Lie Symmetry Analysis and Exact Solutions of General Time Fractional Fifth-order Korteweg-de Vries Equation

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Abstract—**In this paper, using the Lie group analysis method, we study the invariance properties of the general time fractional fifth-order Korteweg-de Vries (KdV) equation. A systematic research to derive Lie point symmetries of the equation is performed. In the sense of point symmetry, all of the geometric vector fields and the symmetry reductions of the equation are obtained, the exact power series solution of equation is constructed, and the convergence of the obtained solution is showed. The derivative corresponding to time fractional in the reduced formula is known as the Erdélyi-Kober fractional derivative.**

Index Terms—**Time-fractional KdV equation; Modified Riemann-Liouville derivative; Erdélyi-Kober operators; Lie symmetry analysis; Exact solution; Convergence.**

I. Introduction

In recent years, mathematics and physics fields have devoted considerable effort to study solutions of partial differential equations (PDEs), such as, the known KdV equation arises in modeling many physical phenomena, it has attracted a great deal of interest as a model for complex spatial-temporal dynamics in capillary-gravity waves [1], [2] and has been studied soliton solutions, solitary wave solutions, periodic wave solutions, rational wave solutions and numerical solutions, etc. Among many powerful methods for solving the equation, Lie symmetry analysis [3]–[7] can provide an effective procedure for conservations laws, explicit and numerical solutions of a wide and general class of differential systems representing real physical problems. Note that the term Lie symmetry analysis refers to consideration of the tangent structural equations under one or several parameter transformation groups in conjunction with the system of differential equations. Lie symmetry analysis for nonlinear PDEs with two independent variables exhibiting solitons helps to study their group theoretical properties and effectively assists to derive several mathematical characteristics related with their complete integrability [8].

In reality, a physical phenomenon may depend not only on time instant but previous history, and so fractional calculus has obtained considerable popularity and importance as generalizations of integer-order evolution equations, which can be successfully modeled by using theory of derivatives and integrals of fractional order. For instance, it used to model problems in neurons, hydrology, image processing, mechanics, mechatronics, physics, finance, control theory, viscoelasticity and rheology, and so on, we can see [9]–[13] and the reference therein. Generally, fractional partial differential equation (FPDE) is obtained by replacing the integer order derivative in PDEs with the fractional order derivative, many known nonlinear analysis methods have been successfully used to solve the FPDE. Recently, a symmetry group of scaling transformations is determined for a FPDE, containing among particular cases of diffusion equation, wave equation and diffusion-wave equation, for its group-invariant solutions, a fractional ordinary differential equation (FODE) with the new independent variable is derived [14]. By making use of the obtained Lie point symmetries, Sahadevan and Bakkyaraj [15] derived Lie point symmetries to time fractional generalized Burgers and KdV equations, and have shown that each of equation has been transformed into a nonlinear FODE with a new independent variable. Wang et al [16] studied invariance properties of time fractional generalized fifth-order KdV equation using Lie group analysis method. It is worth to mention that Djordjevic and Atanackovic [17] analyzed self-similar solutions to a nonlinear fractional diffusion equation and fractional Burgers/KdV equation by using Lie-group scaling transformation, both the equations are reduced into nonlinear FODEs, and solved the resulting ordinary differential equations numerically. Gazizov et al [18] adapted methods of Lie continuous groups for symmetry analysis, the given equations with the fractional order derivative have finite-dimensional groups of admissible transformations, examples of constructing symmetries of FPDE and using these symmetries for constructing exact solutions of the equations under consideration are presented. Liu [19] made complete group classifications on fractional fifth-order KdV type of equation and to investigate symmetry reductions and exact solutions, and so forth. The purpose of this article is to investigate Lie symmetry analysis is useful in analysis of general time fractional fifth-order KdV equation. Taking the advantage of modified Riemann-Liouville calculus approach that initial conditions for fractional differential equation take on the traditional form as for integer-order differential equation, the FPDEs are considered and extent the Lie symmetry analysis to derive their infinitesimals. In present paper, we will investigate the vector fields, symmetry reductions, exact solutions and convergence to general time fractional fifth-order KdV equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + a u_x u_{xx} + b u u_{xxx} + c u^2 u_x + d u_{xxxxx} = 0, \quad (1)$$

where $0 < \alpha < 1$, a, b, c and d are nonzero constant coefficients, $u = u(x,t)$ is a field variable, $x \in \mathbb{R}$ is a

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spatial coordinate in propagation direction of the field and $t \in \mathbb{R}^+$ is temporal coordinate, which occurs in different contexts in mathematical physics, e.g., the dissipative term u_{xxxxx} provides damping at small scales, the nonlinear term u^2u_x stabilizes by transferring energy between large and small scales which have the same form as that in Burgers or one-dimensional Navier-Stokes equation.

The general time fractional fifth-order KdV equation (1) $(0 < \alpha < 1)$ contains the following equation:

When $a = 25, b = 10, c = 20, d = 1$, we have time fractional Kaup-Kupershmidt equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + 25u_xu_{xx} + 10uu_{xxx} + 20u^2u_x + u_{xxxxx} = 0.$$

When $a = 5, b = 5, c = 5, d = 1$, we have time fractional Sawada-Kotera equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + 5u_xu_{xx} + 5uu_{xxx} + 5u^2u_x + u_{xxxxx} = 0.$$

When $a = 30, b = 30, c = 180, d = 1$, we have time fractional Caudrey-Dodd-Gibbon equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + 30u_xu_{xx} + 30uu_{xxx} + 180u^2u_x + u_{xxxxx} = 0.$$

When $a = 20, b = 10, c = 30, d = 1$, we have time fractional Lax equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + 20u_xu_{xx} + 10uu_{xxx} + 30u^2u_x + u_{xxxxx} = 0.$$

When $a = 6, b = 3, c = 2, d = 1$, we have time fractional Ito equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + 6u_xu_{xx} + 3uu_{xxx} + 2u^2u_x + u_{xxxxx} = 0,$$

and so on.

Fractional calculus is defined as the generalization of classical calculus and even can be extended to complex set, many mathematicians pushed forward and developed fractional calculus and various definitions of fractional integration and differentiation [20], [21]. Now we state definition and some important properties for the modified Riemann-Liouville derivative of order α as follows.

Assume that u denotes a continuous function on $\mathbb{R}\times\mathbb{R}^+$, we use the following equality for the integral with respect to $(dt)^\alpha$

$$\begin{aligned} I_t^\alpha u &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}u(x,\tau)d\tau \\ &= \frac{1}{\Gamma(\alpha+1)}\int_0^t u(x,\tau)(d\tau)^\alpha, \quad 0<\alpha<1. \end{aligned}$$

Jumarie's derivative for multivariate function is defined as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(-\alpha)}\frac{\partial}{\partial t}\int_0^t (t-\tau)^{-\alpha-1}(u(x,\tau)-u(x,0))d\tau, \\ \quad \alpha<0, \\ \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t (t-\tau)^{-\alpha}(u(x,\tau)-u(x,0))d\tau, \\ \quad 0<\alpha<1, \\ \left(\frac{\partial^{\alpha-n}}{\partial t^{\alpha-n}}u(x,t)\right)^{(n)}, \quad n\le\alpha<n+1, n\ge 1. \end{cases}$$

The fundamental mathematical operations and results of Jumarie's derivative are given as follows

$$\begin{aligned} &\frac{\partial^\alpha c}{\partial t^\alpha} = 0, \quad \frac{\partial^\alpha t^\beta}{\partial t^\alpha} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}t^{\beta-\alpha}, \\ &\frac{\partial^\alpha}{\partial t^\alpha}(cu(x,t)) = c\frac{\partial^\alpha}{\partial t^\alpha}u(x,t), \quad \int (dt)^\alpha = t^\alpha, \end{aligned}$$

where c is a constant, $0 < \alpha, \beta < 1$.

Leibnitz' formula of the fractional Riemann-Liouville differential takes the form

$$\frac{\partial^\alpha}{\partial t^\alpha}\big(\varphi(x,t)\psi(x,t)\big) = \sum_{n=0}^{\infty}\left(\begin{array}{c}\alpha\\ n\end{array}\right)\frac{\partial^{\alpha-n}}{\partial t^{\alpha-n}}\varphi(x,t)\frac{\partial^n}{\partial t^n}\psi(x,t),$$

where $\left(\begin{array}{c}\alpha\\ n\end{array}\right) = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}$.

Faà di Bruno's formula for multivariate composite functions is given as

$$\frac{\partial^m}{\partial t^m}\psi(\phi) = \sum_{k=0}^{m}\sum_{r=0}^{k}\left(\begin{array}{c}k\\ r\end{array}\right)\frac{1}{k!}(-\phi)^r\frac{\partial^m\phi^{k-r}}{\partial t^m}\frac{d^k\psi(\phi)}{d\phi^k},$$

here $\phi = \phi(x,t)$.

This paper is organized as follows: In Section II and III, the vector fields of equations are presented based on the optimal dynamical system method, and exact solutions to the KdV equation are obtained. Section IV, the exact analytic solutions to the equations are investigated by means of the power series method, respectively, and we show the convergence of the power series solution to equations in Section V. Finally, the conclusions will be given in Section VI.

II. Lie symmetry analysis and similarity reductions

We present brief details of the Lie symmetry analysis for general FPDE with two independent variables. Consider a scalar FPDE having the form

$$\begin{aligned} &\frac{\partial^\alpha u}{\partial t^\alpha} + F(x,t,u,u_x,u_{xx},u_{xxx},u_{xxxx},u_{xxxxx},\ldots) \\ &\quad = 0, \quad \alpha>0, \end{aligned} \tag{2}$$

where $(x,t) \in \mathbb{R}\times\mathbb{R}^+$. Let us assume that the above FPDE, (2) is invariant under a one parameter ϵ continuous transformations

$$\begin{aligned} &\bar{t} = t + \epsilon\tau(x,t,u) + O(\epsilon^2), \\ &\bar{x} = x + \epsilon\xi(x,t,u) + O(\epsilon^2), \\ &\bar{u} = u + \epsilon\eta(x,t,u) + O(\epsilon^2), \\ &\frac{\partial^\alpha\bar{u}}{\partial\bar{t}^\alpha} = \frac{\partial^\alpha u}{\partial t^\alpha} + \epsilon\zeta_\alpha^0 + O(\epsilon^2), \\ &\frac{\partial\bar{u}}{\partial\bar{x}} = \frac{\partial u}{\partial x} + \epsilon\zeta_1^1 + O(\epsilon^2), \\ &\frac{\partial^2\bar{u}}{\partial\bar{x}^2} = \frac{\partial^2 u}{\partial x^2} + \epsilon\zeta_2^1 + O(\epsilon^2), \\ &\frac{\partial^3\bar{u}}{\partial\bar{x}^3} = \frac{\partial^3 u}{\partial x^3} + \epsilon\zeta_3^1 + O(\epsilon^2), \\ &\frac{\partial^4\bar{u}}{\partial\bar{x}^4} = \frac{\partial^4 u}{\partial x^4} + \epsilon\zeta_4^1 + O(\epsilon^2), \\ &\frac{\partial^5\bar{u}}{\partial\bar{x}^5} = \frac{\partial^5 u}{\partial x^5} + \epsilon\zeta_5^1 + O(\epsilon^2), \\ &\ldots, \end{aligned} \tag{3}$$

where τ, ξ and η are infinitesimals and ζ_1^1, ζ_2^1, ... and ζ_α^0 are extended infinitesimals of orders $1, 2, \ldots$ and α, respectively.

The explicit expression for ζ_1^1, ζ_2^1, ..., are given as follows

$$\begin{aligned}
&\zeta_1^1 = D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau),\\
&\zeta_2^1 = D_x(\zeta_1^1) - u_{xx} D_x(\xi) - u_{xt} D_x(\tau),\\
&\zeta_3^1 = D_x(\zeta_2^1) - u_{xxx} D_x(\xi) - u_{xxt} D_x(\tau),\\
&\zeta_4^1 = D_x(\zeta_3^1) - u_{xxxx} D_x(\xi) - u_{xxxt} D_x(\tau),\\
&\zeta_5^1 = D_x(\zeta_4^1) - u_{xxxxx} D_x(\xi) - u_{xxxxt} D_x(\tau),\\
&\quad \dots,
\end{aligned}$$

where D_i denotes total derivative operator and defined as

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \cdots, \quad i = 1, 2, \dots,$$

with infinitesimal generator $X = \tau\partial_t + \xi\partial_x + \eta\partial_u$. Since the lower terminal of the integral in time fractional Riemann-Liouville derivative is fixed and, therefore it should be invariant with respect to the transformations (3). Such invariance condition yields

$$\tau(x, t, u)|_{t=0} = 0. \tag{4}$$

The α-th extended infinitesimal related to the modified Riemann-Liouville derivative with (4) reads

$$\begin{aligned}
\zeta_\alpha^0 = &D_t^\alpha(\eta) + \xi D_t^\alpha u_x - D_t^\alpha(\xi u_x) + D_t^\alpha\big(D_t(\tau)u\big)\\
&- D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1} u,
\end{aligned} \tag{5}$$

where the operator D_t^α denotes the total fractional derivative operator and is defined as

$$\begin{aligned}
D_t^\alpha = &\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\partial^\alpha u}{\partial t^\alpha}\partial_u + \frac{\partial^\alpha u_t}{\partial t^\alpha}\partial_{u_t} + \frac{\partial^\alpha u_x}{\partial t^\alpha}\partial_{u_x}\\
&+ \frac{\partial^\alpha u_{xx}}{\partial t^\alpha}\partial_{u_{xx}} + \dots .
\end{aligned}$$

Using the Leibnitz' formula, (5) can be presented as

$$\begin{aligned}
\zeta_\alpha^0 = &D_t^\alpha \eta - \alpha D_t \tau \frac{\partial^\alpha u}{\partial t^\alpha} - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n \xi D_t^{\alpha-n} u_x\\
&- \sum_{n=1}^{\infty} \binom{\alpha}{n+1} D_t^{n+1} \tau D_t^{\alpha-n} u.
\end{aligned} \tag{6}$$

Further, using Faà di Bruno's formula along with the Leibnitz' formula for the modified Riemann-Liouville derivative with $\varphi(x, t) = 1$, one can be read the first term $D_t^\alpha \eta$ in (6) as

$$\begin{aligned}
D_t^\alpha \eta = &\frac{\partial^\alpha \eta}{\partial t^\alpha} + \eta_u \frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha}\\
&+ \sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{\partial^n \eta_u}{\partial t^n} D_t^{\alpha-n} u + \omega,
\end{aligned}$$

where

$$\begin{aligned}
\omega = &\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1} \binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}\\
&\times (-u)^r \frac{\partial^m u^{k-r}}{\partial t^m}\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}.
\end{aligned}$$

As a consequence the α-th extended infinitesimal presented in (6) becomes

$$\begin{aligned}
\zeta_\alpha^0 = &\frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t \tau)\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha}\\
&+ \sum_{n=1}^{\infty}\Big(\binom{\alpha}{n}\frac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}\tau\Big) D_t^{\alpha-n} u\\
&- \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n} u_x + \omega.
\end{aligned} \tag{7}$$

For invariance of FPDE (2) under transformations (3), we obtain

$$\begin{aligned}
&\frac{\partial^\alpha \bar{u}}{\partial \bar{t}^\alpha} + F(\bar{x}, \bar{t}, \bar{u}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{x}\bar{x}}, \bar{u}_{\bar{x}\bar{x}\bar{x}}, \bar{u}_{\bar{x}\bar{x}\bar{x}\bar{x}}, \bar{u}_{\bar{x}\bar{x}\bar{x}\bar{x}\bar{x}}, \dots)\\
&\quad = 0,
\end{aligned} \tag{8}$$

for any solution $\bar{u} = u(\bar{x}, \bar{t})$ of FPDE (2). Taking into account the higher order of the nonlinear FPDEs, expanding (8) about $\epsilon = 0$ and making use of infinitesimals and their extensions (3) and equating the coefficients of ϵ, and neglecting the terms of higher powers of ϵ, we give the revised invariant equation of FPDE

$$\begin{aligned}
&\zeta_\alpha^0 + \xi\frac{\partial F}{\partial x} + \tau\frac{\partial F}{\partial t} + \eta\frac{\partial F}{\partial u} + \zeta_1^1\frac{\partial F}{\partial u_x} + \zeta_2^1\frac{\partial F}{\partial u_{xx}}\\
&\quad + \zeta_3^1\frac{\partial F}{\partial u_{xxx}} + \zeta_4^1\frac{\partial F}{\partial u_{xxxx}} + \zeta_5^1\frac{\partial F}{\partial u_{xxxxx}} + \cdots\\
&\quad = 0,
\end{aligned} \tag{9}$$

which is known as the invariant equation of FPDE (2). Now solving the invariant equation (9) with (2), we can determine τ, ξ, η explicitly. Notice that the expression for ω given in (8) vanishes when the infinitesimal η is linear in u.

Definition 1: A solution $u = v(x, t)$ is said to be an invariant solution of FPDE (2) if and only if

(i) $u = v(x, t)$ is an invariant surface, i.e. $Xv = 0$,

(ii) $u = v(x, t)$ satisfies FPDE (2).

III. General time fractional fifth-order KdV equation

Let us assume that Eq. (1) is invariant under a one parameter transformations (3), and so the transformed equation is read as

$$\frac{\partial^\alpha \bar{u}}{\partial \bar{t}^\alpha} + a\bar{u}_{\bar{x}}\bar{u}_{\bar{x}\bar{x}} + b\bar{u}\bar{u}_{\bar{x}\bar{x}\bar{x}} + c\bar{u}^2\bar{u}_{\bar{x}} + d\bar{u}_{\bar{x}\bar{x}\bar{x}\bar{x}\bar{x}} = 0. \tag{10}$$

Making use of transformations (3) in (10), we obtain invariant equation to Eq. (1)

$$\begin{aligned}
&\zeta_\alpha^0 + (au_{xx} + cu^2)\zeta_1^1 + au_x\zeta_2^1 + bu\zeta_3^1 + d\zeta_5^1\\
&\quad + \eta(bu_{xxx} + 2cuu_x) = 0,
\end{aligned} \tag{11}$$

which depend on variables u_x, u_t, u_{xx}, u_{xt}, u_{xxx}, u_{xxt}, u_{xxxx}, ..., and $D_t^{\alpha-n}u$, $D_t^{\alpha-n}u_x$ for $n = 1, 2, \dots$ which are considered to be independent. Such a structure of (11) allows one to reduce it into a system of infinitely many linear TFPDEs. Substituting the expressions of $\zeta_1^1, \zeta_2^1, \zeta_3^1, \zeta_5^1$ and ζ_α^0 into (11) and equating various power coefficients of derivatives for u to zero, we obtain an over determined system of linear equations

$$\left\{\begin{array}{l} \xi_u=\xi_t=\tau_u=\tau_x=\eta_{uu}=0, \\ 5\xi_x-\alpha\tau_t=0, \\ 2\xi_{xx}-\eta_{xu}=0, \\ b\eta+bu(\alpha\tau_t-3\xi_x)+d(10\eta_{xxu}-10\xi_{xxx} \\ \quad -3\tau_{xxu})=0, \\ a\eta_x+bu(3\eta_{xu}-3\xi_{xx})+d(10\eta_{xxxu}-5\xi_{xxxx} \\ \quad -3\tau_{xxxu})=0, \\ a\eta_{xx}+bu(3\eta_{xxu}-\xi_{xxx})+cu^2(\alpha\tau_t-\xi_x) \\ \quad +d(5\eta_{xxxxu}-\xi_{xxxxx})+2c\eta u=0, \\ \binom{\alpha}{n}\frac{\partial^n\eta_u}{\partial t^n}-\binom{\alpha}{n+1}D_t^{n+1}\tau=0 \\ \quad \text{for} \quad n=1,2,\ldots, \\ \frac{\partial^\alpha\eta}{\partial t^\alpha}-u\frac{\partial^\alpha\eta_u}{\partial t^\alpha}+bu\eta_{xxx}+cu^2\eta_x+d\eta_{xxxxx} \\ \quad =0. \end{array}\right. \tag{12}$$

Solving system (12) consistently, we obtain a generalized vector field to Eq. (1)

$$X_1=x\partial_x+\frac{5t}{\alpha}\partial_t-2u\partial_u, \quad X_2=\partial_x, \quad X_3=\partial_t^\alpha,$$

and so the underlying Lie algebra to Eq. (1) is two dimensional with generalized basis

$$\Big\{X_1,X_2,X_3,X_3+\frac{v}{\Gamma(1+\alpha)}X_2\Big\},$$

where v is nonzero constant coefficient.

In preceding section, we deal with symmetry reductions [22], [23] and exact solutions to Eq. (1), consider similarity reductions and group-invariant solutions based on optimal dynamical system method. From an optimal system of group-invariant solutions to an equation, every other such solution to the equation can be derived.

For the generator X_1, the similarity variable and similarity transformation corresponding to an infinitesimal generator can be obtained by solving the associated characteristic equation and given as

$$u=t^{-\frac{2\alpha}{5}}f(z), \quad z=xt^{-\frac{\alpha}{5}}. \tag{13}$$

By using the above similarity transformation (13), Eq. (1) can be transformed into a FODE and has the following theorem.

Theorem 1: The similarity transformation $u=t^{-\frac{2\alpha}{5}}f(z)$ along with the similarity variable $z=xt^{-\frac{\alpha}{5}}$ reduces Eq. (1) into a nonlinear constant coefficient ordinary differential equation with variable z

$$\begin{aligned}&\Big(P_{\frac{5}{\alpha}}^{1-\frac{7\alpha}{5},\alpha}f\Big)(z)+af'(z)f''(z)+bf(z)f'''(z)\\&\quad+cf^2(z)f'(z)+df'''''(z)=0,\end{aligned} \tag{14}$$

where $'=\frac{d}{dz}$, with Erdélyi-Kober fractional differential operator

$$\begin{aligned}&\big(P_\delta^{\tau,\alpha}g\big)(z)=\\&\prod_{j=0}^{m-1}\Big(\tau+j-\frac{1}{\delta}z\frac{d}{dz}\Big)\Big(K_\delta^{\tau+\alpha,m-\alpha}g\Big)(z),\\&z>0,\ \alpha>0,\ \delta>0,\\&m=\left\{\begin{array}{ll}[\alpha]+1, & \alpha\notin\mathbb{N},\\ \alpha, & \alpha\in\mathbb{N},\end{array}\right.\end{aligned} \tag{15}$$

and

$$\begin{aligned}&\big(K_\delta^{\tau,\alpha}g\big)(z)=\\&\left\{\begin{array}{ll}\frac{1}{\Gamma(\alpha)}\int_1^\infty(v-1)^{\alpha-1}v^{-\tau-\alpha}g(zv^{\frac{1}{\delta}})dv, & \alpha>0,\\ g(z), & \alpha=0,\end{array}\right.\end{aligned} \tag{16}$$

is Erdélyi-Kober fractional integral operator.

Proof: Let $n-1\leq\alpha\leq n$, $n=1,2,\ldots$. Thus the modified Riemann-Liouville derivative for the similarity transformation (13) becomes

$$\begin{aligned}&\frac{\partial^\alpha u}{\partial t^\alpha}=\\&\quad\frac{\partial^n}{\partial t^n}\Big(\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}s^{-\frac{2\alpha}{5}}f(xs^{-\frac{\alpha}{5}})ds\Big).\end{aligned}$$

Set $v=\frac{t}{s}$. Then the above equation can be written as

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha}&=\frac{\partial^n}{\partial t^n}\Big(t^{n-\frac{7\alpha}{5}}\frac{1}{\Gamma(n-\alpha)}\int_1^\infty(v-1)^{n-\alpha-1}\\&\quad\times v^{-(n+1-\frac{7\alpha}{5})}f(zv^{\frac{\alpha}{5}})dv\Big).\end{aligned}$$

Following the definition of the Erdélyi-Kober fractional integral operator given in (16), we have

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^n}{\partial t^n}\Big(t^{n-\frac{7\alpha}{5}}\big(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha}f\big)(z)\Big). \tag{17}$$

In order to simplify (17), we consider the relation $(z=xt^{-\frac{\alpha}{5}}, \phi\in C^1(0,\infty))$, then $t\frac{\partial}{\partial t}\phi(z)=-\frac{\alpha}{5}z\frac{d}{dz}\phi(z)$ and so, we have

$$\begin{aligned}&\frac{\partial^n}{\partial t^n}\Big(t^{n-\frac{7\alpha}{5}}\big(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha}\big)\Big)\\&\quad=\frac{\partial^{n-1}}{\partial t^{n-1}}\Big(\frac{\partial}{\partial t}\Big(t^{n-\frac{7\alpha}{5}}\big(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha}f\big)(z)\Big)\Big)\\&\quad=\frac{\partial^{n-1}}{\partial t^{n-1}}\Big(t^{n-1-\frac{7\alpha}{5}}\Big(n-\frac{7\alpha}{5}-\frac{\alpha}{5}z\frac{d}{dz}\Big)\\&\qquad\times\Big(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha}f\Big)(z)\Big).\end{aligned}$$

Repeating the similar procedure for $n-1$ times, we have

$$\begin{aligned}&\frac{\partial^n}{\partial t^n}\Big(t^{n-\frac{7\alpha}{5}}\big(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha}\big)\Big)=\\&\quad t^{-\frac{7\alpha}{5}}\prod_{j=0}^{n-1}\Big(1+j-\frac{7\alpha}{5}-\frac{\alpha}{5}z\frac{d}{dz}\Big)\\&\qquad\times\Big(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha}f\Big)(z).\end{aligned}$$

Now using the definition of the Erdélyi-Kober fractional differential operator given in (15), the above equation can be written as

$$\frac{\partial^n}{\partial t^n}\Big(t^{n-\frac{7\alpha}{5}}\big(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha}\big)\Big)=t^{-\frac{7\alpha}{5}}\Big(P_{\frac{5}{\alpha}}^{1-\frac{7\alpha}{5},\alpha}f\Big)(z).$$

Thus we obtain an expression for the time fractional derivative

$$\frac{\partial^\alpha u}{\partial t^\alpha}=t^{-\frac{7\alpha}{5}}\Big(P_{\frac{5}{\alpha}}^{1-\frac{7\alpha}{5},\alpha}f\Big)(z).$$

Continuing further we find that Eq. (1) reduces into FODE (14). ∎

For the generator X_2, we have one trivial solution to Eq. (1) is $u(x,t)=c$, where c is an arbitrary constant.

For the generator X_3, we have

$$u = f(z), \tag{18}$$

where $z = x$. Substituting (18) into Eq. (1), we obtain the following ODE

$$af'(z)f''(z) + bf(z)f'''(z) + cf^2(z)f'(z) + df'''''(z) = 0,$$

where $f' = \frac{df}{dz}$.

For the generator $kX_3 + \frac{v}{\Gamma(1+\alpha)}X_2$, we have

$$u = f(z), \tag{19}$$

where $z = x - \frac{v}{\Gamma(1+\alpha)}t^{\alpha}$, v is regarded as the wave velocity. Substituting (19) into Eq. (1), we have

$$\begin{aligned} &af'(z)f''(z) + bf(z)f'''(z) + cf^2(z)f'(z) \\ &\quad + df'''''(z) - vf'(z) = 0, \end{aligned} \tag{20}$$

where $f' = \frac{df}{dz}$.

IV. Exact solutions of Eqs. (14), (20)

By seeking for exact solutions of the FPDEs, we mean that those can be obtained from some FODEs or, in general, from FPDEs of lower order than the original FPDE. In terms of this definition, the exact solutions to Eq. (1) are obtained actually in both of the preceding Sections II. In spite of this, we still want to detect explicit solutions expressed in terms of elementary or, at least, known functions of mathematical physics, in terms of quadratures, and so on. Notice that power series can be used to solve differential equation, including many complicated differential equations [24]–[27], and so we consider the exact analytic solutions to the reduced equation by using power series method. Once we have the exact analytic solutions of the reduced FODEs, the exact power series solutions to Eq. (1) are obtained.

4.1 Exact power series solution to Eq. (14)

Set

$$f(z) = \sum_{n=0}^{\infty} c_n z^n. \tag{21}$$

Substituting (21) into (14), it yields

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{\Gamma(1-\frac{2\alpha}{5}-\frac{n\alpha}{5})}{\Gamma(1-\frac{7\alpha}{5}-\frac{n\alpha}{5})} c_n z^n \\ &+ a\sum_{n=0}^{\infty}\sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)c_{k+1} \\ &\times c_{n+2-k}z^n \\ &+ b\sum_{n=0}^{\infty}\sum_{k=0}^{n}(n+1-k)(n+2-k)(n+3-k) \\ &\times c_k c_{n+3-k} z^n \\ &+ c\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)c_i c_{k-i} c_{n+1-k} z^n \\ &+ d\sum_{n=0}^{\infty}(n+1)(n+2)(n+3)(n+4)(n+5) \\ &\times c_{n+5} z^n = 0. \end{aligned} \tag{22}$$

From (22), comparing coefficients, for $n = 0$, one has

$$\begin{aligned} c_5 = &-\frac{1}{120d}\Big(\frac{\Gamma(1-\frac{2\alpha}{5})}{\Gamma(1-\frac{7\alpha}{5})}c_0 + 2ac_1c_2 + 6bc_0c_3 \\ &+ cc_0^2c_1\Big). \end{aligned} \tag{23}$$

For $n \geq 1$, we have the following recursion formula

$$\begin{aligned} c_{n+5} = &-\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)d} \\ &\times\Big(\frac{\Gamma(1-\frac{2\alpha}{5}-\frac{n\alpha}{5})}{\Gamma(1-\frac{7\alpha}{5}-\frac{n\alpha}{5})}c_n + a\sum_{k=0}^{n}(k+1) \\ &\times (n+1-k)(n+2-k)c_{k+1}c_{n+2-k} \\ &+ b\sum_{k=0}^{n}(n+1-k)(n+2-k)(n+3-k)c_k \\ &\times c_{n+3-k} + c\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)c_i c_{k-i} \\ &\times c_{n+1-k}\Big). \end{aligned} \tag{24}$$

From (23) and (24), we can obtain all the coefficients c_n $(n \geq 6)$ of the power series (21), e.g.,

$$\begin{aligned} c_6 = &-\frac{1}{720d}\Big(\frac{\Gamma(1-\frac{3\alpha}{5})}{\Gamma(1-\frac{8\alpha}{5})}c_1 + 2a(3c_1c_3 + 2c_2^2) \\ &+ 6b(4c_0c_4 + c_1c_3) + 2c(c_0^2c_2 + c_0c_1^2)\Big), \\ c_7 = &-\frac{1}{2520d}\Big(\frac{\Gamma(1-\frac{4\alpha}{5})}{\Gamma(1-\frac{9\alpha}{5})}c_2 + 6a(2c_1c_4 + 3c_2c_3) \\ &+ 6b(10c_0c_5 + 4c_1c_4 + c_2c_3) + c(3c_0^2c_3 \\ &+ 6c_0c_1c_2 + c_1^3)\Big), \end{aligned}$$

and so on.

Then for arbitrary chosen constants c_i $(i = 0, 1, \ldots, 4)$, the other terms of the sequence $\{c_n\}_{n=0}^{\infty}$ can be determined successively from (23) and (24) in a unique manner. This implies that for Eq. (14), there exists a power series solution (21) with the coefficients given by (23) and (24). Furthermore, it is easy to prove the convergence of the power series (21) with the coefficients given by (23) and (24). Therefore, this power series solution (21) to Eq. (14) is an exact analytic solution. Hence, the power series solution to Eq. (14) can be written as

$$\begin{aligned} f(z) = &c_0 + c_1z + c_2z^2 + c_3z^3 + c_4z^4 \\ &+ c_5z^5 + \sum_{n=1}^{\infty} c_{n+5}z^{n+5} \\ = &c_0 + c_1z + c_2z^2 + c_3z^3 + c_4z^4 \\ &- \frac{1}{120d}\Big(\frac{\Gamma(1-\frac{2\alpha}{5})}{\Gamma(1-\frac{7\alpha}{5})}c_0 + 2ac_1c_2 + 6bc_0c_3 \\ &+ cc_0^2c_1\Big)z^5 \\ &- \sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)d} \end{aligned}$$

$$\times\Big(\frac{\Gamma(1-\frac{2\alpha}{5}-\frac{n\alpha}{5})}{\Gamma(1-\frac{7\alpha}{5}-\frac{n\alpha}{5})}c_n+a\sum_{k=0}^{n}(k+1)$$
$$\times(n+1-k)(n+2-k)c_{k+1}c_{n+2-k}$$
$$+b\sum_{k=0}^{n}(n+1-k)(n+2-k)(n+3-k)c_k$$
$$\times c_{n+3-k}+c\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)c_ic_{k-i}c_{n+1-k}\Big)$$
$$\times z^{n+5}.$$

Thus exact power series solution to Eq. (1) is

$$\begin{aligned}u(x,t)&=c_0t^{-\frac{2\alpha}{5}}+c_1xt^{-\frac{3\alpha}{5}}+c_2x^2t^{-\frac{4\alpha}{5}}+c_3x^3t^{-\alpha}\\&+c_4x^4t^{-\frac{6\alpha}{5}}-\frac{1}{120d}\Big(\frac{\Gamma(1-\frac{2\alpha}{5})}{\Gamma(1-\frac{7\alpha}{5})}c_0\\&+2ac_1c_2+6bc_0c_3+cc_0^2c_1\Big)x^5t^{-\frac{7\alpha}{5}}\\&-\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)d}\\&\times\Big(\frac{\Gamma(1-\frac{2\alpha}{5}-\frac{n\alpha}{5})}{\Gamma(1-\frac{7\alpha}{5}-\frac{n\alpha}{5})}c_n+a\sum_{k=0}^{n}(k+1)\\&\times(n+1-k)(n+2-k)c_{k+1}c_{n+2-k}\\&+b\sum_{k=0}^{n}(n+1-k)(n+2-k)(n+3-k)\\&\times c_kc_{n+3-k}+c\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)c_ic_{k-i}\\&\times c_{n+1-k}\Big)x^{n+5}t^{-\frac{(n+7)\alpha}{5}}.\end{aligned}$$

In mathematical and physical applications, the solution to Eq. (1) will be written as the following approximate form

$$\begin{aligned}u(x,t)&=c_0t^{-\frac{2\alpha}{5}}+c_1xt^{-\frac{3\alpha}{5}}+c_2x^2t^{-\frac{4\alpha}{5}}+c_3x^3t^{-\alpha}\\&+c_4x^4t^{-\frac{6\alpha}{5}}-\frac{1}{120d}\Big(\frac{\Gamma(1-\frac{2\alpha}{5})}{\Gamma(1-\frac{7\alpha}{5})}c_0\\&+2ac_1c_2+6bc_0c_3+cc_0^2c_1\Big)x^5t^{-\frac{7\alpha}{5}}\\&-\frac{1}{720d}\Big(\frac{\Gamma(1-\frac{3\alpha}{5})}{\Gamma(1-\frac{8\alpha}{5})}c_1+2a(3c_1c_3+2c_2^2)\\&+6b(4c_0c_4+c_1c_3)+2c(c_0^2c_2+c_0c_1^2)\Big)\\&\times x^6t^{-\frac{8\alpha}{5}}-\frac{1}{2520d}\Big(\frac{\Gamma(1-\frac{4\alpha}{5})}{\Gamma(1-\frac{9\alpha}{5})}c_2\\&+6a(2c_1c_4+3c_2c_3)+6b(10c_0c_5\\&+4c_1c_4+c_2c_3)+c(3c_0^2c_3+6c_0c_1c_2\\&+c_1^3)\Big)x^7t^{-\frac{9\alpha}{5}}+\cdots.\end{aligned}$$

4.2 Exact power series solution to Eq. (20)

Integrating Eq. (20), we have

$$\begin{aligned}&\frac{a-b}{2}(f'(z))^2+bf(z)f''(z)+\frac{c}{3}f^3(z)\\&\quad+df''''(z)-vf(z)+e=0,\end{aligned}\tag{25}$$

where e is an integration constant. Substituting (21) into (25), we have

$$\begin{aligned}&\frac{a-b}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}(k+1)(n+1-k)c_{k+1}c_{n+1-k}z^n\\&+b\sum_{n=0}^{\infty}\sum_{k=0}^{n}(n+1-k)(n+2-k)c_kc_{n+2-k}z^n\\&+\frac{c}{3}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}c_ic_{k-i}c_{n-k}z^n\\&+d\sum_{n=0}^{\infty}(n+1)(n+2)(n+3)(n+4)c_{n+4}z^n\\&-v\sum_{n=0}^{\infty}c_nz^n+e=0.\end{aligned}\tag{26}$$

From (26), comparing coefficients, for $n=0$, one has

$$c_4=-\frac{1}{24d}\big(\frac{a-b}{2}c_1^2+2bc_0c_2+\frac{c}{3}c_0^3-vc_0+e\big).$$

For $n\geq 1$, we have the following recursion formula

$$\begin{aligned}c_{n+4}=&-\frac{1}{(n+1)(n+2)(n+3)(n+4)d}\\&\times\Big(\frac{a-b}{2}\sum_{k=0}^{n}(k+1)(n+1-k)c_{k+1}c_{n+1-k}\\&+b\sum_{k=0}^{n}(n+1-k)(n+2-k)c_kc_{n+2-k}\\&+\frac{c}{3}\sum_{k=0}^{n}\sum_{i=0}^{k}c_ic_{k-i}c_{n-k}-vc_n\Big).\end{aligned}\tag{27}$$

In view of (27), we can obtain all the coefficients c_n $(n\geq 5)$ of the power series (21), e.g.,

$$\begin{aligned}c_5&=-\frac{1}{120d}\big(2ac_1c_2+6bc_0c_3+cc_0^2c_1-vc_1\big),\\c_6&=-\frac{1}{360b}\big(3(a+b)c_1c_3+2ac_2^2+12bc_0c_4\\&\quad+cc_0(c_0c_2+c_1^2)-vc_2\big),\end{aligned}$$

and so on.

For arbitrary chosen constants c_i $(i=0,1,2,3)$, the other terms of the sequence $\{c_n\}_{n=0}^{\infty}$ can be determined successively from (27) in a unique manner, the power series solution to Eq. (25) can be written as

$$\begin{aligned}f(z)&=c_0+c_1z+c_2z^2+c_3z^3-\frac{1}{24d}\big(\frac{a-b}{2}c_1^2\\&+2bc_0c_2+\frac{c}{3}c_0^3-vc_0+e\big)z^4\\&-\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)(n+4)d}\\&\times\Big(\frac{a-b}{2}\sum_{k=0}^{n}(k+1)(n+1-k)c_{k+1}c_{n+1-k}\\&+b\sum_{k=0}^{n}(n+1-k)(n+2-k)c_kc_{n+2-k}\\&+\frac{c}{3}\sum_{k=0}^{n}\sum_{i=0}^{k}c_ic_{k-i}c_{n-k}-vc_n\Big)z^{n+4},\end{aligned}$$

and so exact power series solution to Eq. (1) is

$$\begin{aligned}
u(x,t) &= c_0 + c_1\big(x - \frac{v}{\Gamma(1+\alpha)}t^\alpha\big)\\
&+ c_2\big(x - \frac{v}{\Gamma(1+\alpha)}t^\alpha\big)^2 + c_3\big(x - \frac{v}{\Gamma(1+\alpha)}t^\alpha\big)^3\\
&- \frac{1}{24d}\big(\frac{a-b}{2}c_1^2 + 2bc_0c_2 + \frac{c}{3}c_0^3 - vc_0 + e\big)\\
&\times \big(x - \frac{v}{\Gamma(1+\alpha)}t^\alpha\big)^4\\
&- \sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)(n+4)d}\\
&\times \Big(\frac{a-b}{2}\sum_{k=0}^{n}(k+1)(n+1-k)c_{k+1}c_{n+1-k}\\
&+ b\sum_{k=0}^{n}(n+1-k)(n+2-k)c_kc_{n+2-k}\\
&+ \frac{c}{3}\sum_{k=0}^{n}\sum_{i=0}^{k}c_ic_{k-i}c_{n-k} - vc_n\Big)\\
&\times \big(x - \frac{v}{\Gamma(1+\alpha)}t^\alpha\big)^{n+4}.
\end{aligned}$$

In applications, it will be given the following approximate form

$$\begin{aligned}
u(x,t) &= c_0 + c_1\big(x - \frac{v}{\Gamma(1+\alpha)}t^\alpha\big)\\
&+ c_2\big(x - \frac{v}{\Gamma(1+\alpha)}t^\alpha\big)^2 + c_3\big(x - \frac{v}{\Gamma(1+\alpha)}t^\alpha\big)^3\\
&- \frac{1}{24d}\big(\frac{a-b}{2}c_1^2 + 2bc_0c_2 + \frac{c}{3}c_0^3 - vc_0 + e\big)\\
&\times \big(x - \frac{v}{\Gamma(1+\alpha)}t^\alpha\big)^4\\
&- \frac{1}{120d}\big(2ac_1c_2 + 6bc_0c_3 + cc_0^2c_1 - vc_1\big)\\
&\times \big(x - \frac{v}{\Gamma(1+\alpha)}t^\alpha\big)^5\\
&- \frac{1}{360b}\big(3(a+b)c_1c_3 + 2ac_2^2 + 12bc_0c_4\\
&+ cc_0(c_0c_2 + c_1^2) - vc_2\big)\big(x - \frac{v}{\Gamma(1+\alpha)}t^\alpha\big)^6\\
&+ \cdots.
\end{aligned}$$

V. Convergence

Now we show that convergence of the power series solution (21) to fifth-order KdV equation. In fact, from (24), we have

$$\begin{aligned}
|c_{n+5}| \le \frac{1}{|d|}\Bigg(&\frac{|\Gamma(1-\frac{2\alpha}{5}-\frac{n\alpha}{5})|}{|\Gamma(1-\frac{7\alpha}{5}-\frac{n\alpha}{5})|}|c_n|\\
&+ |a|\sum_{k=0}^{n}|c_{k+1}||c_{n+2-k}|\\
&+ |b|\sum_{k=0}^{n}|c_k||c_{n+3-k}|\\
&+ |c|\sum_{k=0}^{n}\sum_{i=0}^{k}|c_i||c_{k-i}||c_{n+1-k}|\Bigg),\\
&n = 1,2,\ldots.
\end{aligned} \tag{28}$$

Taking into account some properties of the Γ function, it is no difficulty to find that $\frac{|\Gamma(1-\frac{2\alpha}{5}-\frac{n\alpha}{5})|}{|\Gamma(1-\frac{7\alpha}{5}-\frac{n\alpha}{5})|} < 1$ for arbitrary n. Hence (28) is written as

$$\begin{aligned}
|c_{n+5}| \le M\Big(&|c_n| + \sum_{k=0}^{n}|c_{k+1}||c_{n+2-k}|\\
&+ \sum_{k=0}^{n}|c_k||c_{n+3-k}|\\
&+ \sum_{k=0}^{n}\sum_{i=0}^{k}|c_i||c_{k-i}||c_{n+1-k}|\Big),
\end{aligned}$$

where $M = \max\{\frac{1}{|d|}, \frac{|a|}{|d|}, \frac{|b|}{|d|}, \frac{|c|}{|d|}\}$.

Introduce a power series $A(z) = \sum_{n=0}^{\infty} a_nz^n$. Set

$$a_i = |c_i|, \quad i = 0,1,\ldots,5,$$

and

$$\begin{aligned}
a_{n+5} = M\Big(&a_n + \sum_{k=0}^{n}a_{k+1}a_{n+2-k} + \sum_{k=0}^{n}a_ka_{n+3-k}\\
&+ \sum_{k=0}^{n}\sum_{i=0}^{k}a_ia_{k-i}a_{n+1-k}\Big), \quad n = 1,2,\ldots.
\end{aligned}$$

Then, it is easily seen that

$$|c_n| \le a_n, \quad n = 0,1,\ldots.$$

In other words, the series $A(z) = \sum_{n=0}^{\infty} a_nz^n$ is majorant series of (28). Further, we show that the series $A(z)$ has positive radius of convergence. Indeed, note that by formal calculation, it yields

$$\begin{aligned}
A(z) &= a_0 + a_1z + a_2z^2 + a_3z^3 + a_4z^4\\
&+ M\sum_{n=0}^{\infty}a_nz^{n+5}\\
&+ M\sum_{n=0}^{\infty}\sum_{k=0}^{n}a_{k+1}a_{n+2-k}z^{n+5}\\
&+ M\sum_{n=0}^{\infty}\sum_{k=0}^{n}a_ka_{n+3-k}z^{n+5}\\
&+ M\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}a_ia_{k-i}a_{n+1-k}z^{n+5}.
\end{aligned}$$

Consider now implicit functional system with respect to the independent variable z

$$\begin{aligned}
\mathcal{A}(z,A) &= A - a_0 - a_1z - a_2z^2 - a_3z^3 - a_4z^4\\
&- Mz^2\Big(z^2A^3 + (2 - a_0z^2)A^2 + (z^3\\
&- 3a_0 - 2a_1z - a_2z^2)A + a_0^2 + a_0a_1z\Big).
\end{aligned}$$

Since $\mathcal{A}$ is analytic in a neighborhood of $(0,a_0)$ and $\mathcal{A}(0,a_0) = 0$. Furthermore,

$$\frac{\partial}{\partial A}\mathcal{A}(0,a_0) \ne 0.$$

By implicit function theorem [28], we see that $A(z) = \sum_{n=0}^{\infty} a_nz^n$ is analytic in neighborhood of the point $(0,a_0)$ and with a positive radius. This implies that the power series (28) converge in neighborhood of the point $(0, a_0)$.

Similarly, we can also show that convergence of the power series solution (21) to fifth-order KdV equation. In view of (27), we have

$$\begin{aligned}|c_{n+4}| \leq \frac{1}{|d|}\Big(\frac{|a-b|}{2}\sum_{k=0}^{n}|c_{k+1}||c_{n+1-k}| \\ + |b|\sum_{k=0}^{n}|c_k||c_{n+2-k}| \\ + \frac{|c|}{3}\sum_{k=0}^{n}\sum_{i=0}^{k}|c_i||c_{k-i}||c_{n-k}| + |v||c_n|\Big), \\ \leq M\Big(\sum_{k=0}^{n}|c_{k+1}||c_{n+1-k}| + \sum_{k=0}^{n}|c_k||c_{n+2-k}| \\ + \sum_{k=0}^{n}\sum_{i=0}^{k}|c_i||c_{k-i}||c_{n-k}| + |c_n|\Big), \\ n = 1, 2, \ldots,\end{aligned} \tag{29}$$

where $M = \max\{\frac{|a-b|}{2|d|}, \frac{|b|}{|d|}, \frac{|c|}{3|d|}, \frac{|v|}{|d|}\}$.

Introduce the power series $A(z) = \sum_{n=0}^{\infty} a_n z^n$. Set

$$a_i = |c_i|, \quad i = 0, 1, \ldots, 4,$$

and

$$\begin{aligned}a_{n+4} = M\Big(\sum_{k=0}^{n}a_{k+1}a_{n+1-k} + \sum_{k=0}^{n}a_k a_{n+2-k} \\ + \sum_{k=0}^{n}\sum_{i=0}^{k}a_i a_{k-i}a_{n-k} + a_n\Big), \\ n = 1, 2, \ldots.\end{aligned}$$

Then, it is easily seen that

$$|c_n| \leq a_n, \quad n = 0, 1, \ldots.$$

In other words, the series $A(z) = \sum_{n=0}^{\infty} a_n z^n$ is majorant series of (29). Further, we show that the series $A(z)$ has positive radius of convergence. Indeed, note that by formal calculation, it yields

$$\begin{aligned}A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 \\ + M\sum_{n=0}^{\infty}\sum_{k=0}^{n}a_{k+1}a_{n+1-k}z^{n+5} \\ + M\sum_{n=0}^{\infty}\sum_{k=0}^{n}a_k a_{n+2-k}z^{n+5} \\ + M\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}a_i a_{k-i}a_{n-k}z^{n+5} \\ + M\sum_{n=0}^{\infty}a_n z^{n+5}.\end{aligned}$$

Consider now implicit functional system with respect to the independent variable z

$$\begin{aligned}\mathcal{A}(z, A) = A - a_0 - a_1 z - a_2 z^2 - a_3 z^3 - a_4 z^4 \\ - Mz^3\Big(z^2A^3 + 2A^2 + (z^2 - a_1 z \\ - 3a_0)A + a_0^2\Big).\end{aligned}$$

Since $\mathcal{A}$ is analytic in a neighborhood of $(0, a_0)$ and $\mathcal{A}(0, a_0) = 0$. Furthermore,

$$\frac{\partial}{\partial A}\mathcal{A}(0, a_0) \neq 0.$$

We see that $A(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in neighborhood of the point $(0, a_0)$ and with a positive radius, it implies that the power series (29) converge in neighborhood of the point $(0, a_0)$.

Remark 1: When the coefficients a, b, c, d are suitable chosen, we can obtain exact power series solution and convergence of time fractional Kaup-Kupershmidt, Sawada-Kotera, Caudrey-Dodd-Gibbon, Lax, Ito equation, respectively.

VI. Conclusions

In this paper, the invariance properties of the general time fractional fifth-order KdV equation is presented in the sense of Lie point symmetry. All of the geometric vector fields and the symmetry reductions of the equation are obtained. The reduction of dimension in the symmetry algebra is due to the fact that the time FPDE is invariant under time translation symmetry. We have shown that the equation can be transformed into FODE, and then the exact analytic solution is obtained in terms of the power series method, we also show the convergence of the power series solution. Our results witness that symmetry analysis is very efficient and powerful technique in finding the solution of the proposed equation.

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