On Nonisospectral AKNS System with Infinite Number of Terms and its Exact Solutions

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***Abstract*—In this paper, a new nonisospectral parameter whose varying with time obeys the sine function of spectral parameter is first embedded into the famous Ablowitz–Kaup–Newell–Segur (AKNS) spectral problem. Starting from the AKNS spectral problem equipped with such a nonisospectral parameter and its corresponding time evolution equation, we then derive a new and more general nonisospectral AKNS system including infinite number of terms. Based on a systematic analysis of the time dependence of related scattering data, exact solutions of the derived AKNS system are further formulated through the inverse scattering transform (IST) method. Finally, in the case of reflectionless potentials, the obtained exact solutions are reduced to explicit *n*-soliton solutions. It is graphically shown that dynamical evolutions of the reduced soliton solutions can possess not only time-varying speeds and amplitudes but also singular points.**

***Index Terms*—Nonisospectral AKNS system; Exact solution; Soliton solution; Dynamical evolution; IST method.**

I. Introduction

In soliton theory, nonlinear evolution equations (NLEEs) associated with some linear spectral problems can be generally classified as the isospectral equations which often describe solitary waves in lossless and uniform media and the nonisospectral equations describing the solitary waves in a certain type of nonuniform media. Specifically, when the spectral parameter of the associated linear spectral problem is independent of time, we could construct isospectral NLEEs. While starting from the spectral problem with a time-dependent spectral parameter, nonisospectral NLEEs are usually derived. As early as in 1974, Ablowitz, Kaup, Newell and Segur [1] successfully constructed a hierarchy of isospectral NLEEs here written as

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L^n \begin{pmatrix} -q \\ r \end{pmatrix},\ (n = 0, 1, 2, \cdots), \tag{1}$$

via the compatibility condition

$$M_t - N_x + [M, N] = 0, \tag{2}$$

of the following spectral problem, i.e., the famous AKNS spectral problem

$$\varphi_x = M\varphi,\ M = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix},\ \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{3}$$

and its accompanied time evolution equation

$$\varphi_t = N\varphi,\ N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{4}$$

where the potential functions $q = q(x, t)$, $r = r(x, t)$ and their derivatives of any order with respect to x and t are smooth and vanish as x tends to infinity, the spectral parameter k is a constant, A, B, C are undetermined functions related to t, x, q, r and k, and the operator L is employed as

$$L = \sigma\partial + 2 \begin{pmatrix} q \\ -r \end{pmatrix} \partial^{-1}(r, q),\ \sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5}$$

with the help of $\partial = \partial/\partial x$, $\partial^{-1} = (\int_{-\infty}^{x} \mathrm{d}x - \int_{x}^{+\infty} \mathrm{d}x)/2$.

It is easy to see that (1) includes the following two nontrivial systems ($n = 1, 2$)

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} -q_{xx} + 2q^2 r \\ r_{xx} - 2qr^2 \end{pmatrix}, \tag{6}$$

and

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} q_{xxx} - 6qrq_x \\ r_{xxx} - 6qrr_x \end{pmatrix}. \tag{7}$$

If we set $q = u$, $r = -1$ and $q = v$ and $r = \mp v$, then (7) reduces to the celebrated Korteweg–de Vries (KdV) equation $u_t = u_{xxx} + 6uu_x$ and the modified KdV (mKdV) equation $v_t = v_{xxx} + 6v^2 v_x$, respectively.

Subsequently, in the case of spectral parameter k being dependent of time t, Celogero and Degasperis [2], [3], [4] and Li [5] proposed effective methods to derive different hierarchies of nonisospectral NLEEs. For example, the nonisospectral AKNS hierarchy [6]

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L^n \begin{pmatrix} -xq \\ xr \end{pmatrix},\ (n = 0, 1, 2, \cdots), \tag{8}$$

can be constructed as long as we select $ik_t = (2ik)^n/2$ and use (2)–(4). A direct computation tells that (8) gives the following three nonisospectral systems when $n = 1, 2, 3$

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} q + xq_x \\ r + xr_x \end{pmatrix}, \tag{9}$$

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} -2q_x - xq_{xx} + 2q\partial^{-1}(qr) + 2xq^2 r \\ 2r_x + xr_{xx} - 2r\partial^{-1}(qr) - 2xqr^2 \end{pmatrix}, \tag{10}$$

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} 3q_{xx} + xq_{xxx} - 2q_x\partial^{-1}(qr) \\ -4xqrq_x - 2xq^2 r_x - 8q\partial^{-1}(q_x r) \\ -2xqrq_{xx} + 2xq^2 r_{xx} \\ 3r_{xx} + xr_{xxx} - 2r_x\partial^{-1}(qr) \\ -4xqrr_x - 2xr^2 q_x - 8r\partial^{-1}(qr_x) \\ -2xqrr_{xx} + 2xr^2 q_{xx} \end{pmatrix}. \tag{11}$$

Being appearance of nonisospectral NLEEs, the types of integrable equations were substantially enriched. From then on, constructing nonisospectral NLEEs has attached much

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attention like those in [7], [8], [9], [10] and become one of the most important and significant research directions in nonlinear science. In this paper, we have two motivations: the first one is to embed a nonisospectral parameter k satisfying

$$ik_t = \frac{1}{2}\sin 2ik, \tag{12}$$

into the AKNS spectral problem (3) for constructing a new and more general nonisospectral AKNS system which includes infinite number of terms

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \sum_{j=0}^{+\infty}(-1)^j \frac{1}{(2j+1)!} L^{2j+1}\begin{pmatrix} -xq \\ xr \end{pmatrix}, \tag{13}$$

the other one is to exactly solve such a nonisospectral AKNS system (13) by the IST method [11] and then analyze the dynamical characteristics of the obtained exact soliton solutions in the process of evolutions. It should be noted that the nonisospectral AKNS system (13) cannot be contained by the known AKNS hierarchy (8) and it is more general than (9)–(11).

Since the initial-value problem of the celebrated KdV equation was exactly solved by the IST method in 1967 [11], many effective methods have been proposed for solving NLEEs, such as those in [12], [13], [14], [15], [16], [17], [18], [19], [20]. One of the advantages over other existing methods is that the IST can solve a whole hierarchy of NLEEs associated with the same spectral problem. As a famous technique in mathematical physics, the IST has developed to a systematic method and received a wide range of applications [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38]. However, to the best of our knowledge, the IST has not been extended to such systems associated with a nonisospectral parameter as introduced in (12).

The rest of the paper is organized as follows. Starting from the AKNS spectral problem (3) with the embedded nonisospectral parameter k determined by (12), in Section 2 we derive the nonisospectral AKNS system (13). In Section 3, we exactly solve the AKNS system (13) through the IST method. In more detail, based on a systematic analysis of the time dependence of related scattering data, the uniform formulae of exact solutions of the AKNS system (13) are obtained. Then the obtained exact solutions are reduced to explicit n-soliton solutions in the special case of reflectionless potentials. To analyze the dynamical characteristics of the obtained exact soliton solutions in the process of evolutions, we select $n = 1, 2$ to show by figures that the one-soliton solutions and two-soliton solutions possess not only time-varying speeds and amplitudes but also singular points. In Section 4, we conclude this paper.

II. Derivation of the AKNS system

Firstly, substituting the matrixes M and N of (3) and (4) into (2), then we reduce (2) as

$$A_x = qC - rB - ik_t, \tag{14}$$

$$q_t = B_x + 2ikB + 2qA, \tag{15}$$

$$r_t = C_x - 2ikC - 2rA. \tag{16}$$

Further integrating (14) with respective to x and using (12) yields

$$A = \partial^{-1}(r, q)\begin{pmatrix} -B \\ C \end{pmatrix} - \frac{1}{2}x\sin 2ik + A_0, \tag{17}$$

where A_0 is an arbitrary function of k and t. For convenience, we set $A_0 = 0$ and use Talor series expansion formula to rewrite $\sin 2ik$ as

$$\sin 2ik = \sum_{j=0}^{+\infty}(-1)^j \frac{1}{(2j+1)!}(2ik)^{2j+1}. \tag{18}$$

Then from (15) and (16) we have

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L\begin{pmatrix} -B \\ C \end{pmatrix} - 2ik\begin{pmatrix} -B \\ C \end{pmatrix} + \sum_{j=0}^{+\infty}(-1)^j \frac{1}{(2j+1)!}(2ik)^{2j+1}\begin{pmatrix} -xq \\ xr \end{pmatrix}, \tag{19}$$

Secondly, we suppose that

$$\begin{pmatrix} -B \\ C \end{pmatrix} = \sum_{s=1}^{+\infty}\begin{pmatrix} -b_s \\ c_s \end{pmatrix}(2ik)^{s-1}, \tag{20}$$

where when n is odd and $n \to +\infty$ the following asymptotic condition is assumed

$$\begin{pmatrix} -b_n \\ c_n \end{pmatrix} = (-1)^{\frac{n-1}{2}}\frac{1}{n!}\begin{pmatrix} -xq \\ xr \end{pmatrix}. \tag{21}$$

Substituting (20) into (19) and comparing the coefficients of the same powers of $2ik$ in (19), we have

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L\begin{pmatrix} -b_1 \\ c_1 \end{pmatrix}, \tag{22}$$

$$\begin{pmatrix} -b_{s-1} \\ c_{s-1} \end{pmatrix} = L\begin{pmatrix} -b_s \\ c_s \end{pmatrix} + \frac{(-1)^{\frac{s}{2}+1} - (-1)^{\frac{3s}{2}+2}}{2(s-1)!}\begin{pmatrix} -xq \\ xr \end{pmatrix}, \ s = 2, 3, \cdots. \tag{23}$$

Making use of (23), we have

$$\begin{pmatrix} -b_1 \\ c_1 \end{pmatrix} = \sum_{j=1}^{+\infty}(-1)^j \frac{1}{(2j+1)!} L^{2j}\begin{pmatrix} -xq \\ xr \end{pmatrix}. \tag{24}$$

Finally, substituting (24) into (22) we obtain the AKNS system (13).

III. Exact solutions and soliton dynamics

In this section, the time dependence of scattering data is first determined for the AKNS spectral problem (3) equipped with the nonisospectral k in (12). Based on the determined scattering data, exact solutions of the AKNS hierarchy (13) are then obtained. Finally, the obtained exact solutions are reduced to soliton solutions and the dynamical characteristics of such soliton solutions are analyzed.

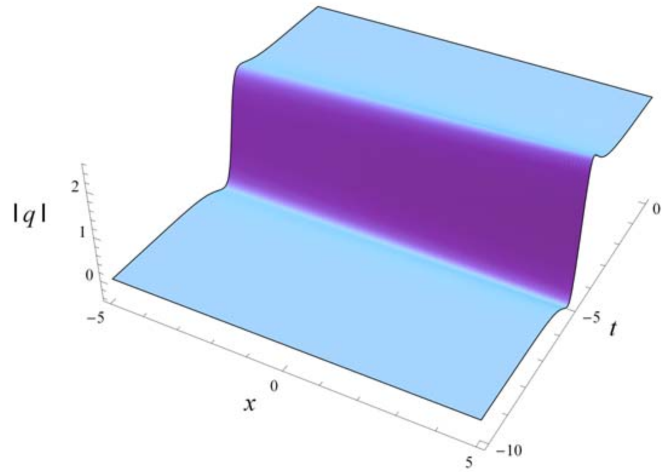

Fig. 1. Spatial structure of bright and dark one-soliton determined by solution (85)

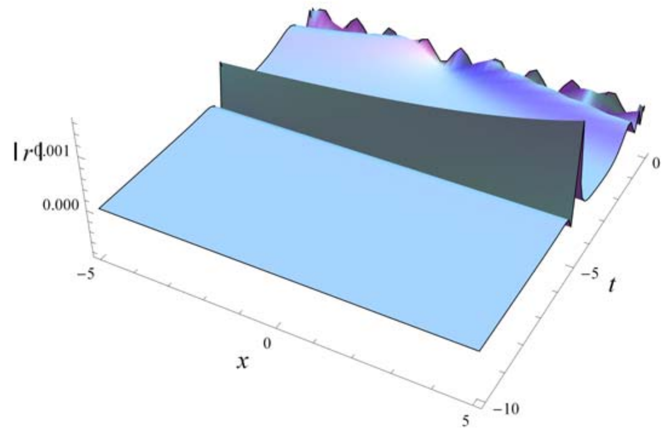

Fig. 2. Spatial structure of bright and dark one-soliton determined by solution (86)

A. The time dependence of the scattering data

Theorem 1: If the spectral problem (3) is equipped with the nonisospectral k in (12), then the scattering data

$$\left\{\kappa_j(t),\ c_j(t),\ R(t,k)=\frac{b(k,t)}{a(k,t)},\ j=1,2,\cdots,n\right\},$$

$$\left\{\bar{\kappa}_m(t),\ \bar{c}_m(t),\ \bar{R}(k,t)=\frac{\bar{b}(k,t)}{\bar{a}(k,t)},\ m=1,2,\cdots,\bar{n}\right\},$$

possess the following time dependence

$$\kappa_j(t)=\frac{1}{2}\ln\frac{(e^{2\kappa_j(0)}+1)e^{-t}+\mathrm{e}^{2\kappa_j(0)}-1}{(e^{2\kappa_j(0)}+1)e^{-t}-e^{2\kappa_j(0)}+1},\tag{25}$$

$$c_j^2(t)=c_j^2(0)e^{\int_0^t \sin(2i\kappa_j(w))\mathrm{d}w},\tag{26}$$

$$a(k,t)=a(k,0),\tag{27}$$

$$b(k,t)=b(k,0),\tag{28}$$

$$\bar{\kappa}_m(t)=\frac{1}{2}\ln\frac{(e^{2\bar{\kappa}_m(0)}+1)e^{-t}+e^{2\bar{\kappa}_m(0)}-1}{(e^{2\bar{\kappa}_m(0)}+1)e^{-t}-e^{2\bar{\kappa}_m(0)}+1},\tag{29}$$

$$\bar{c}_m^2(t)=\bar{c}_m^2(0)e^{-\int_0^t \sin(2i\kappa_m(w))\mathrm{d}w},\tag{30}$$

$$\bar{a}(k,t)=\bar{a}(k,0),\tag{31}$$

$$\bar{b}(k,t)=\bar{b}(k,0),\tag{32}$$

where $c_j^2(0)$, $\bar{c}_m^2(0)$, $\kappa_j(0)$, $\bar{\kappa}_m(0)$, $R(k,0)=b(k,0)/a(k,0)$ and $\bar{R}(k,0)=\bar{b}(k,0)/\bar{a}(k,0)$ are the corresponding scattering data of (3) in the case of $(q(x,0),r(x,0))^T$.

Proof: We can easily see if $\phi(x,k)$ is a solution of (3) equipped with the nonisospectral k in (12) then $P(x,k)=\phi_t(x,k)-N\phi(x,k)$ is another solution of (3). Thus, $P(x,k)$ can be represented by $\phi(x,k)$ and $\widetilde{\phi}(x,k)$ which also satisfies (3) but is independent of $\phi(x,k)$, i.e., there exist two functions $\alpha(k,t)$ and $\beta(k,t)$ so that

$$\phi_t(x,k)-N\phi(x,k)=\alpha(k,t)\phi(x,k)+\beta(k,t)\widetilde{\phi}(x,k).\tag{33}$$

Firstly, we consider the discrete spectral $k=\kappa_j(\mathrm{Im}\kappa_j>0)$. Since when $x\to+\infty$, $\phi(x,\kappa_j)$ decays exponentially while $\widetilde{\phi}(x,\kappa_j)$ must increases exponentially, we then have $\beta(k,t)=0$. Thus, (33) is simplified as:

$$\phi_t(x,\kappa_j)-N\phi(x,\kappa_j)=\alpha(\kappa_j,t)\phi(x,\kappa_j).\tag{34}$$

Using the inner product $(\phi_2(x,\kappa_j),\phi_1(x,\kappa_j))$ to left-multiply (34), we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\phi_1(x,\kappa_j)\phi_2(x,\kappa_j)-[C\phi_1^2(x,\kappa_j)+B\phi_2^2(x,\kappa_j)]=2\alpha(\kappa_j,t)\phi_1(x,\kappa_j)\phi_2(x,\kappa_j).\tag{35}$$

When $\phi(x,\kappa_j)$ is presumed to be a normalization eigenfunction, using

$$2\int_{-\infty}^{\infty}c_j^2(t)\phi_1(x,\kappa_j)\phi_2(x,\kappa_j)\mathrm{d}x=1,\tag{36}$$

we then have

$$\alpha(\kappa_j,t)=-c_j^2(t)\int_{-\infty}^{\infty}[C\phi_1^2(x,\kappa_j)+B\phi_2^2(x,\kappa_j)]\mathrm{d}x,\tag{37}$$

which can be rewritten as:

$$\alpha(\kappa_j,t)=-c_j^2(t)((\phi_2^2(x,\kappa_j),\phi_1^2(x,\kappa_j))^T,(B,C)^T),\tag{38}$$

where the following inner product had been used

$$(f(x),g(x))=\int_{-\infty}^{\infty}[f_1(x)g_1(x)+f_2(x)g_2(x)]\mathrm{d}x\tag{39}$$

for arbitrary two vectors $f(x)=(f_1(x),f_2(x))^T$ and $g(x)=(g_1(x),g_2(x))^T$.

From (3), we have

$$\varphi_{1x}(x,\kappa_j)+i\kappa_j\varphi_1(x,\kappa_j)=q(x)\varphi_2(x,\kappa_j),\tag{40}$$

$$\varphi_{2x}(x,\kappa_j)-i\kappa_j\varphi_2(x,\kappa_j)=r(x)\varphi_1(x,\kappa_j),\tag{41}$$

and then obtain

$$[\varphi_1(x,\kappa_j)\varphi_2(x,\kappa_j)]_x=q(x)\varphi_2^2(x,\kappa_j)+r(x)\varphi_1^2(x,\kappa_j).\tag{42}$$

The integration of (42) with respect to x from $-\infty$ to $+\infty$ gives

$$\int_{-\infty}^{\infty}[q(x)\phi_2^2(x,\kappa_j)+r(x)\phi_1^2(x,\kappa_j)]\mathrm{d}x=\int_{-\infty}^{\infty}[\phi_1(x,\kappa_j)\phi_2(x,\kappa_j)]_x\mathrm{d}x=0.\tag{43}$$

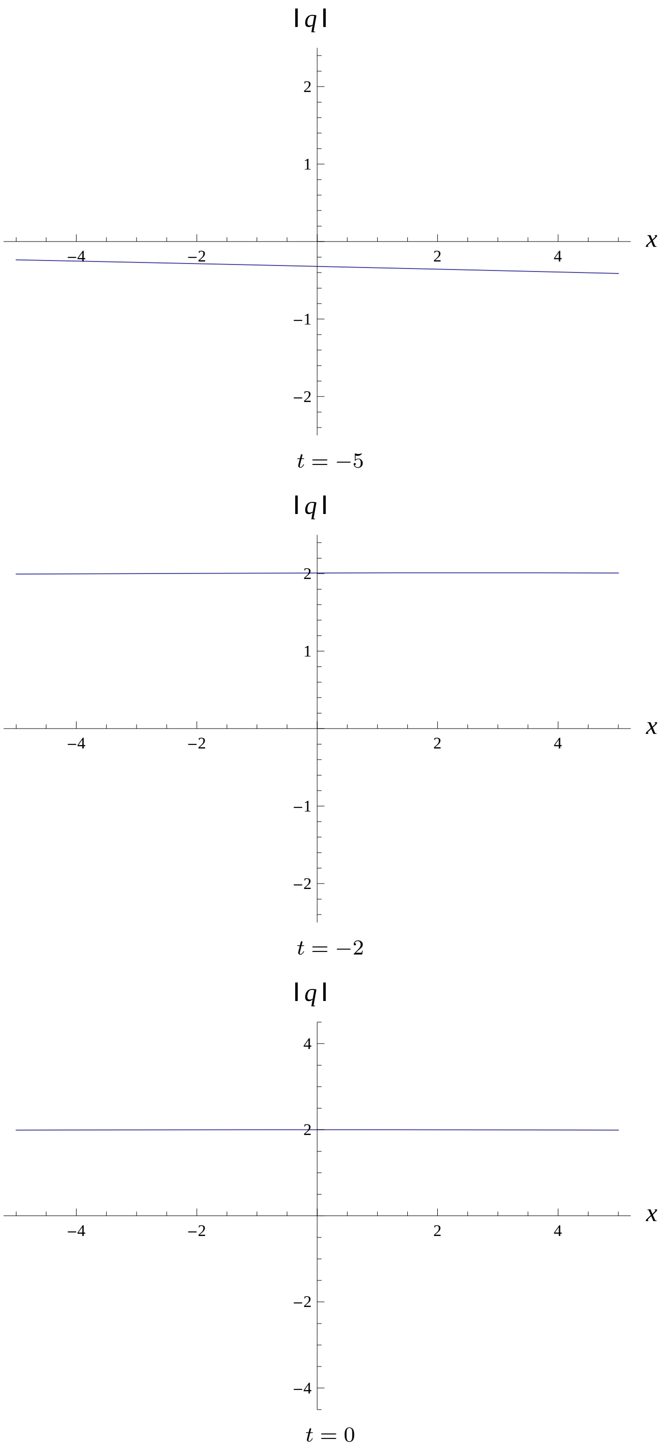

Fig. 3. Dynamical evolutions of one-soliton determined by solution (85).

In the other hand, (20) can be rewritten as

$$\begin{pmatrix} B \\ C \end{pmatrix} = \lim_{n\to+\infty}\sum_{s=1}^{n}\sum_{j=s}^{n}(-1)^{\frac{j-1}{2}}\frac{1}{j!}\bar{L}^{j-s}\begin{pmatrix} xq \\ xr \end{pmatrix}(2i\kappa_j)^{s-1}, \tag{44}$$

by introducing

$$\bar{L} = \sigma\partial - 2\begin{pmatrix} q \\ r \end{pmatrix}\partial^{-1}(-r, q),$$

and then from (38) we obtain

$$\alpha(\kappa_j, t) = -c_j^2(t)\Bigg((\phi_2^2(x,\kappa_j), \phi_1^2(x,\kappa_j))^T,$$

$$\lim_{n\to+\infty}\sum_{s=1}^{n}\sum_{j=s}^{n}(-1)^{\frac{j-1}{2}}\frac{1}{j!}\bar{L}^{j-s}\begin{pmatrix} xq \\ xr \end{pmatrix}(2i\kappa_j)^{s-1}\Bigg)$$

$$= \frac{1}{2}\lim_{n\to+\infty}\sum_{l=0}^{n-1}(-1)^l\frac{1}{(2l+1)!}(2i\kappa_j)^{2l+1} = \frac{1}{2}\sin 2i\kappa_j, \tag{45}$$

by means of the following results

$$\bar{L}^{*j-s}(\phi_2^2(x,\kappa_j), \phi_1^2(x,\kappa_j))^T$$
$$= (2i\kappa_j)^{j-s}(\phi_2^2(x,\kappa_j), \phi_1^2(x,\kappa_j))^T,$$

$$\left((\phi_2^2(x,\kappa_j), \phi_1^2(x,\kappa_j))^T, \begin{pmatrix} xq \\ xr \end{pmatrix}\right)$$
$$= \int_{-\infty}^{\infty} x[\phi_1(x,\kappa_j)\phi_2(x,\kappa_j)]_x \mathrm{d}x = -\frac{1}{2c_j^2(t)},$$

where $\bar{L}^*$ is the conjugation operator of $\bar{L}$ [24]

$$\bar{L}^* = -\sigma\partial + 2\begin{pmatrix} -r \\ q \end{pmatrix}\partial^{-1}(q, r), \ \bar{L} = \sigma L\sigma.$$

In view of (45), (34) is simplify as

$$\phi_t(x,\kappa_j) - N\phi(x,\kappa_j) = \frac{1}{2}\sin 2i\kappa_j\phi(x,\kappa_j). \tag{46}$$

Since

$$N \to \begin{pmatrix} -\frac{1}{2}x\sin 2i\kappa_j & 0 \\ 0 & \frac{1}{2}x\sin 2i\kappa_j \end{pmatrix}, \tag{47}$$

$$\phi(x,\kappa_j) \to c_j(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{i\kappa_j x}, \tag{48}$$

$$\phi_t(x,\kappa_j) \to c_{jt}(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{i\kappa_j x} + i\kappa_{jt}xc_j(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{i\kappa_j x}, \tag{49}$$

$$\kappa_{jt} = -\frac{i}{2}\sin 2i\kappa_j, \tag{50}$$

as $x \to +\infty$, then (46)–(50) give

$$c_{jt}(t) = \frac{1}{2}c_j(t)\sin 2i\kappa_j. \tag{51}$$

Similarly, we have

$$\bar{c}_{mt}(t) = -\frac{1}{2}\bar{c}_m(t)\sin 2i\bar{\kappa}_m. \tag{52}$$

Secondly, we consider k as a real continuous spectral and take a solution $\varphi(x,k)$ of (3) equipped with the non-isospectral k in (12), then $Q(x,k) = \varphi_t(x,k) - N\varphi(x,k)$ is another solution of (3) and hence can be represented linearly by $\varphi(x,k)$ and $\bar{\varphi}(x,k)$ which also satisfies (3) but is independent of $\varphi(x,k)$, i.e., there exist two functions $\omega(k,t)$ and $\vartheta(k,t)$ so that

$$\varphi_t(x,k) - N\varphi(x,k) = \omega(k,t)\varphi(x,k) + \vartheta(k,t)\bar{\varphi}(x,k). \tag{53}$$

Noting the asymptotical properties

$$\varphi_t(x,k) \to -ik_t x\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{-ikx}, \ \varphi(x,k) \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{-ikx}, \tag{54}$$

$$\bar{\varphi}(x,k) \to \begin{pmatrix} 0 \\ -1 \end{pmatrix}e^{ikx}, \tag{55}$$

as $x \to -\infty$, from (53) and (12) we obtain $\vartheta(k,t) = 0$ and $\omega(k,t) = 0$.

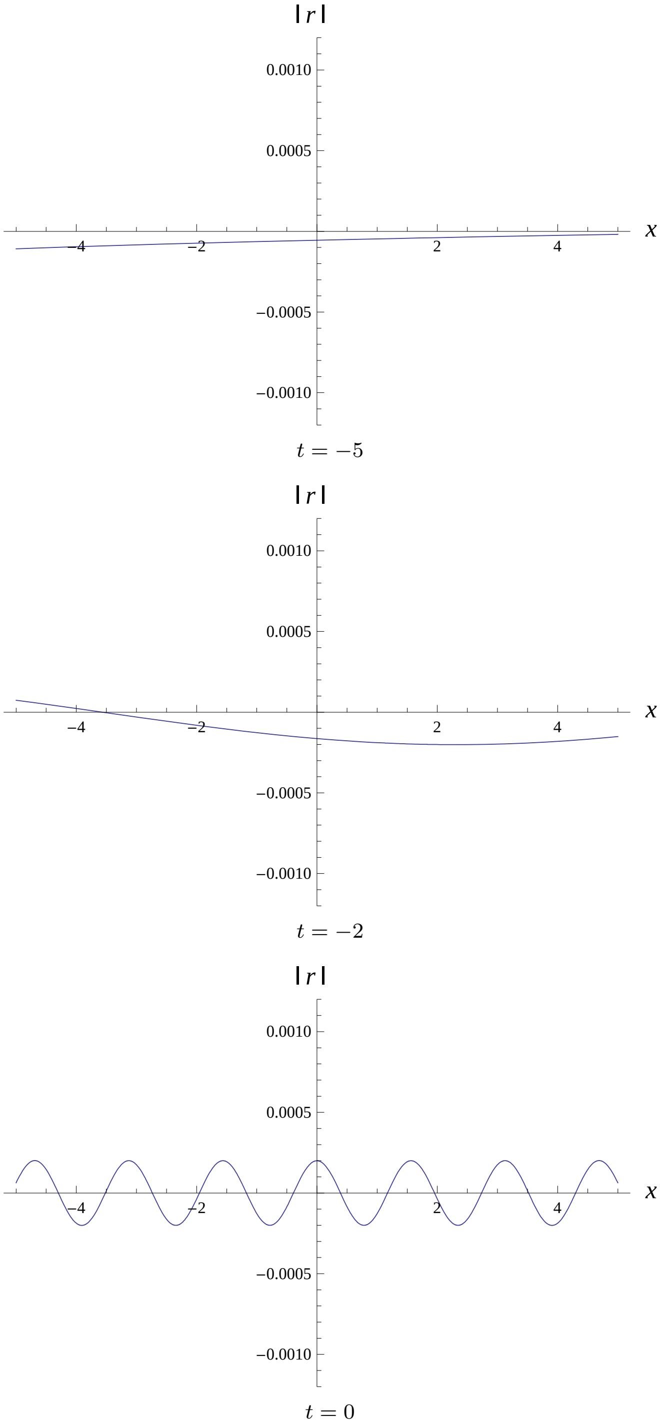

Fig. 4. Dynamical evolutions of one-soliton determined by solution (86).

Substituting Jost relationship $\varphi(x,k) = a(k,t)\bar{\phi}(x,k) + b(k,t)\phi(x,k)$ into (53) yields

$$[a(k,t)\bar{\phi}(x,k) + b(k,t)\phi(x,k)]_t - N[a(k,t)\bar{\phi}(x,k) + b(k,t)\phi(x,k)] = 0. \quad (56)$$

Letting $x \to +\infty$ and using

$$\phi(x,k) \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}, \ \bar{\phi}(x,k) \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}, \quad (57)$$

from (56) we derive

$$\frac{\mathrm{d}a(k,t)}{\mathrm{d}t} = 0, \ \frac{\mathrm{d}b(k,t)}{\mathrm{d}t} = 0. \quad (58)$$

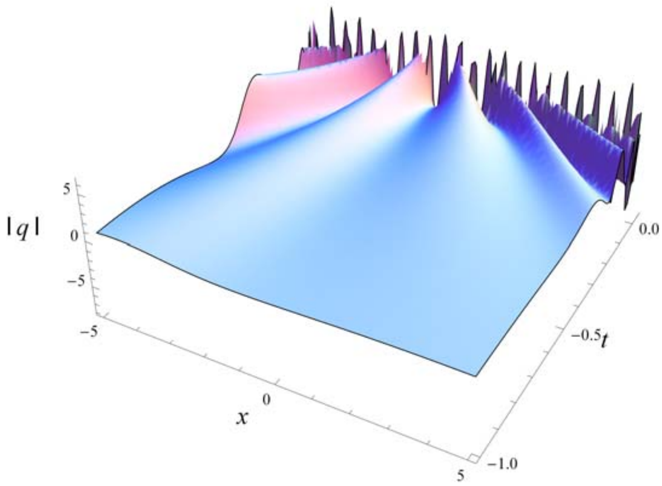

Fig. 5. Spatial structure of bright and dark two-soliton determined by solution (83).

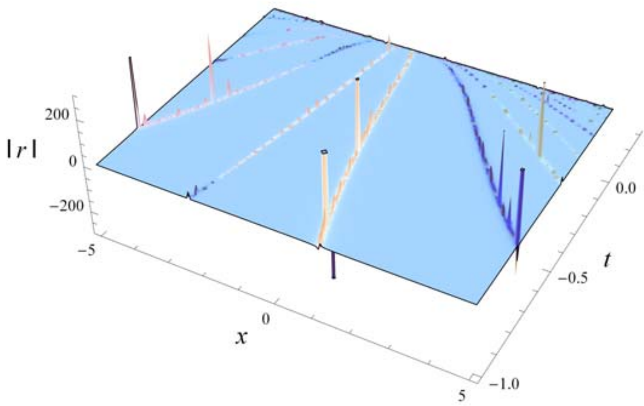

Fig. 6. Spatial structure of bright and dark two-soliton determined by solution (84).

Similarly, we have

$$\frac{\mathrm{d}\bar{a}(k,t)}{\mathrm{d}t} = 0, \ \frac{\mathrm{d}\bar{b}(k,t)}{\mathrm{d}t} = 0. \quad (59)$$

Finally, (25)–(32) can be obtained by directly solving (50)–(52), (58) and (59). Therefore, the proof is end.

Based on Theorem 1, the following Theorem 2 is reached.

Theorem 2: Given the scattering data for the spectral problem (3) equipped with the nonisospectral k in (12), exact solutions of the AKNS system (13) can be determined as follows:

$$q(x,t) = -2K_1(t,x,x), \quad (60)$$

$$r(x,t) = \frac{K_{2x}(t,x,x)}{K_1(t,x,x)}, \quad (61)$$

where $K(t,x,y) = (K_1(t,x,y), K_2(t,x,y))^T$ satisfies the Gel'fand–Levitan–Marchenko (GLM) integral equation:

$$K(t,x,y) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bar{F}(t,x+y) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \int_x^{\infty} F(t,z+x)\bar{F}(t,z+y)\mathrm{d}z$$

$$+\int_x^\infty K(t,x,s)\int_x^\infty F(t,z+s)\bar{F}(t,z+y)\mathrm{d}z\mathrm{d}s=0, \quad (62)$$

with

$$F(t,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}R(t,k)e^{ikx}\mathrm{d}k+\sum_{j=1}^{n}c_j^2e^{i\kappa_jx}, \quad (63)$$

$$\bar{F}(t,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{R}(t,k)e^{-i\bar{k}x}\mathrm{d}k-\sum_{m=1}^{\bar{n}}\bar{c}_m^2e^{i\bar{\kappa}_mx}. \quad (64)$$

In order to give explicit form of solutions (60) and (61), we consider here the reflectionless potentials $q(x,t)$ and $r(x,t)$, i.e., $R(t,k)=\bar{R}(t,k)=0$. In this case, the GLM integral equation (62) can be solved exactly. For the sake of convenience, we use $K(t,x,y)=(K_1(t,x,y),K_2(t,x,y))^T$ to rewrite (62) as:

$$K_1(t,x,y)-\bar{F}_d(t,x+y)$$
$$+\int_x^\infty K_1(t,x,s)\int_x^\infty F_d(t,z+s)\bar{F}_d(t,z+y)\mathrm{d}z\mathrm{d}s=0, \quad (65)$$

$$K_2(t,x,y)-\int_x^\infty F_d(t,z+x)\bar{F}_d(t,z+y)\mathrm{d}z$$
$$+\int_x^\infty K_2(t,x,s)\int_x^\infty F_d(t,z+s)\bar{F}_d(t,z+y)\mathrm{d}z\mathrm{d}s=0. \quad (66)$$

Taking advantage of (63) and (64), we get

$$\int_x^\infty F_d(t,s+z)\bar{F}_d(t,z+y)\mathrm{d}z$$
$$=-\sum_{j=1}^{n}\sum_{m=1}^{\bar{n}}\frac{ic_j^2(t)\bar{c}_m^2(t)}{\kappa_j-\bar{\kappa}_m}e^{i\kappa_j(x+s)-i\bar{\kappa}_m(x+y)}. \quad (67)$$

We suppose that

$$K_1(x,y,t)=\sum_{p=1}^{\bar{n}}\bar{c}_p(t)g_p(t,x)e^{-i\bar{\kappa}_py}, \quad (68)$$

$$K_2(x,y,t)=\sum_{p=1}^{\bar{n}}\bar{c}_p(t)h_p(t,x)e^{-i\bar{\kappa}_py}. \quad (69)$$

and substitute (68) and (69) into (65) and (66), then we have

$$g_m(t,x)+\bar{c}_m(t)e^{-i\bar{\kappa}_mx}$$
$$+\sum_{j=1}^{n}\sum_{p=1}^{\bar{n}}\frac{c_j^2(t)\bar{c}_m(t)\bar{c}_p(t)}{(\kappa_j-\bar{\kappa}_m)(\kappa_j-\bar{\kappa}_p)}e^{i(2\kappa_j-\bar{\kappa}_m-\bar{\kappa}_p)x}g_p(x,t)=0, \quad (70)$$

$$h_m(x,t)-\sum_{j=1}^{n}\frac{ic_j^2(t)\bar{c}_m(t)e^{i(2\kappa_j-\bar{\kappa}_m)x}}{\kappa_j-\bar{\kappa}_m}$$
$$\sum_{j=1}^{n}\sum_{p=1}^{\bar{n}}\frac{c_j^2(t)\bar{c}_m(t)\bar{c}_p(t)}{(\kappa_j-\bar{\kappa}_m)(\kappa_j-\bar{\kappa}_p)}e^{i(2\kappa_j-\bar{\kappa}_m-\bar{\kappa}_p)x}h_p(x,t)=0. \quad (71)$$

We induce the following vectors

$$g(x,t)=(g_1(x,t),g_2(x,t),\cdots,g_{\bar{n}}(x,t))^T, \quad (72)$$

$$h(x,t)=(h_1(x,t),h_2(x,t),\cdots,h_{\bar{n}}(x,t))^T, \quad (73)$$

$$\Lambda=(c_1(t)e^{-i\kappa_1x},c_2(t)e^{-i\kappa_2x},\cdots,c_n(t)e^{-i\kappa_nx})^T, \quad (74)$$

$$\bar{\Lambda}=(\bar{c}_1(t)e^{-i\bar{\kappa}_1x},\bar{c}_2(t)e^{-i\bar{\kappa}_2x},\cdots,\bar{c}_{\bar{n}}(t)e^{-i\bar{\kappa}_{\bar{n}}x})^T, \quad (75)$$

(62) can be written in the matrix from

$$W(x,t)g(x,t)=-\bar{\Lambda}(x,t), \quad (76)$$

$$W(x,t)h(x,t)=iP(x,t)\Lambda(x,t). \quad (77)$$

Supposing $W^{-1}(x,t)$ exists, then one has

$$g(x,t)=-W^{-1}(x,t)\bar{\Lambda}(x,t), \quad (78)$$

$$h(x,t)=iW^{-1}(x,t)P(x,t)\Lambda(x,t), \quad (79)$$

in which $W(x,t)=E+P(x,t)P^T(x,t)$,

$$P(x,t)=\left(\frac{c_j(t)\bar{c}_m(t)}{\kappa_j-\bar{\kappa}_m}e^{i(\kappa_j-\bar{\kappa}_m)x}\right)_{\bar{n}\times n}, \quad (80)$$

and E is a $\bar{n}\times\bar{n}$ unit matrix. Substituting (78) and (79) into (68) and (69) yields

$$K_1(x,y,t)=-\bar{\Lambda}^T(y,t)W^{-1}(x,t)\bar{\Lambda}(x,t), \quad (81)$$

$$K_2(x,y,t)=i\mathrm{tr}(W^{-1}(x,t)P(x,t)\Lambda(y,t)\bar{\Lambda}^T(y,t)), \quad (82)$$

where $\mathrm{tr}(\cdot)$ means the trace of a given matrix.

Substituting (81) and (82) into (60) and (61), we obtain *n*-soliton solutions of the AKNS system (13)

$$q(x,t)=2\mathrm{tr}(W^{-1}(x,t)\bar{\Lambda}(x,t)\bar{\Lambda}^T(x,t)), \quad (83)$$

$$r(x,t)=-\frac{\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{tr}(W^{-1}(x,t)P(x,t)\frac{\mathrm{d}}{\mathrm{d}x}P^T(x,t))}{\mathrm{tr}(W^{-1}(x,t)\bar{\Lambda}(x,t)\bar{\Lambda}^T(x,t))}. \quad (84)$$

Particularly, when $n=\bar{n}=1$ (83) and (84) give one-soliton solutions, the simplified forms of which are listed as follows

$$q=\frac{2\bar{c}_1^2(0)e^{\int_0^t\Psi(w)\mathrm{d}w-2i\bar{\kappa}_1(t)x}}{1+\frac{c_1^2(0)\bar{c}_1^2(0)e^{\int_0^t[\Phi(w)+\Psi(w)]\mathrm{d}w+2i[\kappa_1(t)-\bar{\kappa}_1(t)]x}}{[\kappa_1(t)-\bar{\kappa}_1(t)]^2}}, \quad (85)$$

$$r=\frac{2c_1^2(0)e^{\int_0^t\Phi(w)\mathrm{d}w+2i\kappa_1(t)x}}{1+\frac{c_1^2(0)\bar{c}_1^2(0)e^{\int_0^t[\Phi(w)+\Psi(w)]\mathrm{d}w+2i[\kappa_1(t)-\bar{\kappa}_1(t)]x}}{[\kappa_1(t)-\bar{\kappa}_1(t)]^2}}, \quad (86)$$

where $\kappa_1(t)$ and $\bar{\kappa}_1(t)$ are determined by (25) and (29) respectively, and

$$\Phi(w)=\sin(2i\kappa_1(w)), \quad (87)$$

$$\Psi(w)=-\sin(2i\bar{\kappa}_1(w)). \quad (88)$$

It is easy to see that solutions (85) and (86) possess singularity. In Figs. 1 and 2, two spatial structures of singular bright and dark one-solitons determined by solutions (85) and (86) are shown in the condition of $t\leq\ln[(e^4+1)/(e^4-1)]\approx 0.0366354$, there we select the parameters as $c_1(0)=0.01$, $\bar{c}_1(0)=1$, $\kappa_1(0)=-2$, $\bar{\kappa}_1(0)=0.01$. In Figs. 3 and 4, we describe the corresponding dynamical characteristics of these bright and dark one-solitons at times $t=-5$, $t=-2$ and $t=0$. Figs. 1–4 show that the bright and dark one-solitons determined by solutions (85) and (86) possess time-varying amplitudes and singular points in the process of evolutions. In Figs. 5 and 6, we select $c_1(0)=0.1$, $\bar{c}_1(0)=1$, $c_2(0)=3$, $\bar{c}_2(0)=2$, $\kappa_1(0)=-0.02$, $\bar{\kappa}_1(0)=-0.1$, $\kappa_2(0)=-0.3$, $\bar{\kappa}_2(0)=-0.1$ and show two spatial structures of singular bright and dark two-solitons determined by solutions (83) and (84) in the condition of $t\leq\ln[(e^4+1)/(e^4-1)]\approx 0.0366354$. Figs. 7 and 8 are

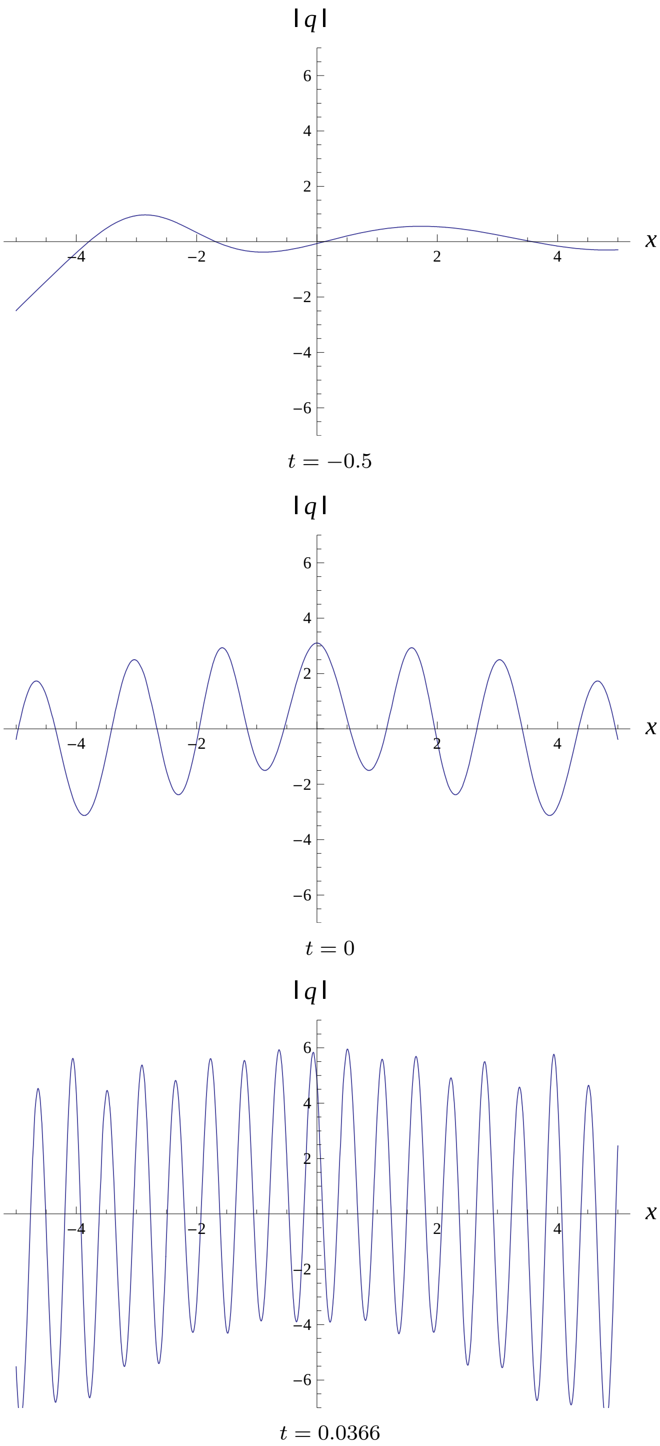

Fig. 7. Dynamical evolutions of two-soliton determined by solution (83).

used to describe the corresponding dynamical characteristics of these bright and dark two-solitons at times $t = -0.5$, $t = 0$ and $t = 0.0366$. From Figs. 5–8 we can see that the bright and dark two-solitons determined by solutions (83) and (84) possess not only singular points but also time-varying velocities and amplitudes in the process of evolutions.

IV. Conclusion

In summary, we have generalized the AKNS spectral problem (3) by embedding a nonisospectral parameter which varies with time obeying sine function of spectral parameter determined in (12). Starting from the generalized AKNS spectral problem (3) and its corresponding time evolution equation (4), together with (5), we constructed a new and more general nonisospectral AKNS system (13) with infinite number of terms. In order to solve the derived AKNS system (13), the IST method is employed. As a result, exact solutions (60) and (61) are formulated and then reduced to explicit n-soliton solutions (83) and (84) in the case of reflectionless potentials. This paper shows by figures that the dynamical evolutions of one-soliton solutions ($n = 1$) and two-soliton solutions ($n = 2$) possess time-varying speeds, amplitudes and singular points. To the best of our knowledge, the derived AKNS system (13) and the obtained n-soliton solutions (83) and (84) have not been reported in literatures. Recently, fractional-order differential calculus and its applications have attached much attention [39], [40], [41], [42], [43], [44], [45]. How to construct hierarchies of fractional-order NLEEs and their exact solutions in the framework of IST method is worthy of study.

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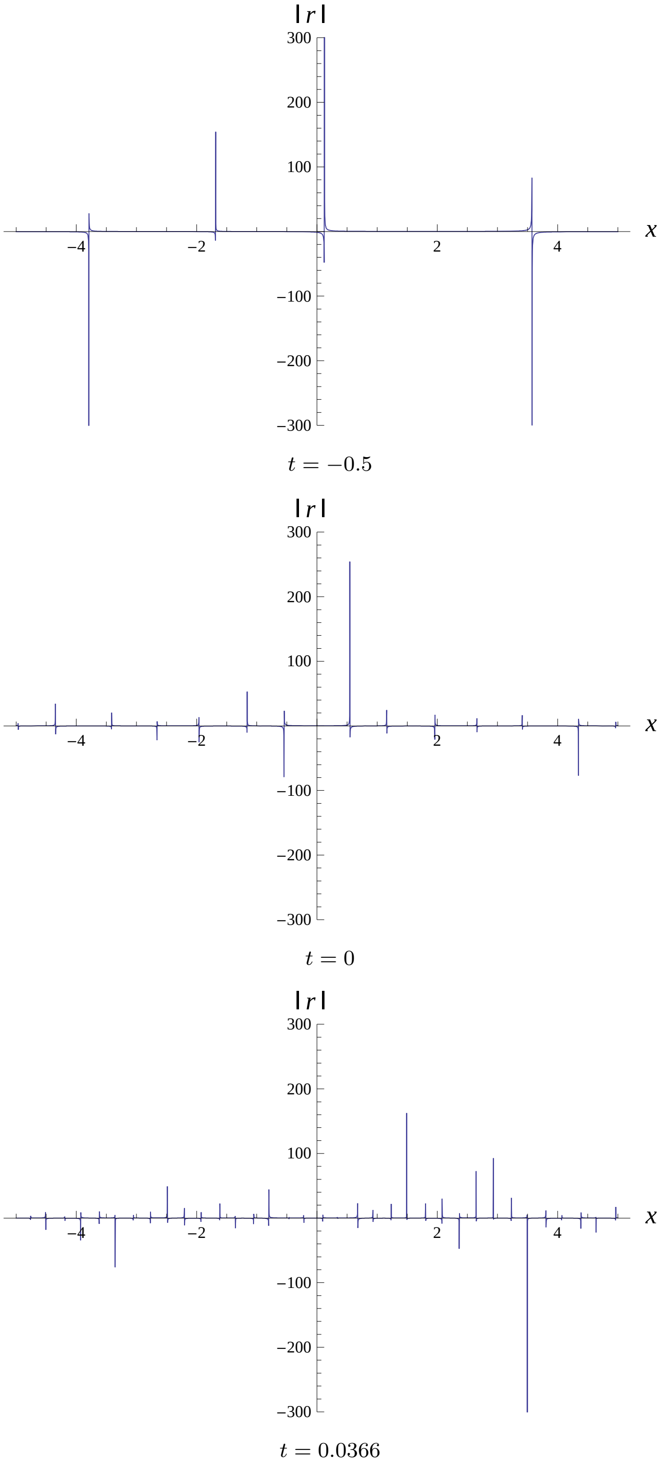

Fig. 8. Dynamical evolutions of two-soliton determined by solution (84).

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