Nonzero Periodic Solutions in Shifts Delta(+/-) for a Higher-Dimensional Nabla Dynamic Equation on Time Scales

Lili Wang, Meng Hu

Abstract—**This paper is concerned with a higher-dimensional neutral nabla dynamic equation on time scales. Based on the theory of calculus on time scales, we first study some properties of the nabla exponential function $\hat{e}_A(t, t_0)$ and shift operators $\delta_\pm$; then by using Krasnoselskii's fixed point theorem and contraction mapping principle as well as the obtained results, sufficient conditions are established for the existence of nonzero periodic solutions in shifts $\delta_\pm$ of the equation as the following form:**

$$x^\nabla(t) = A(t)x(t) + f^\nabla(t, x(t)) + b(t)g(t, x(\tau(t))), t \in \mathbb{T},$$

where $A(t) = (a_{ij}(t))_{n\times n}$ is a nonsingular matrix with continuous real-valued functions as its elements. Finally, numerical examples are presented to illustrate the applicability of the theoretical results.

Index Terms—**periodic solution; neutral nabla dynamic equation; shift operator; time scale.**

I. Introduction

In recent decades, the theory of neutral functional differential equations has been prominent attention due to its tremendous potential of its application in applied mathematics. There are many papers that handle neutral differential equations on regular time scales, such as discrete and continuous cases, but few that deal with general time scales.

A time scale is a nonempty arbitrary closed subset of reals. Stefan Hilger [1] introduced the notion of time scale in 1988 in order to unify the theory of continuous and discrete calculus. The time scales approach not only unifies differential and difference equations, but also solves some other problems such as a mix of stop-start and continuous behaviors [2,3] powerfully. Nowadays the theory on time scales has been widely applied to several scientific fields such as biology, heat transfer, stock market, wound healing and epidemic models.

The existence problem of periodic solutions is an important topic in qualitative analysis of functional dynamic equations. Up to now, there are a few results concerning periodic dynamic equations on time scales; see, for example, [4,5]. In these papers, authors considered the existence of periodic solutions for dynamic equations on time scales satisfying the condition "there exists a $\omega > 0$ such that $t \pm \omega \in \mathbb{T}, \forall t \in \mathbb{T}$." Under this condition all periodic time scales are unbounded above and below. However, there are many time scales such as $\overline{q^{\mathbb{Z}}} = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ and $\sqrt{\mathbb{N}} = \{\sqrt{n} : n \in \mathbb{N}\}$ which do not satisfy the condition. To overcome such difficulties, Adıvar introduced a new periodicity concept on time scales which does not oblige the time scale to be closed under the operation $t \pm \omega$ for a fixed $\omega > 0$. He defined a new periodicity concept with the aid of shift operators $\delta_\pm$ which are first defined in [6] and then generalized in [7].

In recent years, periodic solutions in shifts $\delta_\pm$ for some nonlinear dynamic equations on time scales with delta derivative have been studied by many authors; see, for example, [8-11]. However, to the best of our knowledge, there are few papers published on the existence of periodic solutions in shifts $\delta_\pm$ for a dynamic equation on time scales with nabla derivative, especially for some higher-dimensional nabla dynamic equations on time scales.

Motivated by the above, in the present paper, we consider the following neutral nabla dynamic equation:

$$x^\nabla(t) = A(t)x(t) + f^\nabla(t, x(t)) + b(t)g(t, x(\tau(t))), \quad (1)$$

where $t \in \mathbb{T}$, $\mathbb{T} \subset \mathbb{R}$ be a periodic time scale in shifts $\delta_\pm$ with period $P \in [t_0, \infty)_\mathbb{T}$ and $t_0 \in \mathbb{T}$ is nonnegative and fixed; $A = (a_{ij})_{n\times n}$ is a nonsingular matrix with continuous real-valued functions as its elements, $A \in \mathcal{R}^+$, and $a_{ij} \in C_{ld}(\mathbb{T}, \mathbb{R})$ is ∇-periodic in shifts $\delta_\pm$ with period ω; $b = \mathrm{diag}(b_1, b_2, \cdots, b_n)$, and $b_i \in C_{ld}(\mathbb{T}, \mathbb{R})$ is ∇-periodic in shifts $\delta_\pm$ with period ω; $f = (f_1, f_2, \cdots, f_n)^T, g = (g_1, g_2, \cdots, g_n)^T$, and $f_i, g_i \in C_{ld}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R})$ are periodic in shifts $\delta_\pm$ with period ω with respect to the first variable; $\tau \in C_{ld}(\mathbb{T}, \mathbb{T})$ is periodic in shifts $\delta_\pm$ with period ω.

The main purpose of this paper is to establish some sufficient conditions for the existence of at least one nonzero periodic solution in shifts $\delta_\pm$ of equation (1) using Krasnoselskii's fixed point theorem and contraction mapping principle.

For each $x = (x_1, x_2, \cdots, x_n)^T \in C_{ld}(\mathbb{T}, \mathbb{R}^n)$, the norm of x is defined as $\|x\| = \sup_{t\in[t_0,\delta_+^\omega(t_0)]_\mathbb{T}} |x(t)|_0$, where $|x(t)|_0 = \sum_{i=1}^n |x_i(t)|$, and when it comes to that x is continuous, delta derivative, delta integrable, and so forth; we mean that each element x_i is continuous, delta derivative, delta integrable, and so forth.

II. Preliminaries

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ for all $t \in \mathbb{T}$, while the backward jump

Manuscript received May 22, 2016; revised August 29, 2016. This work was supported in part by the Key Project of Scientific Research in Colleges and Universities in Henan Province (Nos. 15A110004 and 16A110008) and the Basic and Frontier Technology Research Project of Henan Province (No.142300410113).

L. Wang and M. Hu are with the School of Mathematics and Statistics, Anyang Normal University, Anyang, Henan, 455000 China e-mail: ay_wanglili@126.com.

operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ for all $t \in \mathbb{T}$.

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$. The backwards graininess function $\nu : \mathbb{T}_k \to [0, +\infty)$ is defined by $\nu(t) = t - \rho(t)$.

A function $f : \mathbb{T} \to \mathbb{R}$ is ld-continuous provided it is continuous at left-dense point in $\mathbb{T}$ and its right-side limits exist at right-dense points in $\mathbb{T}$.

Definition 1. ([12]) *An $n \times n$-matrix-valued function A on a time scale $\mathbb{T}$ is called ν-regressive (with respect to $\mathbb{T}$) provided*

$$I - \nu(t)A(t)$$

is invertible for all $t \in \mathbb{T}^k$. The set of all ν-regressive and ld-continuous functions $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ will be denoted by $\mathcal{R}_\nu = \mathcal{R}_\nu(\mathbb{T}, \mathbb{R}^{n\times n})$.

Definition 2. ([12]) *Let $t_0 \in \mathbb{T}$ and assume that $A \in \mathcal{R}_\nu$ is an $n \times n$-matrix-valued function. The unique matrix-valued solution of the IVP*

$$Y^\nabla = A(t)Y, \; Y(t_0) = I,$$

where I denotes as usual the $n \times n$-identity matrix, is called the nabla matrix exponential function(at t_0), and is denoted by $\hat{e}_A(\cdot, t_0)$.

Lemma 1. ([12]) *If $A \in \mathcal{R}_\nu$ is an $n \times n$-matrix-valued function on $\mathbb{T}$, then*

(i) $\hat{e}_0(t, s) \equiv I$ *and* $\hat{e}_A(t, t) \equiv I$;

(ii) $\hat{e}_A(\rho(t), s) = (I + \mu(t)A(t))\hat{e}_A(t, s)$;

(iii) $\hat{e}_A(t, s) = \hat{e}_A^{-1}(s, t)$;

(iv) $\hat{e}_A(t, s)\hat{e}_A(s, r) = \hat{e}_A(t, r)$;

(v) $(\hat{e}_A(c, \cdot))^\nabla = -(\hat{e}_A(c, \cdot))^\rho A$ *and* $\int_a^b \hat{e}_A(c, \rho(t))A(t)\Delta t = \hat{e}_A(c, a) - \hat{e}_A(c, b)$.

Lemma 2. ([12]) *If $a, b \in \mathbb{T}$, and $f, f^\nabla, g : \mathbb{T} \to \mathbb{R}$ are ld-continuous, then*

(i) $\int_a^b f(\rho(t))g^\nabla(t)\nabla t = (fg)(b) - (fg)(a) - \int_a^b f^\nabla(t)g(t)\nabla t$;

(ii) $[\int_a^t f(t, s)\nabla s]^\nabla = \int_a^t f^\nabla(t, s)\nabla s + f(\rho(t), t)$;

(ii) $[\int_t^b f(t, s)\nabla s]^\nabla = \int_t^b f^\nabla(t, s)\nabla s - f(\rho(t), t)$.

For more details about the calculus on time scales, see [12].

Lemma 3. ([12]) *Let $A \in \mathcal{R}_\nu$ is an $n \times n$-matrix-valued function on $\mathbb{T}$ and suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is ld-continuous. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$. Then the initial value problem*

$$y^\nabla = A(t)y + f(t), \; y(t_0) = y_0$$

has a unique solution $y : \mathbb{T} \to \mathbb{R}^n$. Moreover, the solution is given by

$$y(t) = \hat{e}_A(t, t_0)y_0 + \int_{t_0}^t \hat{e}_A(t, \rho(\tau))f(\tau)\Delta\tau.$$

Let $\mathbb{T}^*$ be a non-empty subset of the time scale $\mathbb{T}$ and $t_0 \in \mathbb{T}^*$ be a fixed number, define operators $\delta_\pm : [t_0, \infty) \times \mathbb{T}^* \to \mathbb{T}^*$. The operators δ_+ and δ_- associated with $t_0 \in \mathbb{T}^*$(called the initial point) are said to be forward and backward shift operators on the set $\mathbb{T}^*$, respectively. The variable $s \in [t_0, \infty)_\mathbb{T}$ in $\delta_\pm(s, t)$ is called the shift size. The value $\delta_+(s, t)$ and $\delta_-(s, t)$ in $\mathbb{T}^*$ indicate s units translation of the term $t \in \mathbb{T}^*$ to the right and left, respectively. The sets

$$\mathbb{D}_\pm := \{(s, t) \in [t_0, \infty)_\mathbb{T} \times \mathbb{T}^* : \delta_\mp(s, t) \in \mathbb{T}^*\}$$

are the domains of the shift operator $\delta_\pm$, respectively. Hereafter, $\mathbb{T}^*$ is the largest subset of the time scale $\mathbb{T}$ such that the shift operators $\delta_\pm : [t_0, \infty) \times \mathbb{T}^* \to \mathbb{T}^*$ exist.

Definition 3. ([13]) *(Periodicity in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0 \in \mathbb{T}^*$. The time scale $\mathbb{T}$ is said to be periodic in shifts $\delta_\pm$ if there exists $p \in (t_0, \infty)_{\mathbb{T}^*}$ such that $(p, t) \in \mathbb{D}_\pm$ for all $t \in \mathbb{T}^*$. Furthermore, if*

$$P := \inf\{p \in (t_0, \infty)_{\mathbb{T}^*} : (p, t) \in \delta_\pm, \forall t \in \mathbb{T}^*\} \neq t_0,$$

then P is called the period of the time scale $\mathbb{T}$.

Definition 4. ([13]) *(Periodic function in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real-valued function f defined on $\mathbb{T}^*$ is periodic in shifts $\delta_\pm$ if there exists $\omega \in [P, \infty)_{\mathbb{T}^*}$ such that $(\omega, t) \in \mathbb{D}_\pm$ and $f(\delta_\pm^\omega(t)) = f(t)$ for all $t \in \mathbb{T}^*$, where $\delta_\pm^\omega := \delta_\pm(\omega, t)$. The smallest number $\omega \in [P, \infty)_{\mathbb{T}^*}$ is called the period of f.*

Definition 5. *(∇-periodic function in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real-valued function f defined on $\mathbb{T}^*$ is ∇-periodic in shifts $\delta_\pm$ if there exists $\omega \in [P, \infty)_{\mathbb{T}^*}$ such that $(\omega, t) \in \mathbb{D}_\pm$ for all $t \in \mathbb{T}^*$, the shifts $\delta_\pm^\omega$ are ∇-differentiable with ld-continuous derivatives and $f(\delta_\pm^\omega(t))\delta_\pm^{\nabla\omega}(t) = f(t)$ for all $t \in \mathbb{T}^*$, where $\delta_\pm^\omega := \delta_\pm(\omega, t)$. The smallest number $\omega \in [P, \infty)_{\mathbb{T}^*}$ is called the period of f.*

Similar to the proofs of Lemma 2, Corollary 1 and Theorem 2 in [13], we can get the following two lemmas.

Lemma 4. *$\delta_+^\omega(\rho(t)) = \rho(\delta_+^\omega(t))$ and $\delta_-^\omega(\rho(t)) = \rho(\delta_-^\omega(t))$ for all $t \in \mathbb{T}^*$.*

Lemma 5. *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P, and let f be a ∇-periodic function in shifts $\delta_\pm$ with the period $\omega \in [P, \infty)_{\mathbb{T}^*}$. Suppose that $f \in C_{ld}(\mathbb{T})$, then*

$$\int_{t_0}^t f(s)\nabla s = \int_{\delta_\pm^\omega(t_0)}^{\delta_\pm^\omega(t)} f(s)\nabla s.$$

Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$. If one takes $v(t) = \delta_\pm^\omega(t)$, then one has $v(\mathbb{T}) = \mathbb{T}$ and $[f(v(t))]^\nabla = (f^\nabla \circ v)(t)v^\nabla(t)$.

Now, we prove two properties of the nabla exponential functions $\hat{e}_A(t, t_0)$ and shift operators $\delta_\pm$ on time scales.

Lemma 6. *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. Suppose that the shifts $\delta_\pm^\omega$ are ∇-differentiable on $t \in \mathbb{T}^*$ where $\omega \in [P, \infty)_{\mathbb{T}^*}$ and $A \in \mathcal{R}$ is ∇-periodic in shifts $\delta_\pm$ with the period ω. Then*

$$\hat{e}_A(\delta_\pm^\omega(t), \delta_\pm^\omega(t_0)) = \hat{e}_A(t, t_0) \; for \; t, t_0 \in \mathbb{T}^*.$$

Proof: Let $Y(t) = F(\delta_{\pm}^{\omega}(t))$, where $F(t) = \hat{e}_A(t, \delta_{\pm}^{\omega}(t_0))$, then

$$\begin{aligned} Y^{\nabla}(t) &= [F(\delta_{\pm}^{\omega}(t))]^{\nabla} = (F^{\nabla} \circ \delta_{\pm}^{\omega})(t)\delta_{\pm}^{\nabla\omega}(t) \\ &= A(\delta_{\pm}^{\omega}(t))\delta_{\pm}^{\nabla\omega}(t)\hat{e}_A(\delta_{\pm}^{\omega}(t), \delta_{\pm}^{\omega}(t_0)) \\ &= A(t)\hat{e}_A(\delta_{\pm}^{\omega}(t), \delta_{\pm}^{\omega}(t_0)) \\ &= A(t)Y(t), \end{aligned}$$

and $Y(t_0) = \hat{e}_A(\delta_{\pm}^{\omega}(t_0), \delta_{\pm}^{\omega}(t_0)) = I$. Hence Y solves the IVP

$$Y^{\nabla}(t) = A(t)Y(t), Y(t_0) = I,$$

which has exactly one solution according to Lemma 3, and therefore we have

$$\hat{e}_A(\delta_{\pm}^{\omega}(t), \delta_{\pm}^{\omega}(t_0)) = \hat{e}_A(t, t_0) \ for \ t, t_0 \in \mathbb{T}^*.$$

This completes the proof.

Lemma 7. *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with the period P. Suppose that the shifts $\delta_{\pm}^{\omega}$ are ∇-differentiable on $t \in \mathbb{T}^*$ where $\omega \in [P, \infty)_{\mathbb{T}^*}$ and $A \in \mathcal{R}$ is ∇-periodic in shifts $\delta_{\pm}$ with the period ω. Then*

$$\hat{e}_A(\delta_{\pm}^{\omega}(t), \rho(\delta_{\pm}^{\omega}(s))) = \hat{e}_A(t, \rho(s)) \ for \ t, s \in \mathbb{T}^*.$$

Proof: From Lemma 4, we know $\delta_{\pm}^{\omega}(\rho(t)) = \rho(\delta_{\pm}^{\omega}(t))$. By Lemma 6, we can obtain

$$\hat{e}_A(\delta_{\pm}^{\omega}(t), \rho(\delta_{\pm}^{\omega}(s))) = \hat{e}_A(t, \rho(s)) \ for \ t, s \in \mathbb{T}^*.$$

This completes the proof.

Set

$$X = \{x(t) : x \in C_{ld}(\mathbb{T}, \mathbb{R}^n), x(\delta_{+}^{\omega}(t)) = x(t)\}$$

with the norm defined by $\|x\| = \sup_{t \in [t_0, \delta_{+}^{\omega}(t_0)]_{\mathbb{T}}} |x(t)|_0$, where $|x(t)|_0 = \sum_{i=1}^{n} |x_i(t)|$, then X is a Banach space.

Lemma 8. *The function $x(t) \in X$ is an ω-periodic solution in shifts $\delta_{\pm}$ of equation (1) if and only if $x(t)$ is an ω-periodic solution in shifts $\delta_{\pm}$ of*

$$\begin{aligned} x(t) &= f(t, x(t)) + \int_t^{\delta_{+}^{\omega}(t)} G(t, s)[A(s)f(s, x(s)) \\ &\quad + b(s)g(s, x(\tau(s)))]\nabla s, \end{aligned} \tag{2}$$

where

$$G(t, s) = \left[\hat{e}_A(t_0, \delta_{+}^{\omega}(t_0)) - I\right]^{-1}\hat{e}_A(t, \rho(s)) := (G_{ik})_{n\times n}.$$

Proof: If $x(t)$ is an ω-periodic solution in shifts $\delta_{\pm}$ of equation (1). By Lemmas 2.3 and 2.2, for $s \in [t, \delta_{+}^{\omega}(t)]_{\mathbb{T}}$, we have

$$\begin{aligned} x(s) &= \hat{e}_A(s, t)x(t) + \int_t^s \hat{e}_A(s, \rho(\theta))[f^{\nabla}(\theta, x(\theta)) \\ &\quad + b(\theta)g(\theta, x(\tau(\theta)))]\nabla\theta \\ &= \hat{e}_A(s, t)x(t) + \hat{e}_A(s, s)f(s, x(s)) \\ &\quad - \hat{e}_A(s, t)f(t, x(t)) \\ &\quad - \int_t^s [\hat{e}_A(s, \theta)]^{\nabla} f(\theta, x(\theta))\nabla\theta \\ &\quad + \int_t^s \hat{e}_A(s, \rho(\theta))b(\theta)g(\theta, x(\tau(\theta)))\nabla\theta \\ &= \hat{e}_A(s, t)x(t) + f(s, x(s)) - \hat{e}_A(s, t)f(t, x(t)) \\ &\quad \times \int_t^s \hat{e}_A(s, \rho(\theta))A(\theta)f(\theta, x(\theta))\nabla\theta \\ &\quad + \int_t^s \hat{e}_A(s, \rho(\theta))b(\theta)g(\theta, x(\tau(\theta)))\nabla\theta \\ &= \hat{e}_A(s, t)x(t) + f(s, x(s)) - \hat{e}_A(s, t)f(t, x(t)) \\ &\quad + \int_t^s \hat{e}_A(s, \rho(\theta))[A(\theta)f(\theta, x(\theta)) \\ &\quad + b(\theta)g(\theta, x(\tau(\theta)))]\nabla\theta. \end{aligned} \tag{3}$$

Let $s = \delta_{+}^{\omega}(t)$ in (3), we have

$$\begin{aligned} x(\delta_{+}^{\omega}(t)) &= \hat{e}_A(\delta_{+}^{\omega}(t), t)x(t) + f(\delta_{+}^{\omega}(t), x(\delta_{+}^{\omega}(t))) \\ &\quad - \hat{e}_A(\delta_{+}^{\omega}(t), t)f(t, x(t)) \\ &\quad + \int_t^{\delta_{+}^{\omega}(t)} \hat{e}_A(\delta_{+}^{\omega}(t), \rho(\theta))[A(\theta)f(\theta, x(\theta)) \\ &\quad + b(\theta)g(\theta, x(\tau(\theta)))]\nabla\theta. \end{aligned}$$

Noticing that $x(\delta_{+}^{\omega}(t)) = x(t)$, and $f(\delta_{+}^{\omega}(t), x(\delta_{+}^{\omega}(t))) = f(t, x(t))$, then

$$\begin{aligned} x(t) &= \hat{e}_A(\delta_{+}^{\omega}(t), t)x(t) + f(t, x(t)) \\ &\quad - \hat{e}_A(\delta_{+}^{\omega}(t), t)f(t, x(t)) \\ &\quad + \int_t^{\delta_{+}^{\omega}(t)} \hat{e}_A(\delta_{+}^{\omega}(t), \rho(\theta))[A(\theta)f(\theta, x(\theta)) \\ &\quad + b(\theta)g(\theta, x(\tau(\theta)))]\nabla\theta. \end{aligned} \tag{4}$$

Multipling on both sides of (4) by $\hat{e}_A(t, \delta_{+}^{\omega}(t))$, then

$$\begin{aligned} \hat{e}_A(t, \delta_{+}^{\omega}(t))x(t) &= x(t) + (\hat{e}_A(t, \delta_{+}^{\omega}(t)) - I)f(t, x(t)) \\ &\quad + \int_t^{\delta_{+}^{\omega}(t)} \hat{e}_A(t, \rho(\theta))[A(\theta)f(\theta, x(\theta)) \\ &\quad + b(\theta)g(\theta, x(\tau(\theta)))]\nabla\theta, \end{aligned}$$

that is

$$\begin{aligned} x(t) &= f(t, x(t)) \\ &\quad + \int_t^{\delta_{+}^{\omega}(t)} [\hat{e}_A(t, \delta_{+}^{\omega}(t)) - I]^{-1}\hat{e}_A(t, \rho(\theta)) \\ &\quad \times [A(\theta)f(\theta, x(\theta)) + b(\theta)g(\theta, x(\tau(\theta)))]\nabla\theta. \end{aligned}$$

Since $\hat{e}_A(t, \delta_{+}^{\omega}(t)) = \hat{e}_A(t_0, \delta_{+}^{\omega}(t_0))$, then x satisfies (2).

Let $x(t)$ be an ω-periodic solution in shifts $\delta_{\pm}$ of (2). By (2) and Lemmas 1 and 2, we have

$$\begin{aligned} x^{\nabla}(t) &= f^{\nabla}(t, x(t)) + A(t)x(t) - A(t)f(t, x(t)) \\ &\quad + G(\rho(t), \delta_{+}^{\omega}(t))\delta_{+}^{\Delta\omega}(t) \\ &\quad \times [A(\delta_{+}^{\omega}(t))f(\delta_{+}^{\omega}(t), x(\delta_{+}^{\omega}(t))) \\ &\quad + b(\delta_{+}^{\omega}(t))g(\delta_{+}^{\omega}(t), x(\tau(\delta_{+}^{\omega}(t))))] \\ &\quad - G(\rho(t), t)[A(t)f(t, x(t)) + b(t)g(t, x(\tau(t)))] \\ &= A(t)x(t) + f^{\nabla}(t, x(t)) + b(t)g(t, x(\tau(t))). \end{aligned}$$

So, $x(t)$ is an ω-periodic solution in shifts $\delta_{\pm}$ of equation (1). This completes the proof.

By using Lemma 6 and Lemma 7, it is easy to verify that the Green's function $G(t, s)$ satisfies

$$G(\delta_{+}^{\omega}(t), \delta_{+}^{\omega}(s)) = G(t, s), \ \forall t \in \mathbb{T}^*, s \in [t, \delta_{+}^{\omega}(t)]_{\mathbb{T}}. \tag{5}$$

For convenience, we introduce the following notations:

$$G^u := \max_{1\le k\le n} \sup_{s,t \in [t_0, \delta_{+}^{\omega}(t_0)]_{\mathbb{T}}} |\sum_{i=1}^{n} G_{ik}(t, s)|,$$

$$A^u := \max_{1\le i\le n} \int_{t_0}^{\delta_{+}^{\omega}(t_0)} |\sum_{k=1}^{n} a_{ki}(s)|\nabla s,$$

$$B^u := \max_{1\le k\le n}\int_{t_0}^{\delta_+^\omega(t_0)} |b_k(s)|\nabla s.$$

Define an operator $H : X \to X$ by

$$\begin{aligned}(Hx)(t) &= f(t,x(t)) + \int_t^{\delta_+^\omega(t)} G(t,s)[A(s)f(s,x(s)) \\ &\quad +b(s)g(s,x(\tau(s)))]\nabla s, \qquad (6)\end{aligned}$$

and

$$(Hx)(t) = ((H_1x)(t),(H_2x)(t),\cdots,(H_nx)(t))^T,$$

where

$$\begin{aligned}(H_ix)(t) &= \int_t^{\delta_+^\omega(t)} \sum_{k=1}^n G_{ik}[\sum_{i=1}^n a_{ki}f_i(s,x(s)) \\ &\quad +b_k(s)g_k(s,x(\tau(s)))]\nabla s, i=1,2,\cdots,n.\end{aligned}$$

III. Main results

In this section, we shall state and prove our main results about the existence of at least one periodic solution in shifts $\delta_\pm$ of equation (1). In order to prove the existence of a nonzero periodic solution in shifts $\delta_\pm$ of equation (1), we assume that

(H_1) $f^\nabla(t,0) - b(t)g(t,0) \neq 0$ for some $t \in \mathbb{T}^*$.

Lemma 9. [14] *(Krasnoselskii's fixed point theorem) Let M be a closed convex nonempty subset of a Banach space $(X,\|\cdot\|)$. Suppose that B and C map M into X, such that*
(1) $x,y \in M$, implies $Bx + Cy \in M$,
(2) C is continuous and $C(M)$ is contained in a compact set,
(3) B is a contraction mapping.
Then there exists $z \in M$ with $z = Bz + Cz$.

In preparation for the next result, we need to construct two mappings, one is a contraction and the other is compact. Let

$$(Hx)(t) = (Bx)(t) + (Cx)(t),$$

where $B, C : X \to X$ are given by

$$\begin{aligned}(Bx)(t) &= f(t,x(t)), \qquad (7)\\ (Cx)(t) &= \int_t^{\delta_+^\omega(t)} G(t,s)[A(s)f(s,x(s)) \\ &\quad +b(s)g(s,x(\tau(s)))]\nabla s. \qquad (8)\end{aligned}$$

Hereafter, we make the following assumption:

(H_2) There exist positive numbers L_f, L_g such that

$$|f(t,u) - f(t,v)|_0 \le L_f|u-v|_0,$$
$$|g(t,u) - g(t,v)|_0 \le L_g|u-v|_0,$$

for all $t \in \mathbb{T}$, $u, v \in X$.

Lemma 10. [15] *The operator B is a contraction provided $L_f < 1$.*

Lemma 11. *Assume that (H_2) holds. The operator C is continuous and the image $C(M)$ is contained in a compact set, where $M = \{x \in X : \|x\| \le \zeta\}$, ζ is a fixed constant.*

Proof: Firstly, we show that C is continuous. By (H_2), for any $\zeta > 0$ and $\varepsilon > 0$, there exists a $\eta > 0$ such that

$$\big\{\phi,\psi \in C(\mathbb{T},\mathbb{R}^n), \|\phi\| \le \zeta, \|\psi\| \le \zeta, \|\phi-\psi\| < \eta\big\}$$

imply

$$|f(s,\phi(s)) - f(s,\psi(s))|_0 < \frac{\varepsilon}{2G^uA^u},$$

and

$$|g(s,\phi(\tau(s))) - g(s,\psi(\tau(s)))|_0 < \frac{\varepsilon}{2G^uB^u}.$$

Therefore, if $x,y,\in X$ with $\|x\| \le \zeta$, $\|y\| \le \zeta$, $\|x-y\| < \eta$, then

$$\begin{aligned}&|(Cx)(t) - (Cy)(t)|_0 \\ \le\ & \sum_{i=1}^n \Big| \int_t^{\delta_+^\omega(t)} \sum_{k=1}^n G_{ik}[\sum_{i=1}^n a_{ki}f_i(s,x(s)) \\ & +b_k(s)g_k(s,x(\tau(s)))]\nabla s \\ & -\int_t^{\delta_+^\omega(t)} \sum_{k=1}^n G_{ik}[\sum_{i=1}^n a_{ki}f_i(s,y(s)) \\ & +b_k(s)g_k(s,y(\tau(s)))]\nabla s \Big| \\ \le\ & \int_t^{\delta_+^\omega(t)} \sum_{k=1}^n |\sum_{i=1}^n G_{ik}||\sum_{i=1}^n a_{ki}f_i(s,x(s)) \\ & -\sum_{i=1}^n a_{ki}f_i(s,y(s))| \\ & +\sum_{k=1}^n |\sum_{i=1}^n G_{ik}||b_k(s)f_k(s,x(\tau(s))) \\ & -b_k(s)f_k(s,y(\tau(s)))|\nabla s \\ \le\ & G^u\bigg(\int_t^{\delta_+^\omega(t)} |A(s)f(s,x(s)) - A(s)f(s,y(s))|_0 \\ & +|b(s)f(s,x(\tau(s))) - b(s)f(s,y(\tau(s)))|_0\nabla s\bigg) \\ <\ & G^u\bigg(A^u\frac{\varepsilon}{2G^uA^u} + B^u\frac{\varepsilon}{2G^uB^u}\bigg) \\ =\ & \varepsilon,\end{aligned}$$

for all $t \in [t_0, \delta_+^\omega(t_0)]_\mathbb{T}$, which yields

$$\|Cx - Cy\| = \sup_{t\in[t_0,\delta_+^\omega(t_0)]_\mathbb{T}} |(Cx)(t) - (Cy)(t)|_0 \le \varepsilon,$$

that is, C is continuous.

Next, we show that C maps any bounded sets in X into relatively compact sets. We firstly prove that f maps bounded sets into bounded sets. Indeed, let $\varepsilon = 1$, for any $\zeta > 0$, there exists a $\eta > 0$ such that $\big\{x,y \in X, \|x\| \le \zeta, \|y\| \le \zeta, \|x-y\| < \eta, s \in [t_0,\delta_+^\omega(t_0)]_\mathbb{T}\big\}$ imply

$$|f(s,x(s)) - f(s,y(s))|_0 < 1,$$
$$|g(s,x(\tau(s))) - g(s,y(\tau(s)))|_0 < 1.$$

Choose a positive integer N such that $\frac{\zeta}{N} < \eta$. Let $x \in X$ and define $x^k(\cdot) = \frac{x(\cdot)k}{N}$, $k = 0,1,2,\cdots,N$. If $\|x\| < \zeta$, then

$$\begin{aligned}\|x^k - x^{k-1}\| &= \sup_{t\in[t_0,\delta_+^\omega(t_0)]_\mathbb{T}} \left|\frac{x(\cdot)k}{N} - \frac{x(\cdot)(k-1)}{N}\right|_0 \\ &\le \|x\|\frac{1}{N} \le \frac{\zeta}{N} < \eta.\end{aligned}$$

Thus

$$|f(s,x^k(s)) - f(s,x^{k-1}(s))|_0 < 1,$$
$$|g(s,x^k(\tau(s))) - f(s,x^{k-1}(\tau(s)))|_0 < 1,$$

for all $s \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$, and these yield

$$\begin{aligned} |f(s,x(s))|_0 &= |f(s,x^N(s))|_0 \\ &\le \sum_{k=1}^{N} |f(s,x^k(s)) - f(s,x^{k-1}(s))|_0 \\ &\quad +|f(s,0)|_0 \\ &< N + \sup_{s\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} |f(s,0)|_0 =: W. \quad (9) \end{aligned}$$

Similarly, we have

$$|g(s,x(\tau(s)))|_0 < N + \sup_{s\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} g(s,0) =: U. \quad (10)$$

It follows from (8)-(10) that for $t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$,

$$\begin{aligned} \|Cx\| &= \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} \sum_{i=1}^{n} |(C_i x)(t)| \\ &\le G^u(A^u W + B^u U) := D. \end{aligned}$$

Finally, for $t \in \mathbb{T}$, we have

$$\begin{aligned} (Cx)^\nabla(t) &= A(t)(Cx)(t) + A(t)f(t,x(t)) \\ &\quad +b(t)g(t,x(\tau(t))). \end{aligned}$$

So

$$\begin{aligned} \|(Cx)^\nabla\| &= \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} |A(t)(Cx)(t) + A(t)f(t,x(t)) \\ &\quad +b(t)g(t,x(\tau(t)))|_0 \\ &\le \hat{A}(D+W) + \hat{B}U, \end{aligned}$$

where $\hat{A} := \max_{1\le i\le n} \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} |\sum_{k=1}^{n} a_{ki}(t)|$, $\hat{B} := \max_{1\le k\le n} \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} |b_k(t)|$.

To sum up, $\{Cx : x \in X, \|x\| \le \zeta\}$ is a family of uniformly bounded and equicontinuous functionals on $[t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$. By a theorem of Arzela-Ascoli, we know that the functional C is completely continuous, that is, $C(M)$ is compact. This completes the proof.

Theorem 1. *Assume that $(H_1) - (H_2)$ hold. Let $\alpha = \|f(\cdot,0)\|, \beta = \|g(\cdot,0)\|$. Let R_0 be a positive constant satisfies*

$$\begin{aligned} &L_f R_0 + \alpha + G^u[A^u(L_f R_0 + \alpha) \\ &+B^u(L_g R_0 + \beta)] \le R_0. \quad (11) \end{aligned}$$

Then equation (1) has a nonzero periodic solution in shifts $\delta_\pm$ in $M = \{x \in X : \|x\| \le R_0\}$.

Proof: Define $M = \{x \in X : \|x\| \le R_0\}$. By Lemma 11, the mapping C defined by (8) is continuous and $C(M)$ is contained in a compact set. By Lemma 10, the mapping B defined by (7) is a contraction and it is clear that $B : X \to X$.

Next, we show that if $x, y \in M$, we have $\|Bx + Cy\| \le R_0$. In fact, let $x, y \in M$ with $\|x\|, \|y\| \le R_0$. Then

$$\begin{aligned} \|Bx + Cy\| &= \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} |f(t,x(t)) \\ &\quad + \int_t^{\delta_+^\omega(t)} G(t,s)[A(s)f(s,y(s)) \\ &\quad +b(s)g(s,y(\tau(s)))]\nabla s|_0 \\ &\le \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} \Big[|f(t,x(t)) - f(t,0)|_0 \\ &\quad +|f(t,0)|_0 \\ &\quad + \int_t^{\delta_+^\omega(t)} \sum_{k=1}^{n} |\sum_{i=1}^{n} G_{ik}| |\sum_{i=1}^{n} a_{ki} f_i(s,y(s)) \\ &\quad +b_k(s)g_k(s,y(\tau(s)))|\nabla s \Big] \\ &\le L_f\|x\| + \alpha + G^u[A^u(L_f\|y\| + \alpha) \\ &\quad +B^u(L_g\|y\| + \beta)] \\ &\le L_f R_0 + \alpha + G^u[A^u(L_f R_0 + \alpha) \\ &\quad +B^u(L_g R_0 + \beta)] \\ &\le R_0. \end{aligned}$$

Thus $Bx + Cy \in M$. Hence all the conditions of Krasnoselskii's theorem are satisfied, that is, there exists a fixed point $z \in M$, such that $z = Bz + Cz$. By Lemma 9, equation (1) has a nonzero periodic solution in shifts $\delta_\pm$. The proof is completed.

Theorem 2. *Assume that $(H_1) - (H_2)$ hold. If*

$$L_f + G^u(A^u L_f + B^u L_g) < 1, \quad (12)$$

then equation (1) has a unique nonzero periodic solution in shifts $\delta_\pm$.

Proof: Let the mapping H is given by (6). For any $x, y \in X$, we have

$$\begin{aligned} \|Hx - Hy\| &= \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} |(Hx)(t) - (Hy)(t)|_0 \\ &= \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} \Big| f(t,x(t)) \\ &\quad + \int_t^{\delta_+^\omega(t)} G(t,s)[A(s)f(s,x(s)) \\ &\quad +b(s)g(s,x(\tau(s)))] - f(t,y(t)) \\ &\quad - \int_t^{\delta_+^\omega(t)} G(t,s)[A(s)f(s,y(s)) \\ &\quad +b(s)g(s,y(\tau(s)))] \Big|_0 \\ &\le L_f\|x-y\| + G^u(A^u L_f\|x-y\| \\ &\quad +B^u L_g\|x-y\|) \\ &= [L_f + G^u(A^u L_f + B^u L_g)]\|x-y\|. \end{aligned}$$

This completes the proof by invoking the contraction mapping principle.

IV. Numerical examples

Example 1. Let $\mathbb{T} = \mathbb{R}$, $t_0 = 0$, $\omega = 2\pi$, then $\delta_+^\omega(t) = t+2\pi$. For small positive ε_1 and ε_2, we consider the perturbed Van Der Pol equation

$$x^{\nabla\nabla} + (\varepsilon_2 x^2 - 1)x^\nabla + x - \varepsilon_1(\sin t \cdot x^2)^\nabla - \varepsilon_2 \cos t = 0. \quad (13)$$

Using the transformation $x_1^\nabla = x_2$, we can transform the above equation to

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^\nabla &= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon_1 \sin t x_1^2 \end{pmatrix}^\nabla \\ &\quad + \begin{pmatrix} 0 \\ \varepsilon_2 \cos t - \varepsilon_2 x_2 x_1^2 \end{pmatrix}, \end{aligned}$$

that is, $A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, $f(t,x) = \begin{pmatrix} 0 \\ \varepsilon_1 \sin tx_1^2 \end{pmatrix}$, $g(t,x) = \begin{pmatrix} 0 \\ \varepsilon_2 \cos t - \varepsilon_2 x_2 x_1^2 \end{pmatrix}$.

Let $x(t) = (x_1(t), x_2(t)), y(t) = (y_1(t), y_2(t))$. Define $M = \{x \in X : \|x\| \leq R_0\}$, where R_0 is a positive constant.

Then for $x, y \in M$, we have

$$\|f(\cdot, x) - f(\cdot, y)\| \leq 2\varepsilon_1 R_0 \|x - y\|,$$

and

$$\begin{aligned} &\|g(\cdot), x(\cdot)) - g(\cdot, y(\cdot))\| \\ &\leq \varepsilon_2 \sup_{t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}} \Bigg| (x_2(t)(x_1(t) + y_1(t)), y_1^2(t)) \\ &\times \begin{pmatrix} x_1(t) - y_1(t) \\ x_2(t) - y_2(t) \end{pmatrix} \Bigg| \\ &\leq 2\varepsilon_2 R_0^2 \|x - y\|. \end{aligned}$$

Hence, let $L_f = 2\varepsilon_1 R_0, L_g = \varepsilon_2 R_0^2, \alpha = \|f(t,0)\| = 0$ and $\beta = \|g(t,0)\| = \varepsilon_2$. Thus, inequality (11) becomes

$$2\varepsilon_1 R_0^2 + G^u[A^u(2\varepsilon_1 R_0^2) + B^u(\varepsilon_2 R_0^3 + \varepsilon_2)] \leq R_0,$$

which is satisfied for small ε_1 and ε_2. By Theorem 1, (13) has a nonzero periodic solution in shifts $\delta_\pm$ with period $\omega = 2\pi$.

Moreover,

$$2\varepsilon_1 R_0 + G^u(2A^u \varepsilon_1 R_0 + B^u \varepsilon_2 R_0^2) < 1$$

is also satisfied for small ε_1 and ε_2. By Theorem 2, (13) has a unique nonzero periodic solution in shifts $\delta_\pm$ with period $\omega = 2\pi$.

Example 2. Let $\mathbb{T} = 2^{\mathbb{N}_0}$, $\omega = 4$, $t_0 = 1$, then $\delta_+^\omega(t) = 4t$. For small positive ε_1 and ε_2, we consider the following perturbed dynamic equation

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^\nabla &= \begin{pmatrix} 0 & \frac{1}{t} \\ -\frac{1}{t} & \frac{1}{t} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon_1 x_1^2 \end{pmatrix}^\nabla \\ &\quad + \begin{pmatrix} 0 \\ \varepsilon_2 - \varepsilon_2 x_2 x_1^2 \end{pmatrix}, \end{aligned} \tag{14}$$

that is, $A = \begin{pmatrix} 0 & \frac{1}{t} \\ -\frac{1}{t} & \frac{1}{t} \end{pmatrix}$, $f(t,x) = \begin{pmatrix} 0 \\ \varepsilon_1 \sin tx_1^2 \end{pmatrix}$, $g(t,x) = \begin{pmatrix} 0 \\ \varepsilon_2 - \varepsilon_2 x_2 x_1^2 \end{pmatrix}$.

Let $x(t) = (x_1(t), x_2(t)), y(t) = (y_1(t), y_2(t))$. Define $M = \{x \in X : \|x\| \leq R_0\}$, where R_0 is a positive constant.

Then for $x, y \in M$, we have

$$\|f(\cdot, x) - f(\cdot, y)\| \leq 2\varepsilon_1 R_0 \|x - y\|,$$

and

$$\begin{aligned} &\|g(\cdot), x(\cdot)) - g(\cdot, y(\cdot))\| \\ &\leq \varepsilon_2 \sup_{t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}} \Bigg| (x_2(t)(x_1(t) + y_1(t)), y_1^2(t)) \\ &\times \begin{pmatrix} x_1(t) - y_1(t) \\ x_2(t) - y_2(t) \end{pmatrix} \Bigg| \\ &\leq 2\varepsilon_2 R_0^2 \|x - y\|. \end{aligned}$$

Hence, let $L_f = 2\varepsilon_1 R_0, L_g = \varepsilon_2 R_0^2, \alpha = \|f(t,0)\| = 0$ and $\beta = \|g(t,0)\| = \varepsilon_2$. Thus, inequality (11) becomes

$$2\varepsilon_1 R_0^2 + G^u[A^u(2\varepsilon_1 R_0^2) + B^u(\varepsilon_2 R_0^3 + \varepsilon_2)] \leq R_0,$$

which is satisfied for small ε_1 and ε_2. By Theorem 1, (14) has a nonzero periodic solution in shifts $\delta_\pm$ with period $\omega = 4$.

Moreover,

$$2\varepsilon_1 R_0 + G^u(2A^u \varepsilon_1 R_0 + B^u \varepsilon_2 R_0^2) < 1$$

is also satisfied for small ε_1 and ε_2. By Theorem 2, (14) has a unique nonzero periodic solution in shifts $\delta_\pm$ with period $\omega = 4$.

V. Conclusion

This paper developed the theory of nabla exponential function and shift operators on time scales, and studied the existence of nonzero periodic solutions in shifts $\delta_\pm$ for a neutral nabla dynamic equation. It is important to notice that the methods used in this paper can be extended to other types of biological models [16-18]. Future work will include biological dynamic systems modeling and analysis on time scales.

References

[1] S. Hilger, *Ein Masskettenkalkl mit Anwendug auf Zentrumsmanning-faltigkeiten [Ph.D. thesis]*, Universität Würzburg, 1988.

[2] B. Jain and A. Sheng, "An exploration of the approximation of derivative functions via finite differences," *Rose-Hulman Undergraduate Math J.*, Vol. 8, pp. 1-19, 2007.

[3] F. Atici and D. Biles, "Lebedinsky A: An application of time scales to economics," *Math. Comput. Model.*, Vol. 43, pp. 718-726, 2006.

[4] E. Kaufmann and Y. Raffoul, "Periodic solutions for a neutral nonlinear dynamical equation on a time scale," *J. Math. Anal. Appl.*, Vol. 319, No. 1, pp. 315-325, 2006.

[5] X. Liu and W. Li, "Periodic solutions for dynamic equations on time scales," *Nonlinear Anal. TMA.*, Vol. 67, No. 5, pp. 1457-1463, 2007.

[6] M. Adıvar, "Function bounds for solutions of Volterra integro dynamic equations on the time scales," *Electron. J. Qual. Theo.*, Vol. 7, pp. 1-22, 2010.

[7] M. Adıvar, Y. Raffoul, "Existence of resolvent for Volterra integral equations on time scales," *B. Aust. Math. Soc.*, Vol. 82, No. 1, pp. 139-155, 2010.

[8] E. Çetin, "Positive periodic solutions in shifts $\delta_\pm$ for a nonlinear first-order functional dynamic equation on time scales," *Adv. Differ. Equ.*, Vol. 2014, 2014:76.

[9] E. Çetin and F.S. Topal, "Periodic solutions in shifts $\delta_\pm$ for a nonlinear dynamic equation on time scales," *Abstr. Appl. Anal.*, Vol. 2012, Article ID 707319.

[10] M. Hu, L. Wang and Z. Wang, "Positive periodic solutions in shifts $\delta_\pm$ for a class of higher-dimensional functional dynamic equations with impulses on time scales," *Abstr. Appl. Anal.*, Vol. 2014, Article ID 509052.

[11] M. Hu and L. Wang, "Multiple periodic solutions in shifts $\delta_\pm$ for an impulsive functional dynamic equation on time scales," *Adv. Differ. Equ.*, Vol. 2014, 2014:152.

[12] M. Bohner and A. Peterson, *Dynamic equations on time scales: An Introduction with Applications*. Birkhauser, Boston, 2001.

[13] M. Adıvar, "A new periodicity concept for time scales," *Math. Slovaca*, Vol. 63, No. 4, pp. 817-828, 2013.

[14] D. Smart, *Fixed points Theorem*. Cambridge Univ. Press, Cambridge, UK, 1980.

[15] S. Abbas and D. Bahuguna, "Almost periodic solutions of neutral functional differential equations," *Comput. Math. Appl.*, Vol. 55, No. 11, pp. 2593-2601, 2008.

[16] Y. Yang and T. Zhang, "Dynamics of a harvesting schoener's competition model with time-varying delays and impulsive effects," IAENG International Journal of Applied Mathematics, vol. 45, no. 4, pp263-272, 2015.

[17] Y. Guo, "Existence and exponential stability of pseudo almost periodic solutions for Mackey-Glass equation with time-varying delay," IAENG International Journal of Applied Mathematics, vol. 46, no. 1, pp71-75, 2016.

[18] L. Li and B. Du, "Global asymptotical stability in a stochastic predator-prey system with variable delays," IAENG International Journal of Applied Mathematics, vol. 46, no. 2, pp241-246, 2016.