Dynamical Behavior and Bifurcation Analysis of SEIR Epidemic Model and its Discretization

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Abstract—The dynamical behaviors of a SEIR epidemic model are investigated in this paper. More precisely, we presented a new discrete-time SEIR epidemic model by using the Forward-Euler difference method. And the existence, stability and direction of Hopf bifurcation of the SEIR epidemic model and its discretization model are also studied. In addition, the numerical simulations were presented to illustrate the theoretical analysis. Finally, some comparisons of bifurcation between the continuous-time epidemic system and its discrete-time system are given.

Index Terms—Stability, Discretization, Hopf bifurcation, Center manifold theorem, Forward Euler scheme

I. INTRODUCTION

The dynamical system refers to the dynamic system of change over time, which includes continuous dynamical systems and discrete dynamical systems. Despite the simplicity of dynamical systems, these systems have a rich dynamical behavior, ranging from stable equilibrium points to periodic and even chaotic oscillations. And Hopf bifurcation is an important dynamic bifurcation which closely related to some self-excited vibration phenomenon and has a high theoretical value in dynamic bifurcation and limited cycle research. Moreover, the research and application on bifurcation of autonomous systems has become a very popular topic [1-9]. But compared with the continuous systems, the discrete systems possess its unique dynamic characteristics. In the real life, many practical problems can be depicted by the discrete systems, and we can also to discretize the continuous systems. Therefore, the study of discrete system is very important and achieved great development in the field of mathematics, physics and engineering [10-13]. Hu et al. [14] presented a new epidemic model by using the Euler difference method, and discussed the Neimark-Sacker bifurcation of the system based on the center manifold theorem and the bifurcation theory. Elabbasy et al. [15] studied the Pitchfork bifurcation, Flip bifurcation and Neimark-Sacker bifurcation of a two-dimensional discrete Lorenz system. He et al. [16] focus on a third-order rational difference equation with positive parameters, and the existence and direction of the Neimark-Sacker bifurcation of the system are investigated in detail.

Epidemic is caused by the pathogen, which can be spread from human to human, human to animals and animals to animals. And the epidemic can makes a range of biological reduce or lose labor, death and spread rapidly in a certain period of time. Therefore, it has caused great attention of scientists and mathematicians. As early as 1927, Kermark and Mckendrick were established the mathematical model of infectious diseases by using the method of dynamics, and constructed the famous SIR bin model [17]. After the middle of the 20th century, the dynamics of infectious disease has been obtained rapid development, and the epidemic dynamical models have been widely investigated [18, 19]. At present, the bifurcation researches of discrete systems are mostly concentrated in the two-dimensional systems, and there are few studies focusing on the three dimensional discrete system. But compared with the two-dimensional model of infectious diseases, three-dimensional model can better reflect the mechanism of the spread of disease [20].

Wang [21] investigated an SEIR epidemic model with infectious force in latent period as follows:

\[
\begin{align*}
S' &= A - \beta_ES - \beta_IS - (\mu + \rho)S, \\
E' &= \beta_ES + \beta_IS - (\mu + \nu)E, \\
I' &= \nu E - (\mu + \alpha + \delta + \gamma)I, \\
R' &= \rho S + (\delta + \gamma)I - \mu R,
\end{align*}
\]

where \(S, E, I, R\) and \(N\) denote the numbers of susceptible, lurker, infective, recovered individuals and total numbers of the individuals, respectively. Assume that the susceptible crowd has a constant input rate \(A\), and \(\rho\) is the effective vaccination rate. The proportional coefficient of Lurker become infective is \(\nu\), and the lurker crowd possess bilinear incidence rate \(\beta_IS\). The infected people have bilinear incidence rate \(\beta_ES\), and \(\alpha\) is the diseased death rate, \(\delta\) is the cure rate, \(\gamma\) is the natural recovery rate, and \(\mu > 0\) is the natural mortality rate. Here \(N = S + E + I + R\), and all the coefficients are positive.

Due to the first three equations of model (1) is about \((S, E, I)\) not including \(R\), and the fourth equation is the linear equation of \(R\). Therefore, the dynamical behavior of model (1) is equivalent to the dynamical behaviors of the following model:

\[
\begin{align*}
S' &= A - \beta_ES - \beta_IS - (\mu + \rho)S, \\
E' &= \beta_ES + \beta_IS - (\mu + \nu)E, \\
I' &= \nu E - (\mu + \alpha + \delta + \gamma)I.
\end{align*}
\]

(2)

Applying the Forward-Euler difference method to model (2), we obtained the following three-dimensional
discrete-time SEIR epidemic model with infectious force in latent period:

\[
\begin{align*}
S_{n+1} &= S_n + h\left[A - \beta S_n E_n - \beta I_n S_n - (\mu + \rho)S_n\right] \\
E_{n+1} &= E_n + h\left[\beta E_n S_n + \beta I_n S_n - (\mu + v)E_n\right] \\
I_{n+1} &= I_n + h\left[v E_n - (\mu + \alpha + \delta + \gamma)I_n\right]
\end{align*}
\]

(3)

where \(h(0 < h < 1)\) is the step size.

The paper is organized as follows. In section 2, we discuss the stability of fixed points and the Hopf bifurcation by choosing \(\alpha\) as the bifurcation parameter in model (2). The stability of fixed points and the existence, direction and stability of the Hopf bifurcation of the discretized system are investigated in section 3. In section 4, we present the numerical simulations illustrate our results with the theoretical analysis. We have given some comparisons of bifurcation between the continuous-time epidemic system and its discrete-time system in section 5. In section 6, conclusion of the paper is given.

II. DYNAMICAL BEHAVIORS OF THE SEIR EPIDEMIC MODEL

A. Linear Analysis of the Fixed Points and Existence of Hopf Bifurcation

Through a simple calculation, we can easily get the following two fixed points of model (2):

\[ P_1 = \left(\frac{A}{\mu + \rho}, 0, 0\right), \quad P_2 = (S_0, E_0, I_0), \]

(4)

where

\[ S_0 = \left(\mu + v\right)\left(\mu + \alpha + \delta + \gamma\right) + \beta I_0, \]

\[ E_0 = \frac{A\beta v}{\mu + \alpha + \delta + \gamma} + \beta I_0, \]

\[ I_0 = \frac{v\left(A\beta v\right)}{\mu + \alpha + \delta + \gamma}. \]

The Jacobian matrix of model (2) at the fixed point \(P_i\) is given by

\[
J_i = \begin{bmatrix}
-\mu - \rho & -\frac{A\beta}{\mu + \rho} & -\frac{A\beta}{\mu + \rho} \\
0 & \frac{A\beta}{\mu + \rho} - (\mu + v) & \frac{A\beta}{\mu + \rho} \\
0 & v & - (\mu + \alpha + \delta + \gamma)
\end{bmatrix},
\]

(5)

and its characteristic polynomial is

\[
p(\lambda) = (\lambda + \mu + \rho)\left[\lambda^2 + (2\mu + v + \alpha + \delta + \gamma - \frac{A\beta}{\mu + \rho})\lambda + \left(\mu + v - \frac{A\beta}{\mu + \rho}\right)\frac{A\beta v}{\mu + \rho}\right].
\]

(6)

So, we can get the eigenvalues of \(J_i\) as follows

\[
\lambda_1 = -\mu - \rho, \quad \lambda_2 = \sqrt{\left(\mu + v - \frac{A\beta}{\mu + \rho}\right)\left(\mu + v - \frac{A\beta v}{\mu + \rho}\right)} - \frac{A\beta}{\mu + \rho},
\]

\[
\lambda_3 = -\sqrt{\left(\mu + v - \frac{A\beta}{\mu + \rho}\right)\left(\mu + v - \frac{A\beta v}{\mu + \rho}\right)} - \frac{A\beta}{\mu + \rho}.
\]

According to Routh-Hurwitz criterion, it is easy to obtain the following proposition.

**Proposition 1.** If \((\mu + \alpha + \delta + \gamma)\left(\mu + v - \frac{A\beta}{\mu + \rho}\right) - \frac{A\beta v}{\mu + \rho} > 0\), \(\mu + \rho > 0\), and \(\alpha = \alpha_0 = \frac{A\beta}{\mu + \rho} - v - \delta - \gamma - 2\mu\), then the fixed point \(P_i\) is asymptotically stable.

**Proposition 2.** Assume that \(\left(\mu + v - \frac{A\beta}{\mu + \rho}\right)\left(\mu + \alpha + \delta + \gamma\right) - \frac{A\beta v}{\mu + \rho} > 0\). If characteristic polynomial (6) has a pair of purely imaginary eigenvalues \(\lambda_{2,3} = \pm i\omega\), and \(\text{Re}(\lambda')(\alpha_0) \neq 0\), then the Hopf bifurcation occurs at the fixed point \(P_i\) when the bifurcation parameter \(\alpha\) pass through the critical value \(\alpha_0\).

**Proof.** Suppose that \(\lambda = i\omega, (\omega > 0)\) is a root of the equation (6), so we have

\[
-\omega^2 + \left(2\mu + v + \alpha + \delta + \gamma - \frac{A\beta}{\mu + \rho}\right)\omega + \left(\mu + v - \frac{A\beta}{\mu + \rho}\right) = 0,
\]

then separating the real and imaginary parts of above equation, and we get

\[
-\omega^2 + \left(\mu + v - \frac{A\beta}{\mu + \rho}\right)\left(\mu + \alpha + \delta + \gamma\right) - \frac{A\beta v}{\mu + \rho} = 0,
\]

\[
-\omega^2 + \left(2\mu + v + \alpha + \delta + \gamma - \frac{A\beta}{\mu + \rho}\right)\omega = 0.
\]

Through simple calculation, we have

\[
\omega = \omega_0 = \sqrt{\left(\mu + v - \frac{A\beta}{\mu + \rho}\right) - \frac{A\beta v}{\mu + \rho}},
\]

\[
\alpha = \alpha_0 = \frac{A\beta}{\mu + \rho} - v - \delta - \gamma - 2\mu.
\]

Take the derivative of both sides of Eq. (6) with respect to \(\alpha\), we obtain

\[
\frac{d\lambda}{d\alpha} + \frac{\left(\mu + \rho\right)\lambda^2 + \left(2\mu + v + \alpha + \delta + \gamma\right) - \frac{A\beta}{\mu + \rho}\right)\lambda}{\sum} = 0,
\]

and

\[
\frac{d\text{Re}\lambda}{d\alpha} \mid_{\alpha = \alpha_0} = 0, \quad \frac{d\text{Im}\lambda}{d\alpha} \mid_{\alpha = \alpha_0} \neq 0,
\]

where

\[
\sum = 3(\mu + \rho)^2 \lambda^2 + 2(\mu + \rho)^2 \lambda - A\beta + (\mu + \gamma + \alpha + \delta)\left((\mu + v)(\mu + \rho) - A\beta\right).
\]

According to the explicit criterion of Hopf bifurcation [22], we can get \(\alpha_0\) is the critical value of bifurcation, suppose that

\[
\left(\mu + v - \frac{A\beta}{\mu + \rho}\right)\left(\mu + v - \frac{A\beta v}{\mu + \rho}\right) > 0,
\]

when \(\alpha\) pass through the critical value \(\alpha_0\), the model (2) occurs Hopf bifurcation at the fixed point \(P_i\).
B. Direction and Stability of the Hopf Bifurcations

In this section, we study the direction and stability of Hopf bifurcation by using the normal form method and center manifold theory [23]. First, the model (2) can be written as

\[ X_{n+1} = JX_n + \frac{1}{6} B(X_n, X_n) + O(X_n^4), \]  (7)

where \( J \) is the Jacobin matrix at the fixed point \( P \), \( O(X_n^4) \) is the 4 order indefinite small of \( X_n \). And for \( i = 1, 2, 3 \), we have

\[
B_i(x, y, z) = \sum_{j=1}^{\frac{3}{2}} \frac{\partial^2 X_i(x, y, z)}{\partial \xi_j \partial \xi_k} |_{x=X_i, y=Y_i, z=Z_i},
\]

(8)

Let \( p, q \in C^3 \) be vectors such that:

\[ J q = i o_0 q, J^* p = -i o_0 p, \langle p, q \rangle = \sum_{i=1}^{3} \Omega_i q_i = 1, \]

(9)

where \( J^* \) is the transpose of the \( J \). For model (2), we can get

\[
B(x, y, z) = -\beta_1 (x_i y_j + x_j y_i) - \beta_2 (x_i y_j + x_j y_i),
\]

(10)

\[ q = (e_1 + e_2 i, e_1 - e_2 i, 1) \quad p = (0, e_1 i + e_2 i, e_1 - e_2 i), \]

(11)

Thus, we can obtain

\[
B(q, q) = (-2 \beta_1 (e_1 i + e_2 i) + \beta_1 e_1) - 2 \beta_2 (e_1 i + e_2 i) + \beta_2 e_1,\]

(12)

\[
B(q, \bar{q}) = (-2 \beta_1 (e_1 i + e_2 i) - \beta_1 e_1) - 2 \beta_2 (e_1 i + e_2 i) - \beta_2 e_1,\]

(13)

\[
h_{11} = -A^2 B(q, \bar{q}), \]

(14)

\[
h_{12} = 2 i o_0 P - A^2 B(q, \bar{q}),\]

(15)

\[
= (\theta_1 + 2 \theta_2 + (\theta_1 + 2 \theta_2) j, -\theta_1 - 2 \theta_2 - (\theta_1 + 2 \theta_2) j),\]

where

\[
\begin{align*}
\theta_1 &= -2\Omega_3 (\Omega_1 e_1 + \Omega_2 e_2 + \Omega_3 e_3),
\theta_2 &= 2\Omega_3 (\Omega_1 e_1 + \Omega_2 e_2 + \Omega_3 e_3),
\end{align*}
\]

Through direct calculation, we also have

\[
G_{11} = \langle p, H_{11} \rangle = e_1 (2\theta_1 + \theta_2) + e_3 (2\theta_1 + \theta_2) j,\]

(16)

Theorem 1. Consider model (2), the first Lyapunov coefficient associated to the fixed point \( P \) is given by

\[
l_1 = \frac{1}{2} \text{Re} G_{11} = -\frac{1}{2} e_1 (2\theta_1 + \theta_2),\]

(17)

If \( l_1 \) is different from zero, then model (2) has a transversal Hopf point at \( P \). More precisely, if \( l_1 < 0 \), the Hopf bifurcation at the fixed point \( P \) is supercritical and there exists a stable periodic orbit near the asymptotically stable fixed point \( P \); if \( l_1 > 0 \), the Hopf bifurcation at the fixed point \( P \) is subcritical.
III. DYNAMICAL BEHAVIORS OF THE DISCRETIZED SEIR EPIDEMIC MODEL

A. Stability of the Fixed Points of the Discretized Model

Next, we study the asymptotic stability of model (3). The fixed points of model (3) satisfy:

\[
\begin{align*}
S_n &= S_0 + h[A - \beta E_n S_n - \beta I_n S_n - (\mu + \rho) S_n], \\
E_n &= E_0 + h[\beta E_n S_n + \beta I_n S_n - (\mu + \nu) E_n], \\
I_n &= I_0 + h[\nu E_n - (\mu + \alpha + \delta + \gamma) I_n].
\end{align*}
\]

(18)

The Jacobin matrix \( J \) of model (3) at the fixed point \( P^* = (S^*, E^*, I^*) \) is given by

\[
\begin{pmatrix}
-1 - h\beta E^* - h\beta I^* - h(\mu + \rho) & -h\beta S^* - h\beta S^* & 0 \\
1 + h\beta S^* - h(\mu + \nu) & h\beta S^* & 0 \\
0 & 0 & h(\mu + \alpha + \delta + \gamma)
\end{pmatrix}.
\]

(19)

For simplicity, we only consider the fixed point \( P_i \). And the characteristic equation of model (3) at the fixed point \( P_i \) has the form

\[
p(\lambda) = \left[\lambda - 1 - h(\mu + \rho)\right]\left[\lambda^2 + a_1\lambda + a_2\right],
\]

(20)

where

\[
a_1 = h(2\mu + \nu + \alpha + \delta + \gamma) - 2 - \frac{A\beta}{\mu + \rho},
\]

\[
a_2 = \left[1 + \frac{A\beta}{\mu + \rho} - h(\mu + \nu)\right]\left[1 - h(\mu + \alpha + \delta + \gamma)\right] - \frac{A\beta y h^2}{\mu + \rho}.
\]

From the local stability theory of fixed point, it is easy to obtain the following Proposition:

**Proposition 3.**

(1) \( P_i \) is local asymptotically stable if one of the following conditions holds:

(a) \( a_1 - 4a_2 \geq 0, \quad 0 < h(\mu + \rho) < 2, \quad -a_1 < a_2 < \min\{1 + a_1, 1 - a_1\} \);

(b) \( a_1 - 4a_2 < 0, \quad 0 < h(\mu + \rho) < 2, \quad a_2 < 1 \).

(2) \( P_i \) is unstable if one of the following conditions holds:

(a) \( a_1 - 4a_2 \geq 0, \quad h(\mu + \rho) > 2, \quad a_2 < \min\{1 - a_1, 1 - a_1\} \) or \( a_2 > \max\{-1 - a_1, -1 - a_1\} \);

(b) \( a_1 - 4a_2 < 0, \quad h(\mu + \rho) > 2, \quad a_2 > 1 \).

(3) \( P_i \) is non-hyperbolic if one of the following conditions holds:

(a) \( a_1 - 4a_2 \geq 0, \quad h(\mu + \rho) = 2, \quad a_2 = 1 - a_1 \) or \( a_2 = a_1 - 1 \);

(b) \( a_1 - 4a_2 < 0, \quad h(\mu + \rho) = 2, \quad a_2 = 1 \).

B. Existence of Hopf Bifurcation

When \( a_1 - 4a_2 < 0 \), the eigenvalues of model (3) at fixed point \( P_i = (A/\mu + \rho, 0, 0) \) can be written as

\[
\lambda_i = 1 - h(\mu + \rho), \quad \lambda_{2,3} = -\frac{1}{2}\left[a_1 \pm \sqrt{4a_2 - a_1^2}\right].
\]

(21)

Assume that

\[
a_6 = -\frac{A\beta}{\mu + \rho} - h(\mu + \rho) - A\beta y h(\mu + \rho) - (\mu + \alpha + \delta + \gamma),
\]

and \( h(\mu + \rho) \neq 0, 2, \mu + \rho + A\beta y h \neq h(\mu + \rho)(\mu + \rho) \), we can get:

\[
\lambda_i (a_6) = 1 - h(\mu + \rho), \quad \lambda_{2,3} (a_6) = a_2 = 1.
\]

(22)

\[
\frac{d}{da_6} \lambda_i (a_6) = \frac{h(\mu + \nu)(\mu + \rho) - (\mu + \rho + A\beta y h)(\mu + \rho)}{\mu + \rho} \neq 0.
\]

(23)

\[
\lambda_i (a_6) = 1 - h(\mu + \rho), \lambda_{2,3} (a_6) = \frac{a_6}{2} \pm \frac{\sqrt{4a_2 - a_1^2}}{2}.
\]

(24)

and by calculation we can get \( \lambda_{2,3} (a_6) \neq 1, 2, 3, 4 \). According to bifurcation theory [24], the Hopf bifurcation occurs at the fixed point \( P_i = (A/\mu + \rho, 0, 0) \).

C. Direction and Stability of the Hopf Bifurcation

Next, we investigate the stability and direction of the Hopf bifurcation by using the Kuznetsov’s normal form method and center manifold theory [25]. First, the model (3) can be written as

\[
X_{n+1} = JX_n + \frac{1}{2} B(X_n, X_n) + \frac{1}{6} C(X_n, X_n, X_n) + O(X_n^4),
\]

(25)

where \( J \) is the Jacobin matrix at the fixed point \( P_i, O(X_n^4) \) is the 4 order definite small of \( X_n \), and for \( i = 1, 2, 3 \), we can get:

\[
B_i(x, y) = \sum_{j,k=0}^{3} \frac{\partial^2 X_i(x, 0)}{\partial x^j \partial y^k} x^j y^k,
\]

\[
C_i(x, y, z) = \sum_{j,k,l=0}^{3} \frac{\partial^3 X_i(x, 0)}{\partial x^j \partial y^k \partial z^l} x^j y^k z^l,
\]

(26)

Let \( p, q \in \mathbb{C} \) be vectors such that:

\[
J q = \lambda_i q, \quad J^T p = \lambda_i p, \quad \{p, q\} = \frac{1}{3} \sum_{i=1}^{3} q_i = 1,
\]

(27)

where \( J^T \) is the transpose of the \( J \), and \( \lambda_i, \lambda_j \) is a pair of complex conjugate eigenvalues at the fixed point \( P_i \). For the model (3), we can get

\[
B_i(x, y) = \left(-\beta_0 E_i y - x_i y_i\right) + \beta_i \left(x_i y_i + x_i y_i + x_i y_i, 0\right)^T, \quad C_i(x, y, z) = (0, 0, 0)^T,
\]

(28)

and

\[
q = (\varphi_0 + \varphi_i j, \varphi_2 j, \varphi_3 j, 1)^T, \quad p = (0, \varphi_i j, \varphi_0 + \varphi_i j, 1)^T,
\]

(29)

where

\[
\varphi_0 = \frac{(A\beta y h \varphi_0 + A\beta y h) \left[2 - 2h(\mu + \rho) + a_1\right] - A\beta y h \varphi_0 \sqrt{4 - a_1^2}}{2 + 2a_1(2h(\mu + \rho) - 2 - a_1)},
\]

(30)

\[
\varphi_1 = \frac{A\beta y h \varphi_1 + A\beta y h}{{2} + 2a_1(2h(\mu + \rho) - 2 - a_1)},
\]

(31)

\[
\varphi_2 = \frac{2 - 2h(\mu + \alpha + \delta + \gamma) + a_1}{2h}, \quad \varphi_3 = -\sqrt{-a_1^2},
\]

(32)

\[
\varphi_4 = \frac{(\mu + \rho)(2h(\mu + \alpha + \delta + \gamma) - 2 - a_1) \varphi_4 + (\mu + \rho) \varphi_4 \sqrt{4 - a_1^2}}{2A\beta y h},
\]

(33)

\[
\varphi_5 = \frac{2A\beta y h}{2A\beta y h (2h(\mu + \alpha + \delta + \gamma) - 2 - a_1) \varphi_5 + (\mu + \rho) \varphi_5 \sqrt{4 - a_1^2}}{A},
\]

(34)
\[
\varphi_k = \frac{2A\beta h (\mu + \rho) [2h (\mu + \alpha + \delta + \gamma) - 2 - a_i] \varphi_k}{\Delta} + \frac{\varphi_k \sqrt{4 - a_i^2} 2A\beta h (\mu + \rho)}{\Delta},
\]
\[
\Delta = \left[ (\mu + \rho) [2h (\mu + \alpha + \delta + \gamma) - 2 - a_i] \varphi_k + \varphi_k \sqrt{4 - a_i^2} \right]^2 + \left[ 2h (\mu + \alpha + \delta + \gamma) - 2 - a_i \right] \varphi_k + \varphi_k \sqrt{4 - a_i^2} + 2A\beta h \right]^2.
\]

So the coefficients of the normal of model (3) can be computed by the following formulas:

\[
g_{30} = \left\{ p, B (q, q) \right\} = \varphi_0 + \varphi_{30}i, \quad g_{11} = \left\{ p, B (q, \bar{q}) \right\} = \varphi_1 + \varphi_{11}i, \quad g_{02} = \left\{ p, B (\bar{q}, \bar{q}) \right\} = \varphi_3 + \varphi_{02}i.
\]

(30)

where

\[
\varphi_0 = 2\beta h (\varphi_2 - \varphi_0) (\varphi_0 - \varphi_1) - \beta h (\varphi_0 + \varphi_3),
\]
\[
\varphi_0 = 2\beta h (\varphi_0 - \varphi_2) (\varphi_0 - \varphi_1) - \beta h (\varphi_0 + \varphi_3),
\]
\[
\varphi_1 = 2h \varphi_3 \left( \beta (\varphi_0 + \varphi_3) + \beta \varphi_1 \right),
\]
\[
\varphi_2 = 2h \varphi_3 \left( \beta (\varphi_0 + \varphi_3) + \beta \varphi_1 \right),
\]
\[
\varphi_3 = 2h \varphi_3 \left( \beta (\varphi_0 + \varphi_3) + \beta \varphi_1 \right),
\]
\[
\varphi_0 = 2\beta h (\varphi_0 + \varphi_2) (\varphi_0 + \varphi_1) + 2\beta h (\varphi_0 + \varphi_3),
\]
\[
\varphi_0 = 2\beta h (\varphi_0 + \varphi_2) (\varphi_0 + \varphi_1) + 2\beta h (\varphi_0 + \varphi_3),
\]

and

\[
g_{21} = \left\{ p, C (q, \bar{q}) \right\} + 2 \left\{ p, B (I_x - J)^{-1} (q, q) \right\} + \left\{ p, B (\bar{q}, \bar{q}) \right\} + \left( \varphi_0 \right) (1 - 2\lambda_a) \varphi_{21} + \frac{\lambda_a}{1 - \lambda_a} \left| \varphi_{21} \right|^2 + \frac{\lambda_a}{\lambda_a - 1} \left| \varphi_{02} \right|^2
\]
\[
= 2M_2 + N_1 + N_2 + (2M_2 + N_1 + N_2 + N_2) i,
\]

where

\[
M_1 = \left( \varphi_2 (\mu + \alpha + \delta + \gamma) \right)[A\beta h (\mu + \rho)] + \varphi_2 (\mu + \rho) (\mu + \alpha + \delta + \gamma) - 2A\beta h v]
\]
\[
\varphi_2, h (\mu + \rho) (\mu + \alpha + \delta + \gamma) - 2A\beta h v]
\]
\[
M_4 = h (\beta_2 M_3 - \beta_1 (M_1 - M_2)) (\varphi_0 \varphi_0 + \varphi_0 \varphi_0) - \beta_2 h M_3 \varphi_0 \varphi_0 - \beta_1 h M_3 \varphi_0 \varphi_0
\]
\[
N_1 = -\varphi_0 h \left( \beta_0 \varphi_0 + \varphi_0 \varphi_0 + \varphi_0 \varphi_0 + \varphi_0 \varphi_0 + \beta_0 (\varphi_0 + \varphi_0) \right)
\]
\[
N_3 = \left( a_i + 1 \right) (a_i + 2) (\varphi_0 \varphi_0 - \varphi_0 \varphi_0) + \left(3 + 2a_i \right) \sqrt{4 - a_i^2} (\varphi_0 \varphi_1 + \varphi_1 \varphi_2)
\]
\[
\Delta_1 = \left( \frac{3 + 2a_i}{4 + 2a_i} \right) \sqrt{4 - a_i^2} (\varphi_0 \varphi_2 - \varphi_0 \varphi_1)
\]

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\[ n_{z} = \frac{(\mu + \rho) \nu h \left[ f_{1}(\eta_{2} - \eta_{3} + (\mu + \rho) + f_{2} A \beta_{2} \nu h^{2} \right]}{\left[ (\eta_{2} - \eta_{3})(\mu + \rho) - A \beta_{2} \nu h^{2} \right]^{2} + \eta_{2}^{2} (\eta_{2} - \eta_{3})(\mu + \rho)^{2} \right]}
\]

Then, by calculate one has

\[ l_{1}(\alpha_{0}) = Re \left( \frac{\lambda_{z\alpha_{0}}}{2} \right) - Re \left( \frac{\lambda_{z\alpha_{0}}^{2} (1 - 2 \lambda_{z})}{2 (1 - \lambda_{z})} \right) - \frac{1}{2} |\lambda_{z\alpha_{0}}|^{2} - \frac{1}{4} |\lambda_{z\alpha_{0}}|^{2} \]

**Theorem 2.** The direction and stability of Hopf bifurcation at the fixed point \( P_{1} \) is determined by \( l_{1}(\alpha_{0}) \). If \( l_{1}(\alpha_{0}) < 0 > 0 \), the Hopf bifurcation of model (3) at \( \alpha_{0} = \frac{A \beta_{2} - (\mu + \rho)(\mu + \rho) - A \beta_{2} \nu h}{\mu + \rho + A \beta_{h} h - h(\mu + \rho)}(\mu + \delta + \gamma) \) is supercritical (subcritical), and the unique closed invariant curve bifurcating from \( P_{1} \) is asymptotically stable (unstable).

**IV. NUMERICAL SIMULATIONS**

**A. Hop bifurcation simulation of continuous Model**

First, we give a numerical example of model (2). Let \( A = 10, \mu = 0.4, \rho = 0.12, \delta = 0.2, \beta_{1} = 0.1, \beta_{2} = 0.1, \gamma = 0.3, v = 0.1, h = 0.1 \) and by compute we get the critical value \( \alpha_{0} = 0.329 \). The fixed point \( P_{1} \) is stable when \( \alpha = 0.31 < \alpha_{0} \), and unstable when \( \alpha = 0.34 > \alpha_{0} \), as shown in Fig. 3 and Fig. 4, respectively. From the formulas in previous section, we can get \( l_{1}(\alpha_{0}) = 0.0764 > 0 \). Thus, the periodic solutions bifurcating from the fixed point \( P_{1} \) are subcritical and unstable.

**B. Hop bifurcation simulation of discrete Model**

Next, we choose one group parameters: \( A = 10, \mu = 0.4, \rho = 0.12, \delta = 0.2, \beta_{1} = 0.1, \beta_{2} = 0.1, \gamma = 0.3, v = 0.1, h = 0.1 \) and by compute we get the critical value \( \alpha_{0} = 0.329 \). The fixed point \( P_{1} \) is stable when \( \alpha = 0.31 < \alpha_{0} \), and unstable when \( \alpha = 0.34 > \alpha_{0} \), as shown in Fig. 3 and Fig. 4, respectively. Based on the previous conclusions and through complex calculations, we have \( l_{1}(\alpha_{0}) = 0.0764 > 0 \). Therefore the Hopf bifurcation of model (3) at the fixed point \( P_{1} \) is subcritical, and the unique invariant curve which is resulting from the bifurcation at fixed point is unstable.
Fig. 3. Time history and phase diagram of model (3) with $\alpha = 0.31$

**V. COMPARISONS**

For the continuous-time model (2) at the fixed point $P_1$ with

$$\left(\mu + v - \frac{A\beta}{\mu + \rho}\right)^2 + \frac{A\beta v}{\mu + \rho} > 0,$$

we know that the critical value of Hopf bifurcation as follows:

$$\alpha_1^* = \frac{A\beta}{\mu + \rho} - v - \delta - \gamma - 2\mu. \quad (33)$$

For the discrete-time model (3) at the fixed point $P_1$ with $a_1 - 4a_2 < 0$, let $h = 0.4, \beta = 0.5$, we obtain the critical value of Hopf bifurcation as follows:

$$\alpha_2^* = \frac{A\beta - (\mu + v)(\mu + \rho) - 0.2Av}{\mu + \rho + 0.4[A\beta - (\mu + v)(\mu + \rho)]} - (\mu + \delta + \gamma). \quad (34)$$

By simple calculation, we can get the following conclusions.

**Proposition 4.** Hopf bifurcations of continuous-time model (2) and discrete-time model (3) occur simultaneously when

$$0.4A^2\beta^2 + (\mu + \rho)\left[\frac{(\mu + v)(\mu + \rho - 0.4A\beta - 0.2Av)}{(\mu + \rho)(\mu + \rho + 0.4A\beta - 0.4(\mu + v)(\mu + \rho))} = v + \mu, \quad (35)$$

Fig. 4. Time history and phase diagram of model (3) with $\alpha = 0.34$

**Proposition 5.** The continuous-time model (2) undergoes Hopf bifurcation earlier than the discrete-time model (3) when

$$0.4A^2\beta^2 + (\mu + \rho)\left[\frac{(\mu + v)(\mu + \rho - 0.4A\beta - 0.2Av)}{(\mu + \rho)(\mu + \rho + 0.4A\beta - 0.4(\mu + v)(\mu + \rho))} < v + \mu, \quad (36)$$

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Proposition 6. The discrete-time model (3) undergoes Hopf bifurcation earlier than the continuous-time model (2) when
\[
\frac{0.44\beta^2}{(\mu + \rho)} \left[ \frac{\mu + \rho - 0.44\beta - 0.24v}{(\mu + \rho)} \right] > v + \mu.
\]

VI. CONCLUSIONS

In this paper, we introduced a SEIR epidemic model and obtained a new discrete-time epidemic model by using the Forward-Euler difference method. A necessary and sufficient condition for existence of the solution of the SEIR epidemic model is obtained, and we also investigated the local stability of the fixed point of the epidemic model and its discretized counterpart. Besides, the stability and direction of Hopf bifurcation are proved by using the Kuznetsov’s formal method and center manifold theory. And the numerical simulations were presented to illustrate the above main results. Finally we give some comparisons of bifurcation between the discrete-time epidemic model and its continuous-time model..

REFERENCES


