A Compact Difference Scheme for One-dimensional Nonlinear Delay Reaction-diffusion Equations with Variable Coefficient

Jianqiang Xie, Dingwen Deng*, and Huasheng Zheng

Abstract—**First of all, a compact difference scheme (CDS) is established for one-dimensional (1D) nonlinear reaction-diffusion equations (RDEs) with a fixed delay. By the energy method, it is proved that the difference solution converges to exact solution with a convergence order of $\mathcal{O}(\tau^2+h^4)$ in $L^\infty-$ norm. Then, a Richardson extrapolation method (REM) is applied to make the final solution fourth-order accurate in both time and space. Besides, the extensions of the solver to other complex delay problems are studied in detail. Finally, numerical results demonstrate the high accuracy and efficiency of our algorithms.**

Index Terms—**Delay reaction-diffusion equations; Compact difference scheme; Convergence;**

I. Introduction

Delay partial differential equations have been widely applied in economics, physics, ecology, medicine, engineering control, climate model, computer-aided design and many other fields of science (cf. [1]–[6]). However, it is impossible to get their analytical solutions. Hence, it is very significant to develop high-performance numerical algorithms for this kind of equations.

In this paper, we use the CDS (cf. [2]–[4], [7]–[12]) to solve the initial boundary value problems (IBVPs) as follows:

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right)=f(u(x,t),u(x,t-s),x,t),\quad (x,t)\in\Omega\times(0,T],\tag{1a}$$

$$u(x,t)=\varphi(x,t),\qquad (x,t)\in\Omega\times[-s,0],\tag{1b}$$

$$u(a,t)=\alpha(t),\qquad u(b,t)=\beta(t),\qquad t\in(0,T],\tag{1c}$$

where $\Omega:=[a,b]$ and $s>0$ is a constant fixed delay. Sufficiently smooth function $a(x)$ fulfills the boundedness, i.e. $\exists\ c_{11},\ c_{22}>0$, such that, $c_{11}\le a(x)\le c_{22}$. Also, we suppose that function $f(u(x,t),u(x,t-s),x,t)$ is sufficiently smooth, and satisfies

$$|f(\mu+\varepsilon_1,\upsilon+\varepsilon_2,x,t)-f(\mu,\upsilon,x,t)|\le c_1|\varepsilon_1|+c_2|\varepsilon_2|,\tag{2}$$

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in which $\varepsilon_1,\varepsilon_2$ are arbitrary real numbers, c_1 and c_2 are positive constants.

Many numerical methods including finite difference methods (cf. [2], [3]), local discontinuous Galerkin methods (cf. [5]), have been successfully developed for solving IBVPs (1a)–(1c) with $a(x)=1$. However, as we know, little attention has been paid on numerical solutions of IBVPs (1a)–(1c). This study aims at making up for this work.

II. Construction of compact difference scheme

This section concentrates on the derivation of the CDS for problems (1a)–(1c).

A. Partition and Notations

Let $h=(b-a)/M$ $(M\in\mathbb{Z}^+)$ be spatial mesh size. For temporal discretization, constrained temporal grid, namely, $\tau=s/n$ $(n\in\mathbb{Z}^+)$, is used. Setting $x_i=a+ih$, $t_k=k\tau$, the domain $\Omega\times(0,T]$ is covered by $\Omega_h\times\Omega_\tau$, where $\Omega_h=\{x_i|0\le i\le M\}$ and $\Omega_\tau=\{t_k|-n\le k\le N\}$, $N=[T/\tau]$. Let $\{v_i^k|0\le i\le M,-n\le k\le N\}$ be a grid function on $\Omega_{h\tau}$. Then we introduce the following notations: $\delta_t v_i^{k+1/2}=(v_i^{k+1}-v_i^k)/\tau$, $\delta_t^+ v_i^k=(3\delta_t v_i^{k+1/2}-\delta_t v_i^{k-1/2})/2$, $\delta_x v_{i+1/2}^k=(v_{i+1}^k-v_i^k)/h$, $\delta_{0x}v_i^k=(v_{i+1}^k-v_{i-1}^k)/2h$, $\delta_x(a\delta_x v)_i^k=(a_{i+1/2}\delta_x v_{i+1/2}^k-a_{i-1/2}\delta_x v_{i-1/2}^k)/h$.

B. Construction of the compact difference scheme

Let

$$v=\frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right).\tag{3}$$

Define grid functions $U_i^k=u(x_i,t_k)$, $V_i^k=v(x_i,t_k)$, $0\le i\le M$, $-n\le k\le N$. The application of second order backward differentiation formula (BDF2) to approximate (1a) at the point (x_i,t_{k+1}) gives that

$$\delta_t^+U_i^k-V_i^{k+1}=f(U_i^{k+1},U_i^{k+1-n},x_i,t_{k+1})+\tau^2 r_i^k,\tag{4}$$

where

$$r_i^k=-\frac{1}{3}\frac{\partial^3 u}{\partial t^3}(x_i,t_{k+1})+\frac{\tau}{4}\frac{\partial^4 u}{\partial t^4}(x_i,t_{k+1})-\frac{7\tau^2}{60}\frac{\partial^5 u}{\partial t^5}(x_i,\xi_{k+1}),\qquad \xi_{k+1}\in(t_k,t_{k+1}).$$

Using compact finite difference scheme to approximate (3) at the point (x_i,t_{k+1}) yields that

$$\mathcal{A}_h V_i^{k+1}=\delta_x(\hat{a}\delta_x U)_i^{k+1}+\mathcal{O}(h^4),\tag{5}$$

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where the compact operator $\mathcal{A}_h$ (see [11]) is defined as

$$\mathcal{A}_h u_i^{k+1} = \begin{cases} u_i^{k+1} + \dfrac{h^2}{12}(\delta_x^2 u_i^{k+1} - \delta_{0x}(\dfrac{a'}{a}u)_i^{k+1}), \\ \qquad\qquad\qquad 1 \le i \le M-1, \\ u_i^{k+1}, \qquad\qquad i = 0 \ or \ M, \end{cases}$$

where $\tilde{a} = (a')^2/a - a''/2$, $\hat{a} = a - (h^2\tilde{a})/12$, a' and a'' denote the first and second order spatial derivatives of $a(x)$, respectively.

Multiplying $\mathcal{A}_h$ to the both sides of (4), then inserting (5) into the resulting formula, we get

$$\mathcal{A}_h \delta_t^+ U_i^k - \delta_x(\hat{a}\delta_x U)_i^{k+1} = \mathcal{A}_h f(U_i^{k+1}, U_i^{k+1-n}, x_i, t_{k+1}) + R_i^k, \quad 1 \le i \le M-1, \quad 0 \le k \le N-1, \tag{6}$$

where $R_i^k = \mathcal{O}(\tau^2 + h^4)$, by which we can suppose that $\exists$ $c > 0$, such that $|R_i^k| \le c(\tau^2 + h^4)$.

Omitting the small term R_i^k and then replacing U_i^k with its approximations u_i^k in (6), a CDS is devised as follows

$$\mathcal{A}_h \delta_t^+ u_i^k - \delta_x(\hat{a}\delta_x u)_i^{k+1} = \mathcal{A}_h f(u_i^{k+1}, u_i^{k+1-n}, x_i, t_{k+1}), \quad 1 \le i \le M-1, \quad 0 \le k \le N-1, \tag{7}$$

$$u_0^k = \alpha(t_k), \qquad u_M^k = \beta(t_k), \quad 1 \le k \le N, \tag{8}$$

$$u_i^k = \varphi(x_i, t_k), \quad 0 \le i \le M, \quad -n \le k \le 0. \tag{9}$$

At each time level, the CDS (7)–(9) is a linear tridiagonal system with strictly diagonally dominant coefficient matrix, thus it has an unique solution.

III. Analysis of the compact difference scheme

In this section, we study the convergence of CDS (7)–(9). Firstly, we further assume that $\exists$ positive constants c_3, c_4 such that $|a'/a| \le c_4$, $|(a'/a)'| \le c_4$, $c_3 \le \hat{a} \le c_4$.

Let $V = \{v | v = (v_0, v_1, \cdots, v_M), v_0 = v_M = 0\}$ be a grid function space on Ω_h. $\forall$ $v,w \in V$, we define the inner products and discrete norms as follows $(v, w) = h\sum_{i=1}^{M-1} v_i w_i$, $\|v\|^2 = (v, v)$, $\|v\|_\infty = \max_{0\le i\le M} |v_i|$, $|v|_1^2 = h\sum_{i=0}^{M-1}(\delta_x v_{i+\frac{1}{2}})^2$, $|v|_{1*}^2 = h\sum_{i=0}^{M-1} \hat{a}_{i+\frac{1}{2}}(\delta_x v_{i+\frac{1}{2}})^2$.

By the definition of $|v|_{1*}$, it is easy to get the lemma 1.

Lemma 1 $\forall$ $v \in V$, we have $\sqrt{c_3}|v|_1 \le |v|_{1*} \le \sqrt{c_4}|v|_1$.

Lemma 2 (cf. [2]) $\forall$ $v \in V$, we obtain

$$\|v\|_\infty \le (\sqrt{b-a}|v|_1)/2, \ \ \|v\| \le [(b-a)|v|_1]/\sqrt{6}.$$

Lemma 3 $\forall$ $v \in V$, it holds that

$$(\frac{2}{3} - \frac{c_4 h^2}{24})\|v\|^2 \le (\mathcal{A}_h v, v) \le (1 + \frac{c_4 h^2}{24})\|v\|^2.$$

Proof. Noting $\mathcal{A}_h v_i = v_i + \frac{h^2}{12}(\delta_x^2 v_i - \delta_{0x}(\frac{a'}{a}v)_i)$, and utilizing the discrete Green formula yield that

$$(\mathcal{A}_h v, v) = \|v\|^2 - \frac{h^2}{12}\|\delta_x v\|^2 - \frac{h^2}{12}h\sum_{i=1}^{M-1}\frac{(\frac{a'}{a})_{i+1} - (\frac{a'}{a})_i}{2h}v_{i+1}v_i. \tag{10}$$

The use of $\|\delta_x v\|^2 \le 4h^{-2}\|v\|^2$ to (10) deduces the claimed result.

Theorem 1 Let $u(x,t) \in C_{x,t}^{6,3}(\Omega \times (0,T])$ be the exact solution of (1a)–(1c), u^k be the solution of the scheme (7)–(9) at the time level k, respectively. Denote $e_i^k = u(x_i, t_k) - u_i^k$, $0 \le i \le M$, $-n \le k \le N$. Then, as h and τ are sufficiently small, we conclude that

$$\|e^k\|_\infty \le C(\tau^2 + h^4), \quad 0 \le k \le N, \tag{11}$$

where

$$C = \frac{(b-a)}{2}c\sqrt{\frac{3T}{c_3}}\exp\left(\frac{13T(b-a)^2}{12c_3}\max(c_1^2, c_2^2)\right).$$

Proof. Denote $G_i^{k+1} = f(U_i^{k+1}, U_i^{k+1-n}, x_i, t_{k+1}) - f(u_i^{k+1}, u_i^{k+1-n}, x_i, t_{k+1})$. Subtracting (7) from (6), the error equations are derived as follows:

$$\mathcal{A}_h \delta_t^+ e_i^k - \delta_x(\hat{a}\delta_x e)_i^{k+1} = \mathcal{A}_h G_i^{k+1} + R_i^k, \quad 1 \le i \le M-1, \quad 0 \le k \le N-1, \tag{12}$$

$$e_0^k = 0, \qquad e_M^k = 0, \quad 1 \le k \le N, \tag{13}$$

$$e_i^k = 0, \qquad 0 \le i \le M, \quad -n \le k \le 0. \tag{14}$$

In the following, mathematical induction is applied to prove this theorem. From (14), it follows that $\|e^k\|_\infty = 0$, $-n \le k \le 0$. Assuming that (11) is valid for $0 \le k \le l$, we will prove that it is also true for $k = l+1$.

Using inequality $-ab \ge -(a^2 + b^2)/2$ gives that

$$\begin{aligned} & h\sum_{i=1}^{M-1}(\mathcal{A}_h \delta_t^+ e_i^k)\delta_t e_i^{k+\frac{1}{2}} \\ \ge & h\sum_{i=1}^{M-1}(\mathcal{A}_h \delta_t e_i^{k+\frac{1}{2}})\delta_t e_i^{k+\frac{1}{2}} + \frac{1}{4}\{h\sum_{i=1}^{M-1}(\mathcal{A}_h \delta_t e_i^{k+\frac{1}{2}})\delta_t e_i^{k+\frac{1}{2}} \\ & - h\sum_{i=1}^{M-1}(\mathcal{A}_h \delta_t e_i^{k-\frac{1}{2}})\delta_t e_i^{k-\frac{1}{2}}\}. \end{aligned} \tag{15}$$

From the discrete Green formula and inequality $a(a-b) \ge (a^2 - b^2)/2$, it follows that

$$-h\sum_{i=1}^{M-1}(\delta_x(\hat{a}\delta_x e)_i^{k+1})\delta_t e_i^{k+\frac{1}{2}} \ge \frac{(|e^{k+1}|_{1*}^2 - |e^k|_{1*}^2)}{2\tau}. \tag{16}$$

Next, we estimate

$$\begin{aligned} & h\sum_{i=1}^{M-1}(\mathcal{A}_h G_i^{k+1})\delta_t e_i^{k+\frac{1}{2}} \\ & \le h\sum_{i=1}^{M-1}\big(\mathcal{A}_h(c_1|e_i^{k+1}| + c_2|e_i^{k+1-n}|)\big)|\delta_t e_i^{k+\frac{1}{2}}| \\ & := A_1 + A_2 + A_3 + A_4 + A_5. \end{aligned} \tag{17}$$

Applying $ab \le a^2/(2\varepsilon) + (\varepsilon b^2)/2$, we arrive at

$$\begin{aligned} A_1 &= \frac{h}{12}\sum_{i=1}^{M-1}(c_1|e_{i+1}^{k+1}| + c_2|e_{i+1}^{k+1-n}|)|\delta_t e_i^{k+\frac{1}{2}}| \\ &\le \frac{(c_1^2\|e^{k+1}\|^2 + c_2^2\|e^{k+1-n}\|^2)}{12\varepsilon} + \frac{\varepsilon}{24}\|\delta_t e^{k+\frac{1}{2}}\|^2, \end{aligned} \tag{18}$$

$$\begin{aligned} A_2 &= \frac{5h}{6}\sum_{i=1}^{M-1}(c_1|e_i^{k+1}| + c_2|e_i^{k+1-n}|)|\delta_t e_i^{k+\frac{1}{2}}| \\ &\le \frac{5(c_1^2\|e^{k+1}\|^2 + c_2^2\|e^{k+1-n}\|^2)}{6\varepsilon} + \frac{5\varepsilon}{12}\|\delta_t e^{k+\frac{1}{2}}\|^2, \end{aligned} \tag{19}$$

$$A_3 = \frac{h}{12}\sum_{i=1}^{M-1}(c_1|e_{i-1}^{k+1}| + c_2|e_{i-1}^{k+1-n}|)|\delta_t e_i^{k+\frac{1}{2}}|$$
$$\le \frac{(c_1^2\|e^{k+1}\|^2 + c_2^2\|e^{k+1-n}\|^2)}{12\varepsilon} + \frac{\varepsilon}{24}\|\delta_t e^{k+\frac{1}{2}}\|^2, \quad (20)$$

$$A_4 = \frac{h}{24}\sum_{i=1}^{M-1}(c_1|e_{i-1}^{k+1}| + c_2|e_{i-1}^{k+1-n}|)h(\frac{a'}{a})_{i-1}|\delta_t e_i^{k+\frac{1}{2}}|$$
$$\le \frac{(c_1^2\|e^{k+1}\|^2 + c_2^2\|e^{k+1-n}\|^2)}{24\varepsilon} + \frac{c_4^2h^2\varepsilon}{48}\|\delta_t e^{k+\frac{1}{2}}\|^2, \quad (21)$$

$$A_5 = -\frac{h}{24}\sum_{i=1}^{M-1}(c_1|e_{i+1}^{k+1}| + c_2|e_{i+1}^{k+1-n}|)h(\frac{a'}{a})_{i+1}|\delta_t e_i^{k+\frac{1}{2}}|$$
$$\le \frac{(c_1^2\|e^{k+1}\|^2 + c_2^2\|e^{k+1-n}\|^2)}{24\varepsilon} + \frac{c_4^2h^2\varepsilon}{48}\|\delta_t e^{k+\frac{1}{2}}\|^2. \quad (22)$$

The substitution of (18)–(22) into (17) yields that

$$h\sum_{i=1}^{M-1}(\mathcal{A}_h G_i^{k+1})\delta_t e_i^{k+\frac{1}{2}} \le \frac{13(c_1^2\|e^{k+1}\|^2 + c_2^2\|e^{k+1-n}\|^2)}{12\varepsilon}$$
$$+\frac{\varepsilon}{2}\|\delta_t e^{k+\frac{1}{2}}\|^2 + \frac{c_4^2h^2\varepsilon}{24}\|\delta_t e^{k+\frac{1}{2}}\|^2. \quad (23)$$

Besides, using ε-inequality obtains that

$$h\sum_{i=1}^{M-1}R_i^k\delta_t e_i^{k+\frac{1}{2}} \le \frac{1}{2\varepsilon}\|R^k\|^2 + \frac{\varepsilon}{2}\|\delta_t e^{k+\frac{1}{2}}\|^2. \quad (24)$$

Multiplying (12) by $h\delta_t e_i^{k+\frac{1}{2}}$, summing i from 1 to $M-1$, and then substituting (15), (16), (23), (24) into the resulting equation, we have

$$\frac{(|e^{k+1}|_{1*}^2 - |e^k|_{1*}^2)}{2\tau} + h\sum_{i=1}^{M-1}(\mathcal{A}_h\delta_t e_i^{k+\frac{1}{2}})\delta_t e_i^{k+\frac{1}{2}} +$$
$$\frac{1}{4}\{h\sum_{i=1}^{M-1}(\mathcal{A}_h\delta_t e_i^{k+\frac{1}{2}})\delta_t e_i^{k+\frac{1}{2}} - h\sum_{i=1}^{M-1}(\mathcal{A}_h\delta_t e_i^{k-\frac{1}{2}})\delta_t e_i^{k-\frac{1}{2}}\}$$
$$\le \frac{13(c_1^2\|e^{k+1}\|^2 + c_2^2\|e^{k+1-n}\|^2)}{12\varepsilon} + \frac{\varepsilon}{24}c_4^2h^2\|\delta_t e^{k+\frac{1}{2}}\|^2$$
$$+\frac{1}{2\varepsilon}\|R^k\|^2 + \varepsilon\|\delta_t e^{k+\frac{1}{2}}\|^2. \quad (25)$$

Multiplying 2τ to the both sides of (25), summing k from 0 to l, using Lemma 2 and Lemma 3, and then letting $\varepsilon = [2/3 - (c_4h^2)/24]/[1 + (c_4^2h^2)/24]$, we arrive at

$$|e^{l+1}|_{1*}^2 \le 3(b-a)c^2T(\tau^2+h^4)^2$$
$$+\frac{13\tau}{6}(b-a)^2\max(c_1^2,c_2^2)\sum_{k=1}^{l}|e^k|_1^2, \quad (26)$$

in which h is taken to make $\varepsilon \ge 1/3$ because h is sufficiently small. The applications of lemma 1 and the discrete Gronwall inequality to (26) yield

$$|e^{l+1}|_1^2 \le \frac{3}{c_3}(b-a)c^2T(\tau^2+h^4)^2$$
$$\exp\left(\frac{13T(b-a)^2}{6c_3}\max(c_1^2,c_2^2)\right). \quad (27)$$

Applying Lemma 2 infers that

$$\|e^{l+1}\|_\infty \le \frac{\sqrt{b-a}}{2}|e^{l+1}|_1 \le C(\tau^2+h^4). \quad (28)$$

By inductive principle, (11) is valid.

IV. Extension to reaction-diffusion equations with several delays

This section focuses on the extensions of the CDS (7)–(9) and corresponding analytical results to the equations with several delays as follows:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right) = f(u(x,t), u(x,t-s_1), u(x, \\ t-s_2), \ldots, u(x,t-s_K), x, t), (x,t) \in \Omega\times(0,T], \\ u(a,t) = \alpha(t), \quad u(b,t) = \beta(t), \quad t\in(0,T], \\ u(x,t) = \varphi(x,t), \quad (x,t)\in\Omega\times[-\tilde{s},0], \end{cases} \quad (29)$$

where $\tilde{s}_\kappa > 0$ $(\kappa = 1,2,\ldots,K)$, and $\tilde{s} = \max_{1\le\kappa\le K} s_\kappa$, $0 \le c_{11} \le a(x) \le c_{22}$. To preserve high-order accuracy of the algorithms, we should utilize the constrained time integrator: $\tau = s_1/n_1 = s_2/n_2 = \cdots = s_K/n_K$ $(\kappa = 1,2,\ldots,K)$, to make $t = Ls_\kappa$ $(L\in\mathbb{Z}^+)$ locate the temporal grid nodes.

In this case, CDS (7)–(9) for (1a)–(1c) can be easily adapted to the numerical solution of IBVP (29) by replacing $f(U_i^{k+1}, U_i^{k+1-n}, x_i, t_{k+1})$ with $f(U_i^{k+1}, U_i^{k+1-n_1}, U_i^{k+1-n_2}, \ldots, U_i^{k+1-n_K}, x_i, t_{k+1})$ in (7). Moreover, corresponding theoretical results are derived by using the analytical methods similar to that of the solver CDS (7)–(9) as long as $f(u(x,t), u(x,t-s_1), u(x,t-s_2), \ldots, u(x,t-s_K), x, t)$ satisfies Lipschitz condition with respect to its 1st, 2nd, $\ldots$, $K+1-$th variables.

V. Extensions to general nonlinear delay parabolic equations

This section aims to the numerical approximation of the following IBVPs:

$$\begin{cases} \tilde{r}(x,t)\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \tilde{b}(x,t)\frac{\partial u}{\partial x} + \tilde{c}(x,t)u = \tilde{f}(u \\ (x,t), u(x,t-s), x, t), \quad (x,t)\in[a,b]\times(0,T], \\ u(a,t) = \alpha(t), \quad u(b,t) = \beta(t), \quad t\in(0,T], \\ u(x,t) = \varphi(x,t), \quad (x,t)\in[a,b]\times[-s,0]. \end{cases} \quad (30)$$

To begin with, using the skills proposed in [3], multiplying the first equation of (30) by $\exp\big(\int_0^x \tilde{b}(s,t)ds\big)$, then IBVPs (30) is equivalently transformed into

$$\begin{cases} r(x,t)\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) + c(x,t)u = f(u \\ (x,t), u(x,t-s), x, t), \quad (x,t)\in[a,b]\times(0,T], \\ u(a,t) = \alpha(t), \quad u(b,t) = \beta(t), \quad t\in(0,T], \\ u(x,t) = \varphi(x,t), \quad (x,t)\in[a,b]\times[-s,0], \end{cases} \quad (31)$$

where $r(x,t) = \tilde{r}(x,t)\exp\big(\int_0^x \tilde{b}(s,t)ds\big)$, $a(x,t) = \exp\big(\int_0^x \tilde{b}(s,t)ds\big)$, $c(x,t) = \tilde{c}(x,t)\exp\big(\int_0^x \tilde{b}(s,t)ds\big)$, $f(u(x,t), u(x,t-s), x, t) = \exp\big(\int_0^x \tilde{b}(s,t)ds\big)\tilde{f}(u(x,t), u(x,t-s), x, t)$.

Analogous to (1a)–(1c), the CDS for IBVP (31) is designed as follows

$$\begin{cases} \mathcal{A}_h(r_i^{k+1}\delta_t^+ u_i^k) - \delta_x(\hat{a}\delta_x u)_i^{k+1} + c_i^{k+1}u_i^{k+1} = \mathcal{A}_h f(u_i^{k+1}, \\ u_i^{k+1-n}, x_i, t_{k+1}), \quad 1\le i\le M-1,\ 0\le k\le N-1, \\ u_0^k = \alpha(t_k), \quad u_M^k = \beta(t_k), \quad 1\le k\le N, \\ u_i^k = \varphi(x_i,t_k), \quad 0\le i\le M, \quad -n\le k\le 0, \end{cases} \quad (32)$$

which is also adapted to solve IBVPs (31) with several delays using techniques developed in section (IV). It is easy to obtain the corresponding theoretical results similar to Theorem 1, too.

Remark Also, our CDS can be slightly modified to solve the following nonlinear parabolic equations with proportional delay [13]–[15]:

$$r(x,t)\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right) = f(u(x,t),u(x,pt),x,t), \tag{33}$$

where $(x,t) \in [a,b]\times[0,T]$, $0<p<1$. Let $w(x,t) = u(x,e^t)$ for $t \geq t_0 + ln(p)$, where $t_0 \geq 0$. Then $w(x,t)$ satisfies the problem:

$$\begin{cases} \hat{r}(x,t)\frac{\partial w}{\partial t} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial w}{\partial x}\right) = f_1(w(x,t), \\ w(x,t-s),x,t), \quad x\in[a,b], \quad t\in[t_0,T], \\ w(x,t) = u(x,e^t), \quad t\in[-s,t_0], \end{cases} \tag{34}$$

where $s=-ln(p)$, $\hat{r}(x,t) = e^{-t}r(x,e^t)$, $f_1(w(x,t),w(x,t-s),x,t) = f(w(x,t),w(x,t-s),x,e^t)$.

Therefore, w_i^n is firstly obtained by using CDS (32). Then, u_i^n is provided by applying transformation $u(x,e^t) = w(x,t)$.

VI. Numerical Examples

In this section, three numerical examples are solved to illustrate the performance of the algorithms. L^∞-norm errors, which are defined by $er_\infty(h,\tau) = \|e^k\|_\infty$, $\hat{er}_\infty(h,\tau) = \|\hat{e}^k\|_\infty$, respectively, and CPU time are applied to measure the accuracy and efficiency of the algorithms. Here, $\hat{e}_i^k = u(x_i,t_k) - \hat{u}_i^k$, and $\hat{u}_i^k$ is computed by the following REM

$$\begin{cases} \hat{u}_i^k = \frac{1}{21}u_i^k(h,\tau) - \frac{4}{7}u_i^{2k}(h,\frac{\tau}{2}) + \frac{32}{21}u_i^{4k}(h,\frac{\tau}{4}), \\ \hat{u}_0^k = \alpha(t_k), \quad \hat{u}_M^k = \beta(t_k). \end{cases} \tag{35}$$

Convergence rates in L^∞-norm are defined as follows: $r_1 = \log_2\frac{er_\infty(2h,4\tau)}{er_\infty(h,\tau)}$, $r_2 = \log_2\frac{er_\infty(2h,2\tau)}{er_\infty(h,\tau)}$, $r_3 = \log_2\frac{\hat{er}_\infty(2h,2\tau)}{\hat{er}_\infty(h,\tau)}$, respectively. All computer programs are carried out by Matlab 7.0.

Example 1 Consider the following equation with two constant delays:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right) + \frac{u(x,t-s_1)}{1+u^2(x,t-s_2)} + h(x,t),$$

where $(x,t)\in(0,1)\times(0,1]$, $a(x) = x^2+2$, $h(x,t) = x\cos(t) - 2x\sin(t) - [x\sin(t-s_1)]/[1+x^2\sin^2(t-s_2)]$. Initial-boundary conditions are determined by its exact solution $u(x,t) = x\sin(t)$. The proposed solvers are applied to solve example 1 with $s_1=s_2=0.2$ or $s_1=0.1$, $s_2=0.2$.

Numerical results testify the following conclusions.

(1) Table I confirms that CDS (7)–(9) is second-order accurate in time and fourth-order accurate in space. Table II shows that the combination of CDS (7)–(9) with REM (35) can obtain the approximation solution of order 4 in both time and space.

(2) Approximation solution U and errors $|\hat{e}|$ are plotted in Figure 1, from which we find that numerical oscillation does not appear, and CDS (7)–(9) combined with REM (35) has a good resolution.

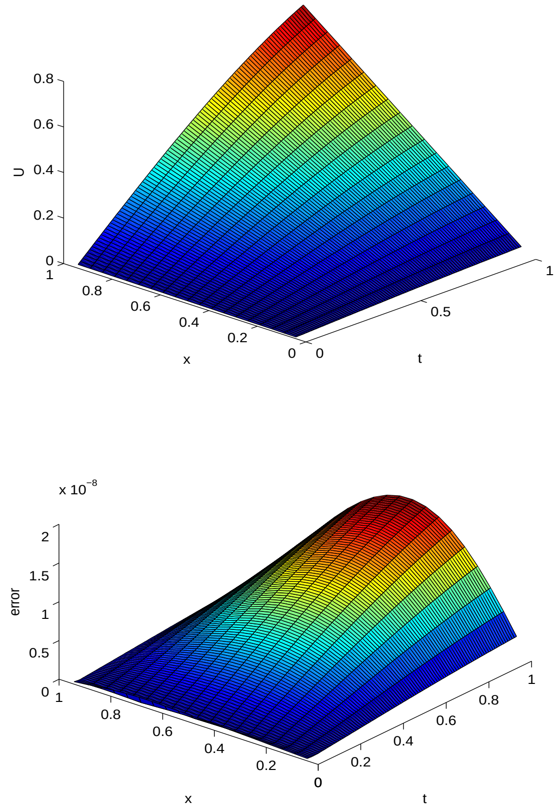

Fig. 1. Example 1 with $s_1=0.1, s_2=0.2$ (solved by the CDS (7)–(9) combined with REM (35) with $h=\tau=1/20$):approximation solution U (top) and error $|\hat{e}|$ (bottom), respectively.

Example 2 Let $\Omega=[0,1]$. Then, on $\Omega\times[-0.2,1]$, in order to further exhibit the validity of our proposed solver, we consider the following IBVPs:

$$e^{-x}\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} = t^2 f(u(x,t),u(x,t-0.2),x,t),$$

where

$$f = -u^2(x,t) + u^2(x,t-0.2) + e^{-x}t^{-2}\cos(\pi x) + \pi t^{-1} \sin\pi x + \pi^2 t^{-1}\cos(\pi x) + t^2\cos^2(\pi x) - (t-0.2)^2\cos^2(\pi x).$$

Initial and boundary conditions are determined by its exact solution $u(x,t) = t\cos(\pi x)$.

In view of section V, this problem can be solved using CDS (32). Table III displays the errors and convergence orders for different step-sizes. Figure 2 depicts the error surface for Example 2. From Table III and Figure 2, it is observed that numerical results are in accordance with theoretical results.

Example 3 Finally, the combination of CDS (32) with transformation $w(x,t) = u(x,e^t)$ is suggested for solving the general parabolic equation with proportional delay as follows:

$$t\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right) = u(x,pt) + h(x,t),$$

where $p=3/4$, $a(x) = e^x$, $h(x,t) = x^2te^t - 2e^{x+t}(x+1) - x^2e^{pt}$. Its initial and boundary values can be determined by the exact solution $u(x,t) = x^2e^t$.

TABLE I
NUMERICAL RESULTS OBTAINED USING CDS (7)–(9) AT $T = 1$ ($\tau = h^2$).

(h, τ)	$s_1 = s_2 = 0.2$			$s_1 = 0.1, s_2 = 0.2$		
	er_∞	r_1	CPU	er_∞	r_1	CPU
$(1/10, 1/100)$	$2.779e-7$	$*$	0.16	$2.785e-7$	$*$	0.16
$(1/20, 1/400)$	$1.736e-8$	4.001	1.04	$1.740e-8$	4.000	0.98
$(1/40, 1/1600)$	$1.087e-9$	3.997	13.63	$1.090e-9$	3.997	13.8
$(1/80, 1/6400)$	$6.794e-11$	4.001	321.1	$6.809e-11$	4.001	317.7

TABLE II
NUMERICAL RESULTS PROVIDED USING CDS (7)–(9) AND CDS (7)–(9) COMBINED WITH REM (35) AT $T = 1$ ($\tau = h$).

h	$s_1 = s_2 = 0.2$						$s_1 = 0.1, s_2 = 0.2$					
	er_∞	r_2	CPU	$\hat{er}_\infty$	r_3	CPU	er_∞	r_2	CPU	$\hat{er}_\infty$	r_3	CPU
$\frac{1}{20}$	$8.771e-7$	$*$	0.04	$1.609e-8$	$*$	0.24	$8.919e-7$	$*$	0.06	$1.614e-8$	$*$	0.28
$\frac{1}{40}$	$2.183e-7$	2.006	0.15	$1.007e-9$	3.999	1.16	$2.221e-7$	2.005	0.19	$1.010e-9$	3.999	1.03
$\frac{1}{80}$	$5.452e-8$	2.002	0.61	$6.293e-11$	4.000	4.29	$5.549e-8$	2.001	0.72	$6.311e-11$	4.000	4.12
$\frac{1}{160}$	$1.363e-8$	2.000	2.34	$3.962e-12$	3.989	18.3	$1.387e-8$	2.000	2.65	$3.974e-12$	3.989	18.3

TABLE III
NUMERICAL RESULTS FOR EXAMPLE 2 WITH DIFFERENT STEP-SIZES ($\tau = 10h^2$).

h	τ	er_∞	r_1
$1/20$	$1/40$	$2.500e-6$	$*$
$1/40$	$1/160$	$1.561e-7$	4.001
$1/80$	$1/640$	$9.764e-9$	3.999
$1/160$	$1/2560$	$6.102e-10$	4.000
$1/320$	$1/10240$	$3.803e-11$	4.004

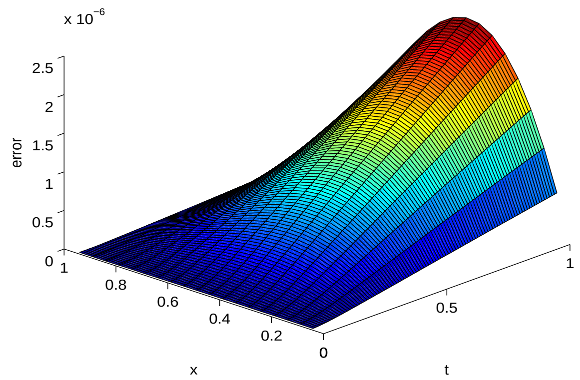

Fig. 2. Errors for Example 2 when $h = \tau = 1/20$.

TABLE IV
NUMERICAL RESULTS FOR EXAMPLE 3 WITH DIFFERENT STEP-SIZES.

h	τ	er_∞	r_2	$\hat{er}_\infty$	r_3
$1/20$	$1/20$	$6.998e-4$	$*$	$4.574e-7$	$*$
$1/40$	$1/40$	$1.860e-4$	1.912	$2.926e-8$	3.966
$1/80$	$1/80$	$4.711e-5$	1.981	$1.832e-9$	3.997
$1/160$	$1/160$	$1.186e-5$	1.990	$1.143e-10$	4.002

From Table IV and V, we can observe that the CDS (32) combined with transformation $u(x, e^t) = w(x, t)$ has a convergence rate of $\mathcal{O}(\tau^2 + h^4)$ in L^∞, and the combination of CDS (32), REM (35) and transformation $u(x, e^t) = w(x, t)$ can provided the numerical solution with an order of $\mathcal{O}(\tau^4 + h^4)$ in L^∞.

TABLE V
NUMERICAL RESULTS FOR EXAMPLE 3 ($\tau = h^2$).

h	τ	er_∞	r_1
$1/10$	$1/100$	$3.658e-5$	$*$
$1/20$	$1/400$	$2.316e-6$	3.981
$1/40$	$1/1600$	$1.455e-7$	3.993
$1/80$	$1/6400$	$9.101e-9$	3.999

VII. CONCLUSIONS

In this paper, a mixed numerical solver which combines CDS (7)–(9) with REM (35), has been constructed for IBVPs (1a)–(1c). Also, this method can be generalized to solve several complex delay problems, such as, IBVPs (29), IBVPs (30), and IBVPs (33). Numerical results testify the exactness of theoretical results and the practicability of the algorithms.

In future, we will develop high-order and efficient computational methods for high-dimensional IBVPs with viscous complex delays.

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