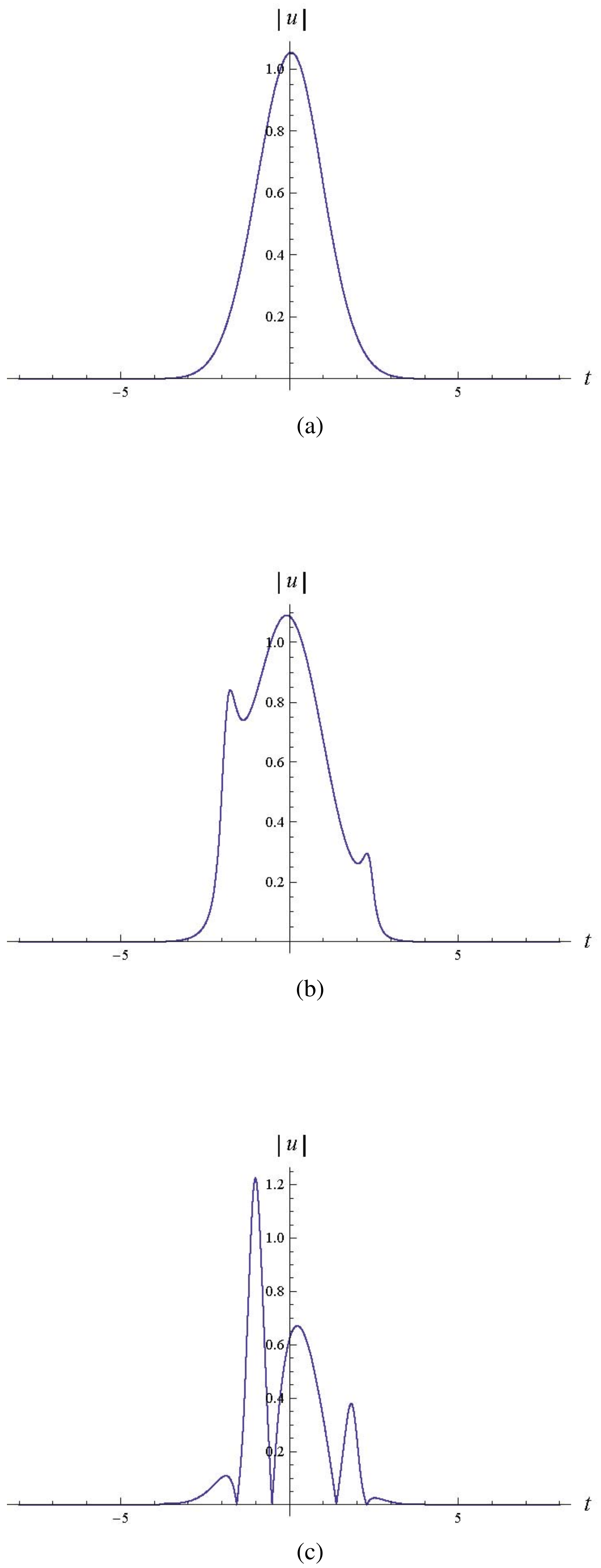


Fig. 9. Profiles of three-soliton solution (42) at different locations (a) $x = -5$, (b) $x = 0$ and (c) $x = 5$.

$$+B\mathrm{e}^{-\int f(t)\mathrm{d}t}, \tag{46}$$

under the constraints of Eqs. (6) and (41). Here the summation $\Sigma_{\mu=0,1}$ refers to all possible combination of each $\mu_i = 0,\ 1$ for $i = 1, 2, \cdots, n$.

III. Conclusion

We have obtained one-soliton solution (29), two-soliton solution (36), three-soliton solution (42) and n-soliton solution (46) of the time-dependent-coefficient Gardner equation (1) by improving two key steps of the HBM [7]. Though the kink one-soliton solution (17) is obtained by using the idea of HBM [7], these obtained multi-soliton solutions cannot been constructed by the HBM [7] without further improvement. For a given nonlinear PDE, say in two independent variables x and t

$$P(u, u_t, u_x, u_{xx}, u_{xt}, \cdots) = 0, \tag{47}$$

where P is in general a polynomial function of the indicated variables, the subscripts denote the partial derivatives, the steps of the improved HBM for n-soliton solutions can be outlined as follows:

Step 1: Supposing the solution of Eq. (47) is of the form

$$u(x,t) = \frac{\partial^{m+s} f(w)}{\partial x^m \partial t^s} + \sum_{i=0}^{m+s-1} a_i(x,t) f^{(i)}[w(x,t)], \tag{48}$$

where $m \geq 0$, $s \geq 0$ are integers which can be determined by balancing the highest nonlinear terms and the highest order partial derivative terms, and $a_i(x,t)$ is a polynomial of the partial derivative terms of $w(x,t)$ generated in the process of calculating the derivatives of $f(w)$ with respective to $w(x,t)$.

Step 2: Substituting Eq. (48) into Eq. (47), collecting all terms with the highest degree of derivatives of $w(x,t)$ and setting its coefficients to zero, one obtains an ODE for $f(w)$ and then solves it, in most cases $f(w)$ is a logarithm function.

Step 3: Supposing

$$w(x,t) = \frac{\sum_{i_1=0}^{p_1}\sum_{i_2=0}^{p_2}\cdots\sum_{i_n=0}^{p_n} a_{i_1 i_2 \cdots i_n} \mathrm{e}^{\sum_{g=1}^{n} i_g \xi_g}}{\sum_{j_1=0}^{q_1}\sum_{j_2=0}^{q_2}\cdots\sum_{j_n=0}^{q_n} b_{j_1 j_2 \cdots j_n} \mathrm{e}^{\sum_{g=1}^{n} j_g \xi_g}}, \tag{49}$$

where $\xi_g = k_g x + c_g t$, p_1, p_2, $\cdots$, p_n, q_1, q_2, $\cdots$, q_n are selected integers, $a_{i_1 i_2 \cdots i_n}$, $b_{j_1 j_2 \cdots j_n}$, k_g and c_g are undetermined constants.

Step 4: Substituting $f(w)$ determined in Step 2 along with Eq. (49) into Eq. (47) and equating each coefficient of the same powers of the exponential functions to zeros, then a set of nonlinear ODEs are derived. Solving the set of nonlinear ODEs, one determines all the constants $a_{i_1 i_2 \cdots i_n}$, $b_{j_1 j_2 \cdots j_n}$, k_g and c_g.

Step 5: Substituting $f(w)$, $w(x,t)$, m and s determined in above Steps into Eq. (48), the n-solution of Eq. (47) is finally determined.

Obviously, Steps 1, 2 and 5 are same as the corresponding Steps in the HBM [7, 24] but Steps 3 and 4 are different. This paper shows that the improved HBM can also be used to construct n-soliton solutions of some other nonlinear PDEs in mathematical physics.

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