Some Inequalities for L_p-geominimal Surface Area

Yibin Feng

***Abstract*—In this article, we first investigate two affine isoperimetric inequalities for L_p-geominimal surface area. Then some Blaschke-Santaló type inequalities for L_p-geominimal surface area are established.**

***Index Terms*—L_p-geominimal surface area, L_p-centroid body, L_p-curvature image, L_p-projection body.**

I. Introduction

LET $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbf{R}^n$. For the set of convex bodies containing the origin in their interiors, the set of convex bodies whose centroid lie at the origin and the set of origin-symmetric convex bodies in $\mathbf{R}^n$, we write $\mathcal{K}_o^n$, $\mathcal{K}_c^n$ and $\mathcal{K}_{os}^n$, respectively. $\mathcal{S}_o^n$ denotes the set of star bodies (about the origin) in $\mathbf{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbf{R}^n$, and $V(K)$ denotes the n-dimensional volume of a body K. For the standard unit ball B in $\mathbf{R}^n$, we denote its volume by $\omega_n = V(B)$.

The concept of geominimal surface area was introduced by Petty [22] about 40 years ago. The study of affine surface area goes back to Blaschke [1] and is about one hundred years old. In [10], Lutwak demonstrated that there were natural extensions of affine and geominimal surface area in the Brunn-Minkowski-Firey theory. It motivates extensions of some known inequalities for affine surface area and geominimal surface areas to L_p-affine surface area and L_p-geominimal surface area, respectively. Since then, considerable attention has been paid to the L_p-affine surface area and the L_p-geominimal surface area, which is now at the core of the rapidly developing L_p-Brunn-Minkowski theory (see articles [4], [5], [18], [19], [20], [21], [32], [33], [34], [38]).

For $K \in \mathcal{K}_o^n$, the geominimal surface area, $G(K)$, of K is defined by (see [22])

$$\omega_n^{\frac{1}{n}} G(K) = \inf\{nV_1(K,Q)V(Q^*)^{\frac{1}{n}} : Q \in \mathcal{K}_o^n\}.$$

Here Q^* denotes the polar of body Q and $V_1(M,N)$ denotes the mixed volume of $M, N \in \mathcal{K}_o^n$.

According to L_p-mixed volume, Lutwak [10] introduced the notion of L_p-geominimal surface area. For $K \in \mathcal{K}_o^n$ and $p \geq 1$, the L_p-geominimal surface area, $G_p(K)$, of K is defined by

$$\omega_n^{\frac{p}{n}} G_p(K) = \inf\{nV_p(K,Q)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\}. \quad (1)$$

Here $V_p(M,N)$ denotes the L_p-mixed volume of $M, N \in \mathcal{K}_o^n$.

Manuscript received January 18, 2016; revised February 20, 2016. This work was supported by the National Natural Science Foundations of China (Grant No.11561020 and 11371224), and was supported by the Young Foundation of Hexi University (Grant No.QN2015-02).

Yibin Feng is with the School of Mathematics and Statistics, Hexi University, Zhangye, 734000, China. E-mail: fengyibin001@163.com.

Obviously, if $p = 1$, $G_p(K)$ is just the geominimal surface area $G(K)$. Further, Lutwak in [10] proved a affine isoperimetric inequality for L_p-geominimal surface area.

Theorem 1.A. If $K \in \mathcal{K}_o^n$ and $p \geq 1$, then

$$G_p(K) \leq n\omega_n^{\frac{p}{n}} V(K)^{\frac{n-p}{n}}, \quad (2)$$

with equality if and only if K is an ellipsoid.

The following result [37] is also a affine isoperimetric inequality, which is related to L_p-projection body.

Theorem 1.B. If $K \in \mathcal{K}_c^n$ and $p \geq 1$, then

$$G_p(K) \leq n\omega_n^{\frac{n-p}{n}} V(\Pi_p K)^{\frac{p}{n}}, \quad (3)$$

with equality if and only if K is an ellipsoid.

In this paper, we first give the polar form of Theorem 1.A for L_p-geominimal surface area.

Theorem 1.1. If $K \in \mathcal{K}_c^n$ and $p \geq 1$, then

$$G_p(K) \leq n\omega_n^{\frac{2n-p}{n}} V(K^*)^{\frac{p-n}{n}}, \quad (4)$$

with equality if and only if K is an ellipsoid.

Next, we establish the polar form of Theorem 1.B for L_p-geominimal surface area.

Theorem 1.2. If $K \in \mathcal{K}_o^n$ and $p \geq 1$, then

$$G_p(K) \leq n\omega_n^{\frac{n+p}{n}} V(\Pi_p^* K)^{-\frac{p}{n}}, \quad (5)$$

with equality if and only if K is an ellipsoid centered at the origin.

As some applications of Theorem 1.1 and Theorem 1.A, we establish the following Blaschke-Santaló type inequalities for L_p-geominimal surface area.

Theorem 1.3. If $K \in \mathcal{F}_c^n$ and $1 \leq p \leq n$, then

$$G_p(\Lambda_p K) G_p(\Pi_p^* K) \leq (n\omega_n)^2, \quad (6)$$

with equality if and only if K is an ellipsoid centered at the origin.

Theorem 1.4. If $K \in \mathcal{K}_o^n$ and $1 \leq p \leq n$, then

$$G_p(K) G_p(\Gamma_p^* K) \leq (n\omega_n)^2, \quad (7)$$

with equality if and only if $\Gamma_p K$ is an ellipsoid where K is an ellipsoid centered at the origin.

Theorem 1.5. If $K \in \mathcal{K}_c^n$ and $p \geq n$, then

$$G_p(K)^{p-n} G_p(\Pi_p K)^p \leq (n\omega_n)^{2p-n}, \quad (8)$$

with equality if and only if K is an ellipsoid centered at the origin.

Finally, we use a method different from the above to also show a Blaschke-Santaló type inequality for L_p-geominimal surface area.

Theorem 1.6. If $K \in \mathcal{F}_{os}^n$ and $1 \leq p \leq n$, then

$$G_p(K)^{n-p} G_p(\Lambda_p^* K)^p \leq (n\omega_n)^n, \quad (9)$$

with equality for $1 < p \leq n$ if and only if $\Lambda_p^* K$ and K are dilates; for $p = 1$ if and only if $\Lambda_p^* K$ and K are homothetic.

(Advance online publication: 23 February 2017)

Please see the next section for the above interrelated background materials. The proofs of Theorems 1.1-1.6 will be given in Section 3 of this paper.

II. Preliminaries

A. Support function, radial function and polar of convex bodies

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K,\cdot) : \mathbf{R}^n \to (-\infty, \infty)$, is defined by (see[2], [24])

$$h(K,x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbf{R}^n, \qquad (10)$$

where $x \cdot y$ denotes the standard inner product of x and y.

If K is a compact star-shaped (about the origin) set in $\mathbf{R}^n$, then its radial function, $\rho_K = \rho(K,\cdot) : \mathbf{R}^n \setminus \{0\} \to [0,\infty)$, is defined by (see[2], [24])

$$\rho(K,u) = \max\{\lambda \geq 0 : \lambda \cdot u \in K\}, \quad u \in S^{n-1}. \qquad (11)$$

If ρ_K is continuous and positive, then K will be called a star body. Two star bodies K, L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If $K \in \mathcal{K}_o^n$, the polar body, K^*, of K is defined by (see [2], [24])

$$K^* = \{x \in \mathbf{R}^n : x \cdot y \leq 1, y \in K\}. \qquad (12)$$

From (12), we easily have $(K^*)^* = K$, and

$$h_{K^*} = \frac{1}{\rho_K}, \quad \rho_{K^*} = \frac{1}{h_K}. \qquad (13)$$

For $K \in \mathcal{K}_c^n$ and its polar body, the well-known Blaschke-Santaló inequality is stated that (see [23])

Theorem 2.A. If $K \in \mathcal{K}_c^n$, then

$$V(K)V(K^*) \leq \omega_n^2, \qquad (14)$$

with equality if and only if K is an ellipsoid.

B. L_p-mixed volume

For $K, L \in \mathcal{K}_o^n$, $p \geq 1$ and $\varepsilon > 0$, the Firey L_p-combination, $K +_p \varepsilon \cdot L \in \mathcal{K}_o^n$, is defined by (see [11])

$$h(K +_p \varepsilon \cdot L, \cdot)^p = h(K,\cdot)^p + \varepsilon h(L,\cdot)^p,$$

where " $\cdot$ " in $\varepsilon \cdot L$ denotes the Firey scalar multiplication.

If $K, L \in \mathcal{K}_o^n$, then for $p \geq 1$, the L_p-mixed volume, $V_p(K,L)$, of K and L is defined by (see [11])

$$\frac{n}{p} V_p(K,L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

For $K, L \in \mathcal{K}_o^n$ and $p \geq 1$, there is a positive Borel measure, $S_p(K,\cdot)$, on S^{n-1} such that (see [11])

$$V_p(K,L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(u) dS_p(K,\cdot). \qquad (15)$$

From (15), we have

$$V_p(K,K) = V(K). \qquad (16)$$

The Minkowski inequality for L_p-mixed volume is called L_p-Minkowski inequality. The L_p-Minkowski inequality was given by Lutwak (see [10], [11]):

Theorem 2.B. If $K, L \in \mathcal{K}_o^n$ and $p \geq 1$, then

$$V_p(K,L) \geq V(K)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}}, \qquad (17)$$

with equality for $p = 1$ if and only if K and L are homothetic; for $p > 1$ if and only if K and L are dilates.

C. L_p-dual mixed volume

For $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the L_p-harmonic radial combination, $\lambda \star K +_{-p} \mu \star L \in \mathcal{S}_o^n$, of K and L is defined by (see [10])

$$\rho(\lambda \star K +_{-p} \mu \star L, \cdot)^{-p} = \lambda \rho(K,\cdot)^{-p} + \mu \rho(L,\cdot)^{-p}. \qquad (18)$$

Associated with the L_p-harmonic radial combination of star bodies, Lutwak in [10] introduced the notion of L_p-dual mixed volume as follows: For $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and $\varepsilon > 0$, the L_p-dual mixed volume, $\widetilde{V}_{-p}(K,L)$, of K and L is defined by (see [10])

$$\frac{n}{-p} \widetilde{V}_{-p}(K,L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_{-p} \varepsilon \star L) - V(K)}{\varepsilon}.$$

The definition above and Hospital's role give the following integral representation of L_p-dual mixed volume (see [10]):

$$\widetilde{V}_{-p}(K,L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p}(u) \rho_L^{-p}(u) dS(u), \qquad (19)$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1}.

From formula (19), we get

$$\widetilde{V}_{-p}(K,K) = V(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^n(u) dS(u). \qquad (20)$$

D. L_p-curvature image

For $K \in \mathcal{K}_o^n$ and $p \geq 1$, the L_p-surface area measure, $S_p(K)$, of K is defined by (see [11])

$$\frac{dS_p(K,\cdot)}{dS(K,\cdot)} = h(K,\cdot)^{1-p}. \qquad (21)$$

Equation (21) is also called Radon-Nikodym derivative, it turns out that the measure $S_p(K,\cdot)$ is absolutely continuous with respect to surface area measure $S(K,\cdot)$.

A convex body $K \in \mathcal{K}_o^n$ is said to have L_p-curvature function (see [10]), $f_p(K,\cdot) : S^{n-1} \to \mathbf{R}$, if its L_p-surface area measure $S_p(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measures, and

$$f_p(K,\cdot) = \frac{dS_p(K,\cdot)}{dS}. \qquad (22)$$

Let $\mathcal{F}_o^n$, $\mathcal{F}_{os}^n$ and $\mathcal{F}_c^n$ denote the set of all bodies in $\mathcal{K}_o^n$, $\mathcal{K}_{os}^n$ and $\mathcal{K}_c^n$ that have a positive continuous curvature function, respectively.

Lutwak showed the notion of L_p-curvature image in [10] as follows: For $K \in \mathcal{F}_o^n$ and $p \geq 1$, defined $\Lambda_p K \in S_o^n$, L_p-curvature image of K, by

$$\rho(\Lambda_p K, \cdot)^{n+p} = \frac{V(\Lambda_p K)}{\omega_n} f_p(K,\cdot). \qquad (23)$$

Note that for $p = 1$, this definition differs from the definition of classical curvature image. For the studies of classical curvature image and L_p-curvature image, see articles [7], [12], [23], [25], [26], [27], [28], [30].

E. L_p-projection body and L_p-centroid body

The notion of L_p-projection body is shown by Lutwak (see [14]). For $K \in \mathcal{K}_o^n$ and $p \geq 1$, the L_p-projection body, $\Pi_p K$, of K is the origin-symmetric convex body whose support function is given by

$$h_{\Pi_p K}^p(u) = \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |u \cdot v|^p dS_p(K, v), \qquad (24)$$

where $u, v \in S^{n-1}$, and $S_p(K, v)$ is a positive Borel measure on S^{n-1}.

In 1997, Lutwak and Zhang in [13] introduced the concept of L_p-centroid body as follows: For each compact star-shaped about the origin $K \subset \mathbf{R}^n$ and real number $p \geq 1$, the L_p-centroid body, $\Gamma_p K$, of K is the origin-symmetric convex body whose support function is defined by

$$h_{\Gamma_p K}^p(u) = \frac{1}{c_{n,p} V(K)} \int_K |u \cdot x|^p dx. \qquad (25)$$

Here the integration is with respect to Lebesgue and $c_{n,p} = \omega_{n+p}/\omega_2 \omega_n \omega_{p-1}$.

Using polar coordinates in (25), we easily get

$$h_{\Gamma_p K}^p(u) = \frac{1}{(n+p)c_{n,p} V(K)} \int_{S^{n-1}} |u \cdot v|^p \rho_K(v)^{n+p} dv, \qquad (26)$$

for any $u \in S^{n-1}$.

Since Lutwak, Yang, and Zhang's seminal work, there are many papers on L_p-centroid body and L_p-projection body, see e.g., [3], [6], [9], [13], [14], [15], [16], [17], [29], [30], [31], [35], [36].

III. The Proofs of Theorems 1.1-1.6

Proof of Theorem 1.1. From (1) and Theorem 2.B, we get

$$\begin{aligned} &\omega_n^{\frac{p}{n}} V(K^*)^{\frac{n-p}{n}} G_p(K) \\ &= \inf\{nV_p(K, Q) V(K^*)^{\frac{n-p}{n}} V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\} \\ &\leq \inf\{nV_p(K, Q) V_p(K^*, Q^*) : Q \in \mathcal{K}_o^n\}. \end{aligned} \qquad (27)$$

Taking $Q = K$ in (27), it follows from (14) and (16) that

$$\begin{aligned} &\omega_n^{\frac{p}{n}} V(K^*)^{\frac{n-p}{n}} G_p(K) \\ &\leq \inf\{nV(K)V(K^*) : K \in \mathcal{K}_c^n\} \leq n\omega_n^2. \end{aligned}$$

That is

$$G_p(K) \leq n\omega_n^{\frac{2n-p}{n}} V(K^*)^{\frac{p-n}{n}}.$$

According to the equality conditions of (14) and (17), we see that equality holds in (4) if and only if K is an ellipsoid.

In order to prove Theorem 1.2, we need the following Lemmas.

Lemma 3.1.([14]) If $K \in \mathcal{S}_o^n$ and $p \geq 1$, then for any $Q \in \mathcal{K}_o^n$,

$$V_p(Q, \Gamma_p K) = \frac{\omega_n}{V(K)} \widetilde{V}_{-p}(K, \Pi_p^* Q). \qquad (28)$$

Lemma 3.2.([13]) If $K \in \mathcal{S}_o^n$, then for $p \geq 1$

$$V(K)V(\Gamma_p^* K) \leq \omega_n^2, \qquad (29)$$

with equality if and only if K is an ellipsoid centered at the origin.

Proof of Theorem 1.2. From (1), we have

$$\omega_n^{\frac{p}{n}} G_p(K) \leq nV_p(K, Q) V(Q^*)^{\frac{p}{n}}. \qquad (30)$$

For $L \in \mathcal{S}_o^n$, Taking $Q = \Gamma_p L$ in (30), it follows from (28) that

$$\begin{aligned} \omega_n^{\frac{p}{n}} G_p(K) &\leq nV_p(K, \Gamma_p L) V(\Gamma_p^* L)^{\frac{p}{n}} \\ &= \frac{n\omega_n}{V(L)} \widetilde{V}_{-p}(L, \Pi_p^* K) V(\Gamma_p^* L)^{\frac{p}{n}}. \end{aligned} \qquad (31)$$

Taking $L = \Pi_p^* K$ in (31), we obtian

$$G_p(K) \leq n\omega_n^{\frac{n-p}{n}} V(\Gamma_p^*(\Pi_p^* K))^{\frac{p}{n}}. \qquad (32)$$

Together (29) with (32), we get

$$V(\Pi_p^* K)^{\frac{p}{n}} G_p(K) \leq n\omega_n^{\frac{n+p}{n}}. \qquad (33)$$

Namely,

$$G_p(K) \leq n\omega_n^{\frac{n+p}{n}} V(\Pi_p^* K)^{-\frac{p}{n}}.$$

From the equality condition of (29), we know that equality holds in (5) if and only if K is an ellipsoid centered at the origin.

Lemma 3.3.([28]) If $K \in \mathcal{F}_o^n$ and $p \geq 1$, then

$$V(\Pi_p K) \geq V(\Lambda_p K), \qquad (34)$$

with equality if and only if K is an ellipsoid centered at the origin.

Proof of Theorem 1.3. If $1 \leq p \leq n$, then from Theorem 1.A and Lemma 3.3, we get

$$G_p(\Lambda_p K) \leq n\omega_n^{\frac{p}{n}} V(\Lambda_p K)^{\frac{n-p}{n}} \leq n\omega_n^{\frac{p}{n}} V(\Pi_p K)^{\frac{n-p}{n}}. \qquad (35)$$

By Theorem 1.1, we also have

$$G_p(\Pi_p^* K) \leq n\omega_n^{\frac{2n-p}{n}} V(\Pi_p K)^{\frac{p-n}{n}}. \qquad (36)$$

Combining (35) with (36), this yields

$$G_p(\Lambda_p K) G_p(\Pi_p^* K) \leq (n\omega_n)^2.$$

According to the equality conditions of Theorem 1.A, Theorem 1.1 and Lemma 3.3, we easily see that equality holds in (6) if and only if K is an ellipsoid centered at the origin.

Proof of Theorem 1.4. From Theorem 1.A, it follows that

$$G_p(\Gamma_p^* K) \leq n\omega_n^{\frac{p}{n}} V(\Gamma_p^* K)^{\frac{n-p}{n}}. \qquad (37)$$

By Lemma 3.2, we get that for $1 \leq p \leq n$,

$$G_p(\Gamma_p^* K) \leq n\omega_n^{\frac{2n-p}{n}} V(K)^{\frac{p-n}{n}}. \qquad (38)$$

Associated (2) with (38), this yields

$$G_p(K) G_p(\Gamma_p^* K) \leq (n\omega_n)^2.$$

According to the equality conditions of (2) and (29), we easily see that equality holds in (7) if and only if $\Gamma_p K$ is an ellipsoid where K is an ellipsoid centered at the origin.

Lemma 3.4.([14]) If $K \in \mathcal{K}_o^n$, then for $p \geq 1$,

$$V(K)^{\frac{n-p}{p}} V(\Pi_p^* K) \leq \omega_n^{\frac{n}{p}}, \qquad (39)$$

with equality if and only if K is an ellipsoid centered at the origin.

Proof of Theorem 1.5. It follows from (4) that,

$$G_p(\Pi_p K) \leq n\omega_n^{\frac{2n-p}{n}} V(\Pi_p^* K)^{\frac{p-n}{n}}. \tag{40}$$

By Lemma 3.4 and (40), we obtain that for $p \geq n$

$$G_p(\Pi_p K) \leq n\omega_n^{\frac{3np-n^2-p^2}{pn}} \left[V(K)^{\frac{p-n}{n}}\right]^{\frac{p-n}{p}}. \tag{41}$$

Theorem 1.A implies that for $p \geq n$

$$V(K)^{\frac{p-n}{n}} \leq n\omega_n^{\frac{p}{n}} G_p(K)^{-1}. \tag{42}$$

Combining (41) and (42), we get

$$G_p(K)^{p-n} G_p(\Pi_p K)^p \leq (n\omega_n)^{2p-n}.$$

Together with the equality conditions of (2), (4) and (39), we know that equality holds in (8) if and only if K is an ellipsoid centered at the origin.

Lemma 3.5.([10]) If $K \in \mathcal{F}_{os}^n$ and $p \geq 1$, then

$$V(\Lambda_p K) \leq \omega_n^{\frac{2p-n}{p}} V(K)^{\frac{n-p}{p}}, \tag{43}$$

with equality if and only if K is an ellipsoid.

Lemma 3.6.([28]) If $K \in \mathcal{F}_o^n$ and $p \geq 1$, then

$$V(\Lambda_p^* K) \leq \omega_n^{\frac{n}{p}} V(K)^{\frac{p-n}{p}}, \tag{44}$$

with equality for $p > 1$ if and only if $\Lambda_p^* K$ and K are dilates; for $p = 1$ if and only if $\Lambda_p^* K$ and K are homothetic.

Proof of Theorem 1.6. From (1), we get for any $Q \in \mathcal{K}_o^n$,

$$\omega_n^{\frac{p}{n}} G_p(\Lambda_p^* K) \leq nV_p(\Lambda_p^* K, Q) V(Q^*)^{\frac{p}{n}}. \tag{45}$$

Taking $Q = \Lambda_p^* K$ in (45), it follows that

$$\omega_n^{\frac{p}{n}} G_p(\Lambda_p^* K) \leq nV(\Lambda_p^* K) V(\Lambda_p K)^{\frac{p}{n}}. \tag{46}$$

From (43), (44) and (46), we obtain

$$G_p(\Lambda_p^* K) \leq n\omega_n^{\frac{n^2+p^2-pn}{pn}} V(K)^{-\frac{(n-p)^2}{pn}}. \tag{47}$$

Combining (47) with Theorem 1.A, this implies that for $1 \leq p \leq n$,

$$G_p(K)^{n-p} G_p(\Lambda_p^* K)^p \leq (n\omega_n)^n.$$

By the equality conditions of (2), (43) and (44), we see that equality holds in (9) for $1 < p \leq n$ if and only if $\Lambda_p^* K$ and K are dilates; for $p = 1$ if and only if $\Lambda_p^* K$ and K are homothetic.

Acknowledgment

The author is indebted to the editors and the anonymous referees for many valuable suggestions and comments.

References

[1] W. Blaschke, *Vorlesungen über Differentialgeometrie II, Affine Differentialgeometrie*, Springer-Verlag, Berlin, 1923.
[2] R. J. Gardner, *Geometric Tomography*, 2nd ed., Cambridge University Press, Cambridge, 2006.
[3] E. Grinberg and G. Y. Zhang, "Convolutions transforms and convex bodies," *Proceedings of the London Mathematical Society*, vol. 78, no. 1, pp. 77-115, Jan. 1999.
[4] Y. Y. Guo, T. Y. Ma and L. Gao, "Orlicz mixed geominimal surface area," *IAENG International Journal of Applied Mathematics*, vol. 46, no. 3, pp. 398-404, 2016.
[5] J. S. Guo and Y. B. Feng, "L_p-dual geominimal surface area and general L_p-centroid bodies," *Journal of Inequalities and Applications*, vol. 2015, no. 358, pp. 1-9, Nov. 2015.
[6] Q. Huang and B. He, "An asymmetric Orlicz centroid inequality for probability measures," *Science China Mathematics*, vol. 57, no. 6, pp. 1193-1202, Jun. 2014.
[7] G. S. Leng, "Affine surface area of curvature for convex body," *Acta Mathematica Sinica Chinese Series*, vol. 45, no. 4, pp. 792-802, Apr. 2002.
[8] K. Leichtweiß, *Affine Geometry of Convex Bodies*, Johann Ambrosius Barth, Heidelberg, 1998.
[9] A. Li and G. Leng, "A new proof of the Orlicz Busemann-Petty centroid inequality," *Proceedings of the American Mathematical Society*, vol. 139, no. 4, pp. 1473-1481, Apr. 2011.
[10] E. Lutwak, "The Brunn-Minkowski-Firey theory II: Affine and geominimal surface areas," *Advances in Mathematics*, vol. 118, no. 2, pp. 244-294, Mar. 1996.
[11] E. Lutwak, "The Brunn-Minkowski-Firey theory I: Mixed volumes and the Minkowski problem," *Journal Difierential Geometry*, vol. 38, no. 1, pp. 131-150, Jul. 1993.
[12] E. Lutwak, "On some affine isoperimetric inequalities," *Journal Difierential Geometry*, vol. 23, no. 1, pp. 1-13, Jan. 1986.
[13] E. Lutwak and G. Y. Zhang, "Blaschke-Santaló inequalities," *Journal Difierential Geometry*, vol. 47, no. 1, pp. 1-16, Jan. 1997.
[14] E. Lutwak, D. Yang and G. Y. Zhang, "L_p-affine isoperimetric inequalities," *Journal Difierential Geometry*, vol. 56, no. 1, pp. 111-132, Jan. 2000.
[15] E. Lutwak, D. Yang and G. Y. Zhang, "The Cramer-Rao inequality for star bodies," *Duke Mathematical Journal*, vol. 112, no. 1, pp. 59-81, Mar. 2002.
[16] E. Lutwak, D. Yang and G. Zhang, "Orlicz projection bodies," *Advances in Mathematics*, vol. 223, no. 4, pp. 220-242, Mar. 2010.
[17] E. Lutwak, D. Yang and G. Zhang, "Orlicz centroid bodies," *Journal Difierential Geometry*, vol. 84, no. 2, pp. 365-387, Feb. 2010.
[18] T. Y. Ma and Y. B. Feng, "Dual L_p-mixed geominimal surface area and related inequalities," *Journal of Function Spaces*, vol. 2016, no. 2, pp. 1-10, Jul. 2016.
[19] T. Y. Ma and Y. B. Feng, "Some inequalities for p-geominimal surface area and related results," *IAENG International Journal of Applied Mathematics*, vol. 46, no. 1, pp. 92-96, 2016.
[20] T. Y. Ma and W. D. Wang, "Dual Orlicz geominimal surface area," *Journal of Inequalities and Applications*, vol. 2016, no. 56, pp. 1-13, Dec. 2016.
[21] T. Y. Ma and W. D. Wang, "Some inequalities for generalized L_p-mixed affine surface areas," *IAENG International Journal of Applied Mathematics*, vol. 45, no. 4, pp. 321-326, 2015.
[22] C. M. Petty, "Geominimal surface area," *Geometry Dedicata*, vol. 3, no. 1, pp. 77-97, May 1974.
[23] C. M. Petty, "Affine isoperimetric problems," *Discrete Geometry and Convexity*, vol. 440, no. 1, pp. 113-127, May 1985.
[24] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, 2nd ed., Cambridge University Press, Cambridge, 2014.
[25] W. D. Wang and G. S. Leng, "Some affine isoperimetric inequalities associated with L_p-affine surface area," *Houston Journal of Mathematics*, vol. 34, no. 2, pp. 443-453, Apr. 2008.
[26] W. D. Wang and C. Qi, "L_p-dual geominimal surface area," *Journal of Inequalities and Applications*, vol. 2011, no. 6, pp. 1-10, Jun. 2011.
[27] W. D. Wang and G. S. Leng, "On the inequalities for L_p-curvature image of convex bodies," *Chinese Annals of Mathematics*, vol. 27A, no. 6, pp. 829-836, Nov. 2006. (in Chinese)
[28] W. D. Wang, D. J. Wei and Y. Xiang, "Some inequalities for the L_p-curvature image," *Journal of Inequalities and Applications*, vol. 2009, no. 3, pp. 1-12, Mar. 2009.
[29] W. D. Wang, F. H. Lu and G. S. Leng, "A type of monotonicity on the L_p-centroid body and L_p-projection body," *Mathematical Inequalities Applications*, vol. 8, no. 4, pp. 735-742, Oct. 2005.
[30] W. D. Wang and G. S. Leng, "On the monotonicity of L_p-centroid body," *Journal of Systems Science and Mathematical Sciences*, vol. 28, no. 2, pp. 154-163, Feb. 2008. (in Chinese)
[31] W. D. Wang and Y. P. Zhou, "Reverses of the Blaschke-Santaló inequality for convex bodies," *Chinese Quarterly Journal of Mathematics*, vol. 28, no. 4, pp. 605-611, Oct. 2013.
[32] L. Yan, W. D. Wang and L. Si, "L_p-dual mixed geominimal surface areas," *Journal of Nonlinear Science and Applications*, vol. 9, no. 3, pp. 1143-1152, May 2016.
[33] D. P. Ye, "New Orlicz affine isoperimetric inequalities," *Journal of Mathematical Analysis and Applications*, vol. 427, no. 2, pp. 905-929, Jul. 2015.
[34] D. P. Ye, B. C. Zhu and J. Z. Zhou, "The mixed L_p-geominimal surface areas for multiple convex bodies," *Indiana University Mathematics Journal*, to be published.
[35] J. Yuan, L. Z. Zhao and G. S. Leng, "Inequalities for L_p-centroid body," *Taiwanese Journal of Mathematics*, vol. 11, no. 5, pp. 1315-1325, Dec. 2007.

[36] G. Zhu, "The Orlicz centroid inequality for star bodies," *Advances in Applied Mathematics*, vol. 48, no. 2, pp. 432-445, Feb. 2012.
[37] B. C. Zhu, N. Li and J. Z. Zhou, "Isoperimetric inequalities for L_p-geominimal surface area," *Glasgow Mathematical Journal*, vol. 53, no. 3, pp. 717-726, Sep. 2011.
[38] B. C. Zhu, J. Z. Zhou and W. X. Xu, "L_p-mixed geominimal surface area," *Journal of Mathematical Analysis and Applications*, vol. 422, no. 2, pp. 1247-1263, Feb. 2015.