On Minimal Energy of a Class of Bicyclic Graphs with given Cycles' Length and Pendent Vertices*

Yan Mi, Liancui Zuo† and Chunhong Shang

Abstract

The energy of G is defined as the sum of absolute value of all eigenvalues of the adjacency matrix $A(G)$. Let $B^2(n,a,b,p)$ be the set of all bicyclic graphs on n vertices with p pendent vertices and two cycles C_a and C_b which have unique common vertex u_0, $B^2_\theta(n,a,b,p)$ the graph class obtained by coinciding the common vertex u_0 of C_a and C_b with the center of the star $S_{n-(a+b-1)+1}$, and $B^2_\mu(n,a,b,p)$ the set obtained by coinciding the common vertex u_0 of C_a and C_b with the center of the star S_p and connecting a pendent path $P_{n-(a+b-1)-(p-1)}$ on point u_0. In this paper, it is obtained that $B^2_\theta(n,a,b,p)$ has the minimal energy in all graphs which have only pendent vertices except two cycles, and $B^2_\mu(n,a,b,p)$ has the minimal energy in all graphs which have prescribed cycles' length and pendent vertices.

Keywords: Bicyclic graphs; Quasi-Order method; Minimal energy

1 Introduction

In this paper, all graphs are finite, connected, undirected and simple. In recent years, many parameters and classes of graphs were studied. For example, in [1], the restricted connectivity of Cartesian product graphs were obtained, and in [2, 12], some results on 3-equitable labeling and the n-dimensional cube-connected complete graph were gained. Let G be a graph with order n and adjacency matrix $A(G)$. The characteristic polynomial of G, denoted by $\phi(G)$, is defined as

$$\phi(G,x)=\det(xI-A(G))=\sum_{i=0}^{n}a_ix^{n-i},$$

where I is the identity matrix of order n. The roots of the equation $\phi(G)=0$, denoted by $\lambda_1,\lambda_2,\cdots,\lambda_n$, are the eigenvalues of $A(G)$. It's easy to see that each λ_i is real since $A(G)$ is real symmetric. The energy of G, denoted by $E(G)$, is defined as

$$E(G)=\sum_{i=1}^{n}|\lambda_i|.$$

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For the coefficients $a_i(G)$ of $\phi(G)$, set $b_i(G)=|a_i(G)|\,(i=0,1,\cdots,n)$. The following formula is given in Sachs theorem,

$$a_i(G)=\sum_{S\in L_i}(-1)^{\omega(S)}2^{c(S)},$$

where L_i denotes the set of Sachs subgraphs (the subgraphs in which every component is either a K_2 or a cycle) of G that contain i vertices, $\omega(S)$ is the number of connected components of S, and $c(S)$ is the number of cycles contained in S.

It is well known that the Coulson integral formula of the energy is expressed as the following form

$$\begin{aligned}E(G)= &\ \tfrac{1}{2\pi}\int_{-\infty}^{+\infty}\tfrac{1}{x^2}\ln[(\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^ib_{2i}x^{2i})^2\\ &+(\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^ib_{2i+1}x^{2i+1})^2]dx.\end{aligned}\qquad(1)$$

Obviously, the formula (1) is a strictly monotonically increasing function of $b_i(G)$, that is to say, for any two graphs G_1 and G_2 with the same order, there exists the following relationship:

$$b_i(G_1)\ge b_i(G_2)\ hold\ for\ i\ge 0$$
$$\Rightarrow E(G_1)\ge E(G_2).$$

In order to compare the energy of graphs with a better way, we need to define the following Quasi-Order: if $b_i(G_1)\ge b_i(G_2)$ hold for all $i\ge 0$, we can write $G_1\succeq G_2$ or $G_2\preceq G_1$; if $G_1\succeq G_2$ and there exists some i_0 such that $b_{i_0}(G_1)>b_{i_0}(G_2)$, then we write $G_1\succ G_2$. Combining with the formula (1), the increasing property $G_1\succ G_2\Rightarrow E(G_1)>E(G_2)$ of graph energy is obtained.

In theoretical chemistry, the energy of a molecular graph can be approximately used to represent π-electron energy of the molecule, which is an important application of the energy of graphs in the chemical field, and has been widely studied by scholars. For a more detailed explanation can refer to the literature [3-4,10-11,13-15]. In addition, for some special class of graphs, searching for their extreme energy becomes an interesting topic. For the graphs with cycles, many researching conclusions have been obtained, for example, [7,9] give unicycle graphs with minimal energy and maximal energy, respectively. In [8,10], bicyclic graphs with minimal energy and maximal energy are gotten. On this basis, the relevant conclusions about the extreme energy of the graphs with

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given parameters are also gained. For example, [13] obtains unicycle graphs with given diameter and minimal energy. In [5], the minimal energy of unicyclic graphs with prescribed girth and pendent vertices is given. In [6], the minimal energy of bicyclic graphs with a given diameter is obtained.

Bicyclic graphs are defined as connected graphs with n vertices and $n+1$ edges. According to the characteristics of the number of common vertices in two circles in the bicyclic graphs, they can be divided into three classes: two cycles without any common vertex, two cycles with common edges, and two cycles with an unique common vertex. Here we discuss the last case, which is denoted by $B^2(n, a, b, p)$, where a, b are the two cycles' length, and p is the number of pendent vertices. In addition, $B^2_\theta(n, a, b, p)$ is the graph class obtained by coinciding the common vertex u_0 of C_a and C_b with the center of the star $S_{n-(a+b-1)+1}$. Let $B^2_\mu(n, a, b, p)$ be the graph class obtained by coinciding the common vertex u_0 of C_a and C_b with the center of the star S_p, and connecting a pendent path $P_{n-(a+b-1)-(p-1)}$ on point u_0. For sake of the convenience, we write $B^2_\theta(n, a, b, p)$ and $B^2_\mu(n, a, b, p)$ as $B^2_\theta(n, p)$ and $B^2_\mu(n, p)$ simply (see Fig.1-2,20 and 28), subgraphs of G without pendent vertices is called the base graph of G, and the base graph class of graph class mentioned above is represented by B^2_∞ (see Fig.3). Let $V'(G)$ denote the set of all pendent vertices of G. $d_G(x, y)$ is defined as the distance between two vertices x and y of a graph G, and

$$d_G(x, C_{a,b}) = min\{d_G(x, y)| \quad y \in V(C_{a,b}), \\ x \notin V(C_{a,b})\}.$$

In this paper, using Quasi-Order method, we discuss the graphs with minimal energy in $B^2(n, a, b, p)$ that are given length of two cycles with unique common point and pendent vertices by mathematical induction. It is obtained that $B^2_\theta(n, a, b, p)$ has the minimal energy in all graphs which have only pendent vertices except two cycles, and $B^2_\mu(n, a, b, p)$ has the minimal energy in all graphs which have prescribed cycles' length and pendent vertices (please see Fig.1-2).

Lemma 1 ([10]) Let G be a simple graph and $e = uv$ be a pendent edge of G with pendent vertex v, then

$$b_i(G) = b_i(G - v) + b_{i-2}(G - v - u).$$

Lemma 2 ([10]) Let G be a simple graph. If H is a subgraph (proper subgraph) of G, then $G \succeq H (G \succ H)$.

2 Graphs with only pendent vertices except two cycles in $B^2(n, a, b, p)$

Theorem 3. Let $G \in B^2(n, a, b, p)$ with $n = a+b+p-1$ and $p \geq 1$. Assume that the cycles' length a and b are fixed, and u_0 is the common vertex of two cycles. If $G \neq B^2_\theta(n, a, b, p)$, then $G \succ B^2_\theta(n, a, b, p)$.

Proof. We will show the theorem by induction on p.

(1) Suppose that $p = 1$, i. e., there is only one pendent vertex. Assume that $e = u_0v$ is a pendent edge and v is its pendent vertex, as shown in Fig.2. By Lemma 1, for B^2_∞ as in Fig.3, we can get

$$\begin{aligned} b_i(B^2_\theta(n, 1)) &= b_i(G - v) + b_{i-2}(G - u_0 - v) \\ &= b_i(B^2_\infty) + b_{i-2}(P_{a-1} \cup P_{b-1}). \end{aligned}$$

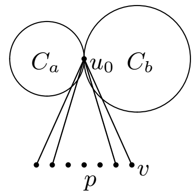

Fig.1 $B^2_\theta(n, a, b, p)$

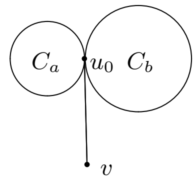

Fig.2 $B^2_\theta(n, a, b, 1)$

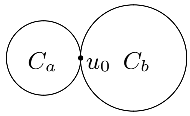

Fig.3 B^2_∞

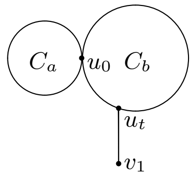

Fig.4 A_{11}

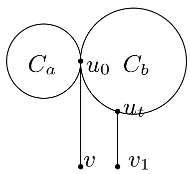

Fig.5 B_{11}

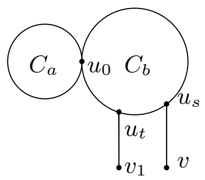

Fig.6 B_{12}

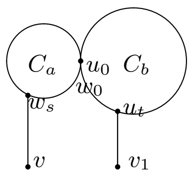

Fig.7 B_{13}

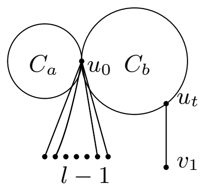

Fig.8 C_{11}

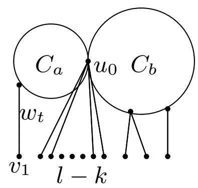

Fig.9 C_{12}

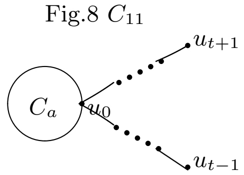

Fig.10 H_1

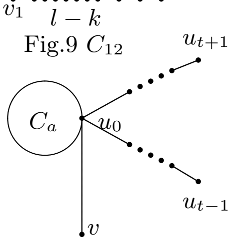

Fig.11 H_2

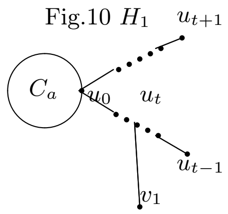

Fig.12 H_3

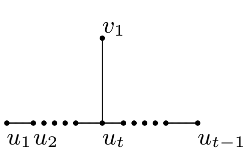

Fig.13 H_4

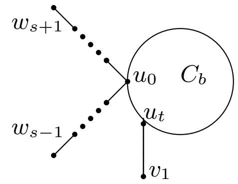

Fig.14 H_5

Assume that v is adjacent to u_t but u_0 belongs to cycle C_b, as shown in Fig.4. By Lemma 1, for graphs A_{11} in

Fig.4 and H_1 in Fig.10, we obtain that

$$\begin{aligned} b_i(A_{11}) &= b_i(A_{11}-v_1)+b_{i-2}(A_{11}-u_t-v_1) \\ &= b_i(B^2_\infty)+b_{i-2}(H_1). \end{aligned}$$

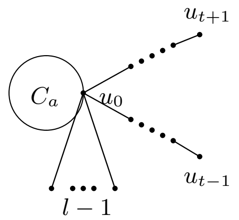

Fig.15 H_6

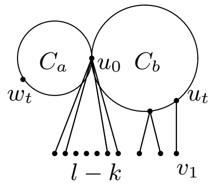

Fig.16 H_7

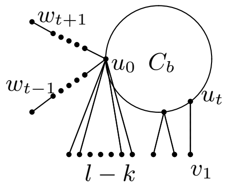

Fig.17 H_8

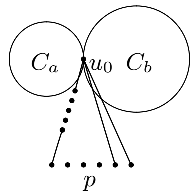

Fig.18 H_9

Obviously, $P_{a-1}\cup P_{b-1}$ is a proper subgraph of H_1, so $P_{a-1}\cup P_{b-1}\prec H_1$, and then $B^2_\theta(n,1)\prec A_{11}$.

(2) For $p=2$ and $B^2_\theta(n,2)$, by Lemma 1, we can get

$$\begin{aligned} b_i(B^2_\theta(n,2)) &= b_i(B^2_\theta(n,2)-v) \\ &\quad +b_{i-2}(B^2_\theta(n,2)-u_0-v) \\ &= b_i(B^2_\theta(n-1,1))+b_{i-2}(P_{a-1}\cup P_{b-1}). \end{aligned}$$

If $G\ncong B^2_\theta(n,2)$, then there is at least one pendent vertex which is not adjacent to u_0. The problem can be divided into the following cases.

Case 1. There is at least one pendent point which is adjacent to u_0. Suppose that one point is such one and the other is adjacent to u_t on cycle C_b, as B_{11} shown in Fig.5. By Lemma 1, we have

$$\begin{aligned} b_i(B_{11}) &= b_i(B_{11}-v_1)+b_{i-2}(B_{11}-u_t-v_1) \\ &= b_i(B^2_\theta(n-1,1))+B_i(H_2), \end{aligned}$$

where H_2 is shown in Fig.11. Since $P_{a-1}\cup P_{b-1}$ is a proper subgraph of H_2, $P_{a-1}\cup P_{b-1}\prec H_2$. Thus, $B^2_\theta(n,2)\prec B_{11}$.

Case 2. There is no pendent vertex which is adjacent to u_0, and both of them are adjacent to u_t and u_s on one cycle C_b, respectively, as B_{12} shown in Fig. 6. Then by Lemma 1, we can get

$$\begin{aligned} b_i(B^2_\theta(n,2)) &= b_i(B^2_\theta(n,2)-v) \\ &\quad +b_{i-2}(B^2_\theta(n,2)-u_0-v) \\ &= b_i(B^2_\theta(n-1,1))+b_{i-2}(P_{a-1}\cup P_{b-1}), \end{aligned}$$

$$\begin{aligned} b_i(B_{12}) &= b_i(B_{12}-v)+b_{i-2}(B_{12}-u_s-v) \\ &= b_i(H_3)+b_{i-2}(H_4), \end{aligned}$$

where H_3,H_4 are shown in Fig. 12-13.

We can obtain that $B^2_\theta(n-1,1)\prec H_3$ from (1). In addition, it is obvious that $P_{a-1}\cup P_{b-1}$ is a proper subgraph of H_4, so $P_{a-1}\cup P_{b-1}\prec H_4$. Hence, $B^2_\theta(n,2)\prec B_{12}$.

Case 3. There is no pendent vertex that is adjacent to u_0, and two points are adjacent to u_t and w_s on two cycles C_b and C_a, respectively, as B_{13} shown in Fig. 7. By Lemma 1, we obtain that

$$\begin{aligned} b_i(B_{11}) &= b_i(B_{11}-v)+b_{i-2}(B_{11}-u_0-v) \\ &= b_i(H_3)+b_{i-2}(P_{a-1}\cup H_5), \end{aligned}$$

and

$$\begin{aligned} b_i(B_{13}) &= b_i(B_{13}-v)+b_{i-2}(B_{13}-w_s-v) \\ &= b_i(H_3)+b_{i-2}(H_6), \end{aligned}$$

where H_5,H_6 are shown in Fig.14-15, respectively.

Comparing $P_{a-1}\cup H_5$ with H_6, since the former is a proper subgraph of the latter, $P_{a-1}\cup H_5\prec H_6$, that is, $B_{11}\prec B_{13}$. Combining the conclusion $B^2_\theta(n,2)\prec B_{11}$ from (1), we get $B^2_\theta(n,2)\prec B_{13}$.

Thus, the theorem holds for $p=2$.

(3) Assume that the theorem holds for $p=l-1$. In the sequel, we prove that the theorem still holds for $p=l$. For $B^2_\theta(n,l)$, we have

$$\begin{aligned} b_i(B^2_\theta(n,l)) &= b_i(B^2_\theta(n,l)-v)+b_{i-2}(B^2_\theta(n,l)-u_0-v) \\ &= b_i(B^2_\theta(n-1,l-1))+b_{i-2}(P_{a-1}\cup P_{b-1}). \end{aligned}$$

If there is at least one pendent vertex whose neighbor is not u_0, as C_{11} shown in Fig.8, then by Lemma 1, we can get

$$\begin{aligned} b_i(C_{11}) &= b_i(C_{11}-v_1)+b_{i-2}(C_{11}-u_t-v_1) \\ &= b_i(B^2_\theta(n-1,l-1))+b_{i-2}(H_7), \end{aligned}$$

where H_7 is shown in Fig.16. Since H_7 contains $P_{a-1}\cup P_{b-1}$ as its proper subgraph, $P_{a-1}\cup P_{b-1}\prec H_7$, so we have $B^2_\theta(n,a,b,l)\prec C_{11}$.

Now suppose that there are $p-k$ pendent vertices that are adjacent to u_0 $(l\geq k\geq 2)$, as C_{12} shown in Fig.9 (the pendent vertices are not shown all). At this time, choose a pendent edge w_tv_1 with $w_t\neq u_0$, then

$$\begin{aligned} b_i(C_{12}) &= b_i(C_{12}-v_1)+b_{i-2}(C_{12}-w_t-v_1) \\ &= b_i(H_8)+b_{i-2}(H_9), \end{aligned}$$

where H_8,H_9 are shown in Fig.17-18. Note that H_8 and H_9 display a part of the pendent vertices except the deformation by deleting one or two vertices from C_{12}. Since $B^2_\theta(n-1,l-1)$ and H_8 are graphs which have only pendent vertices except two cycles with $n-1$ vertices, by induction hypothesis, we can get $B^2_\theta(n-1,l-1)\prec H_8$, and $P_{a-1}\cup P_{b-1}\prec H_9$ because $P_{a-1}\cup P_{b-1}$ is a proper subgraph of H_9. Thus $B^2_\theta(n,p)\prec C_{12}$, and the theorem holds for $p=l$.

In a word, we have shown that $B^2_\theta(n,p)$ is the graph with minimal energy for $1\leq p\leq l, l\geq 2$ and $n=a+b+p-1$.

3 Bicyclic graphs with prescribed cycles' length and pendent vertices in $B^2(n,a,b,p)$

On the basis of Theorem 3, we study the case of $n>a+b+p-1$. Let $Q_n^{a,b,p}$ be graphs shown in Fig. 19.

Theorem 4. Let $G \in B^2(n,a,b,p)$ with $n \geq a+b+p-1$ and $p \geq 1$. Assume that the cycles' length a and b are fixed, and u_0 is the common vertex of two cycles. If $G \ncong B^2_\mu(n,a,b,p)$ and $G \ncong Q_n^{a,b,p}$, then $E(G) > E(B^2_\mu(n,a,b,p))$.

Proof. For $n = a+b-1+p$, by Lemma 1, we have $B^2_\mu(n,a,b,p) \cong B^2_\theta(n,a,b,p)$, the result holds for this case. We show that the result is also true for $n-(a+b-1+p) \geq 1$ in the following.

Assume that $V'(G)$ is the set of all pendent vertices of G, v is the point in $V'(G)$ such that $d_G(v, C_{a,b}) = \max\{d_G(x, C_{a,b}), x \in V'(G)\}$, and u is its unique neighbor.

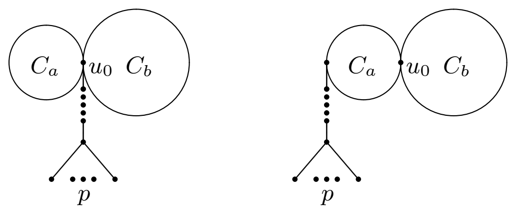

Fig.19 $Q_n^{a,b,p}$

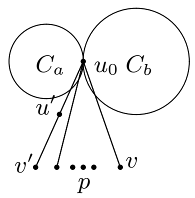

Fig.20 $B^2_\mu(n,a,b,p)$

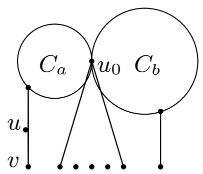

Fig.21 D_{11}

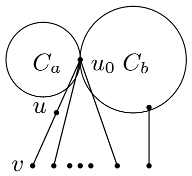

Fig.22 D'_{11}

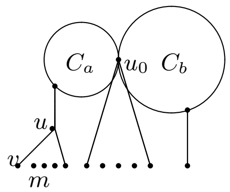

Fig.23 D_{12}

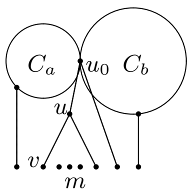

Fig.24 D'_{12}

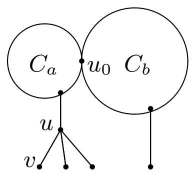

Fig.25 $D_{12}1$

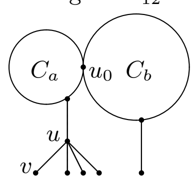

Fig.26 $D_{12}2$

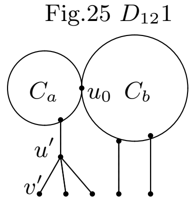

Fig.27 $D_{12}2'$

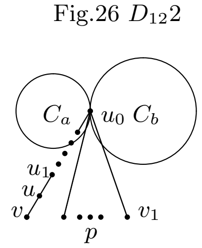

Fig.28 $B^2_\mu(n,p)$

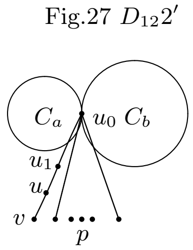

Fig.29 $B^2_\mu(a+b+p+1,p)$

Based on the value of $n-(a+b-1+p)$, there are two cases that need to be discussed.

Case 1. $n-(a+b-1+p) = 1$.

By the definition, we can get $d_G(v, C_{a,b}) = 2$.

Subcase 1. 1. $d(u) = 2$. By Lemma 1, it is not difficult to verify that

$$b_i(B^2_\mu(n,p)) = b_i(B^2_\mu(n,p)-v') + b_{i-2}(B^2_\mu(n,p)-v'-u').$$

Suppose that $G \ncong B^2_\mu(n,p)$, then we must have $G \cong D_{11}$ or $G \cong D'_{11}$ (both D_{11} and D'_{11} are graphs with n vertices and p pendent vertices, and a part of pendent vertices which are adjacent to vertices on cycles are shown in Fig.21-22). Take D_{11} as an example, we have

$$b_i(D_{11}) = b_i(D_{11}-v) + b_{i-2}(D_{11}-v-u),$$

where $D_{11}-v \in B^2(n-1,p)$ and $D_{11}-v-u \in B^2(n-2,p-1)$. Since

$$B^2_\mu(n,p) - v' \cong B^2_\theta(n-1,p)$$

and

$$B^2_\mu(n,p) - v' - u' \cong B^2_\theta(n-2,p-1),$$

we have

$$B^2_\theta(n-1,p) \prec B^2(n-1,p),$$
$$B^2_\theta(n-2,p-1) \prec B^2(n-2,p-1)$$

by Theorem 3. On account of $G \ncong B^2_\mu(n,p)$, we can get

$$B^2_\mu(n,p) \ -v' \prec D_{11} - v$$

and

$$B^2_\mu(n,p) \ -v'-u' \prec D_{11} - v - u.$$

Thus, $B^2_\mu(n,p) \prec D_{11}$. Similarly, we can obtain that $B^2_\mu(n,p) \prec D'_{11}$.

Subcase 1. 2. $d(u) \geq 3$.

We get firstly

$$b_i(B^2_\mu(n,p)) = b_i(B^2_\mu(n,p)-v) + b_{i-2}(B^2_\mu(n,p)-v-u_0)$$

$$= b_i(B^2_\mu(n-1,p-1)) + b_{i-2}(P_{a-1} \cup P_{b-1} \cup P_2), \quad (2)$$

Assume that $G \ncong B^2_\mu(n,a,b,p)$, we must get $G \cong D_{12}$ or $G \cong D'_{12}$ (both D_{12} and D'_{12} are graphs with n vertices and p pendent vertices, and a part of pendent vertices that are adjacent to vertices on cycles are shown in Fig.23-24). Take D_{12} as an example. Clearly,

$$b_i(D_{12}) = b_i(D_{12}-v) + b_{i-2}(D_{12}-v-u), \quad (3)$$

and $D_{12} - v \in B^2(n-1,p-1)$. All points in $N(u)$ are pendent vertices except one on a cycle. Suppose that there are m pendent vertices adjacent to u, then $|N(u)| = m+1$. Since $G \ncong Q_n^{a,b,p}$, we have $p \geq m+1$. Set $D_{12}-v-u = G' \cup (m-1)P_1$, then $G' \in B^2(n-m-1,p-m)$.

Now we show the result by induction n and p in the following. Assume that the result holds for small n and p, then $B^2_\mu(n-1,p-1) \prec B^2(n-1,p-1)$. Because

$p \geq m+1$, $d(u) \geq 3$, and $d_G(v, C_{a,b}) = 2$, there are at least four pendent vertices in G, so $n \geq a+b+4$.

(1) Assume that $n-(a+b) = 4$, i. e., $n = a+b+4$, and $G \in B^2(a+b+4, 4)$. Suppose that $G \cong D_{12}1$, $|V(D_{12}1-v)| - (a+b) = 3$ and $D_{12}1 - v \in B^2(a+b+3, 3)$ (see Fig.25). In the following, we use Quasi-Order method to compare $B^2_\mu(a+b+3, 3)$ and $D_{12}1 - v$. By Lemma 1, we get

$$b_i(B^2_\mu(a+b+3,3)) \\ = b_i(B^2_\mu(a+b+2,2)) + b_{i-2}(P_{a-1} \cup P_{b-1} \cup P_2),$$

and

$$b_i(D_{12}1 - v) = b_i(B^2(a+b+2,2)) + b_{i-2}(B^2(a+b,1)).$$

By Subcase 1.1, we have $B^2_\mu(a+b+2,2) \prec B^2(a+b+2,2)$, and

$$P_{a-1} \cup P_{b-1} \cup P_2 \prec B^2(a+b,1)$$

since $P_{a-1} \cup P_{b-1} \cup P_2$ is a proper subgraph of $B^2(a+b,1)$, so we get

$$B^2_\mu(a+b+3,3) \prec D_{12}1 - v$$

and

$$B^2_\mu(a+b+3,3) \prec B^2(a+b+3,3).$$

Thus, for $n = a+b+4$ and $p = 4$, we have

$$B^2_\mu(a+b+3,3) \prec B^2(a+b+3,3),$$

that is,

$$B^2_\mu(n-1,p-1) \prec B^2(n-1,p-1).$$

(2) Suppose that $n-(a+b) = 5$, i. e., $n = a+b+5$.

On the basis of (1), add one vertex to graph $D_{12}1$ such that the number of pendent vertices increasing, then the added pendent vertex is adjacent to u or other vertex on cycles, and we obtain $D_{12}2, D_{12}2' \in B^2(a+b+5,5)$ that are shown in Fig.26-27.

Suppose that $G \cong D_{12}2$ or $G \cong D_{12}2$, then $|V(D_{12}2 - v)| = |V(D_{12}2' - v')| = a+b+4$ and $D_{12}2 - v, D_{12}2' - v' \in B^2(a+b+4, 4)$. By Lemma 1, we get

$$b_i(B^2_\mu(a+b+4,4)) \\ = b_i(B^2_\mu(a+b+3,3)) + b_{i-2}(P_{a-1} \cup P_{b-1} \cup P_2),$$

$$b_i(D_{12}2 - v) = b_i(B^2(a+b+3,3)) + b_{i-2}(B^2(a+b,1)),$$

and

$$b_i(D_{12}2' - v') = b_i(B^2(a+b+3,3)) + b_{i-2}(B^2(a+b+1,2)).$$

Combining with the result of (1), we have

$$B^2_\mu(a+b+3,3) \prec B^2(a+b+3,3).$$

Since $P_{a-1} \cup P_{b-1} \cup P_2$ is a proper subgraph of both $B^2(a+b,1)$ and $B^2(a+b+1,2)$, we obtain that

$$P_{a-1} \cup P_{b-1} \cup P_2 \prec B^2(a+b,1), \\ P_{a-1} \cup P_{b-1} \cup P_2 \prec B^2(a+b+1,2),$$

$$B^2_\mu(a+b+4,4) \prec D_{12}2 - v$$

and

$$B^2_\mu(a+b+4,4) \prec D_{12}2' - v',$$

so we have

$$B^2_\mu(a+b+4,4) \prec B^2(a+b+4,4).$$

Thus, for $n = a+b+5$ and $p = 5$, we have

$$B^2_\mu(a+b+4,4) \prec B^2(a+b+4,4),$$

that is,

$$B^2_\mu(n-1,p-1) \prec B^2(n-1,p-1).$$

(3) Assume that the theorem holds for $n-(a+b) = p-1$, i. e., $B^2_\mu(a+b+p-2, p-2) \prec B^2(a+b+p-2, p-2)$.

In the sequel, we prove that the theorem still holds for $n-(a+b) = p$. Suppose that $G \cong D_{12}$ or $G \cong D'_{12}$ (see Fig. 23-24). Take D_{12} as an example. Clearly, $|V(D_{12} - v)| = a+b+p-1$ and $D_{12} - v \in B^2(a+b+p-1, p-1)$.

A comparison between $B^2_\mu(a+b+p-1, p-1)$ and $D_{12} - v$ will be accomplished in the following. By Lemma 1, we have

$$b_i(B^2_\mu(a+b+p-1,p-1)) = \\ b_i(B^2_\mu(a+b+p-2,p-2)) + b_{i-2}(P_{a-1} \cup P_{b-1} \cup P_2),$$

and

$$b_i(D_{12} - v) = b_i(B^2(a+b+p-2,p-2)) \\ + b_{i-2}(B^2(a+b+p-m-1,p-m)).$$

By induction assumption, we obtain that

$$B^2_\mu(a+b+p-2,p-2) \prec B^2(a+b+p-2,p-2).$$

Because $p-m \geq 1$, it is not difficult to see that $P_{a-1} \cup P_{b-1} \cup P_2$ is a proper subgraph of $B^2(a+b+p-m-1, p-m)$, so

$$P_{a-1} \cup P_{b-1} \cup P_2 \prec B^2(a+b+p-m-1,p-m),$$

and

$$B^2_\mu(a+b+p-1,p-1) \prec D_{12} - v.$$

Thus, for $n = a+b+p$, we have

$$B^2_\mu(a+b+p-1,p-1) \prec B^2(a+b+p-1,p-1),$$

that is,

$$B^2_\mu(n-1,p-1) \prec B^2(n-1,p-1).$$

By induction, we have

$$B^2_\mu(n-1,p-1) \prec B^2(n-1,p-1),$$

and the relationship between the first item on right of equations (2) and (3): $B^2_\mu(n-1,p-1) \prec D_{12} - v$. The problem is turned into a proof of the relationship between the second:$P_{a-1} \cup P_{b-1} \cup P_2 \prec D_{12} - v - u$.

As what mentioned before, $D_{12}-v-u \in B^2(n-m-1, p-m)$. What remains is to prove that $P_{a-1} \cup P_{b-1} \cup P_2 \prec B^2(n-m-1, p-m)$. Since $p-m \geq 1$, it is clear that $P_{a-1} \cup P_{b-1} \cup P_2$ is a proper subgraph of $B^2(n-m-1, p-m)$, so we get

$$P_{a-1} \cup P_{b-1} \cup P_2 \prec B^2(n-m-1, p-m).$$

Thus, we have $B^2_\mu(n,p) \prec D_{12}$. Similarly, we can obtain that $B^2_\mu(n,p) \prec D'_{12}$. Hence the result in Case 1 is proved.

Case 2. $n-(a+b-1+p) \geq 2$.

In the following, we deal with the problem in two subcases.

Subcase 2.1. $d_G(v, C_{a,b}) \geq 3$.

Subcase 2.1.1. $d(u) = 2$.

Subcase 2.1.1.1. $d(u_1) = 2$ and u_1 is the neighbor vertex of u.

Assume that $G \ncong B^2_\mu(n,a,b,p)$, we must have $G \cong D_{13}$ which has n vertices and p pendent vertices, where a part of pendent vertices and pendent pathes are shown in Fig.31.

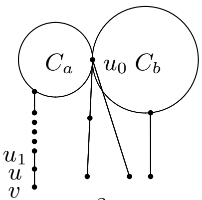

Fig.30 $B^2_\mu(a+b+p+2, p+1)$

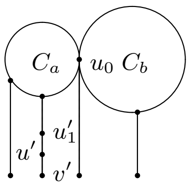

Fig.31 D_{13}

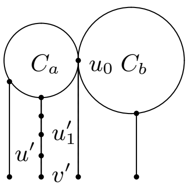

Fig.32 $D_{13}1$

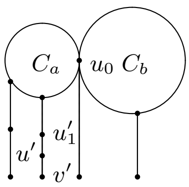

Fig.33 $D_{13}2$

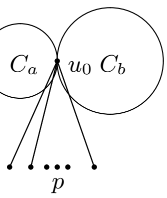

Fig.34 $D_{13}2'$

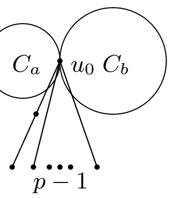

Fig.35 D_{14}

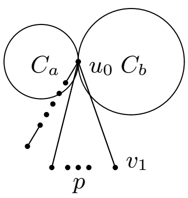

Fig.36 $B^2_\mu(a+b+p-1, p)$

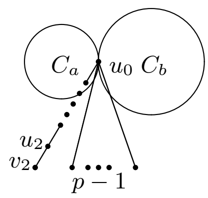

Fig.37 $B^2_\mu(a+b+p-1, p-1)$

Obviously, $D_{13}-v \in B^2(n-1,p)$ and $D_{13}-v-u \in B^2(n-2,p)$. In the following, the two items on the right of equal will be compared, respectively. If we want to get $B^2_\mu(n-1,p) \prec D_{13}-v$ and $B^2_\mu(n-2,p) \prec D_{13}-v-u$, we need to prove that $B^2_\mu(n-1,p) \prec B^2(n-1,p)$ and $B^2_\mu(n-2,p) \prec B^2(n-2,p)$. We show the result by induction on n. Assume that the result holds for small n, then $B^2_\mu(n-1,p) \prec B^2(n-1,p)$ and $B^2_\mu(n-2,p) \prec B^2(n-2,p)$. Note that $d_G(v, C_{a,b}) \geq 3$, and $d(u) = d(u_1) = 2$. Assume that the number of pendent vertices as p will not change even though the order of graph G changes constantly. There are at least two vertices except two cycles and pendent vertices, so $n \geq a+b+p+1$.

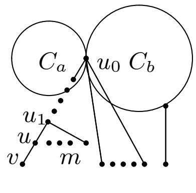

Fig.38 D_{15}

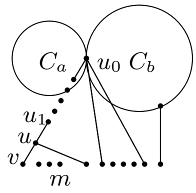

Fig.39 D'_{15}

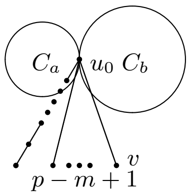

Fig.40 $B^2_\mu(n-m-1, p-m+1)$

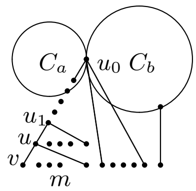

Fig.41 D_{16}

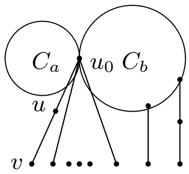

Fig.42 D_{17}

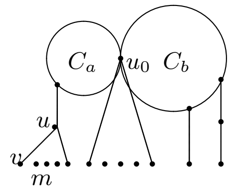

Fig.43 D_{18}

For $B^2_\mu(n,a,b,p)$ and D_{13}, by Lemma 1, we have

$$b_i(B^2_\mu(n,p)) = b_i(B^2_\mu(n-1,p)) + b_{i-2}(B^2_\mu(n-2,p)),$$

$$b_i(D_{13}) = b_i(D_{13}-v) + b_{i-2}(D_{13}-v-u).$$

(1)If $n-(a+b-1)-p = 2$, then $n = a+b+p+1$.

Suppose that $G \cong D_{13}1$ (the pendent edges are not shown all in Fig.32) since $G \ncong B^2_\mu(a+b+p+1, p)$. We can get $V(D_{13}-v') = a+b+p$ and $D_{13}-v' \in B^2(a+b+p, p)$. We use Quasi-Order method to compare $B^2_\mu(a+b+p, p)$ and $D_{13}1-v'$. By Lemma 1, we get

$$b_i(B^2_\mu(a+b+p,p)) = b_i(B^2_\mu(a+b+p-1,p)) + b_{i-2}(B^2_\mu(a+b+p-2,p-1))$$

and

$$b_i(D_{13}1-v') = b_i(B^2(a+b+p-1,p)) + b_{i-2}(B^2(a+b+p-2,p-1).$$

Since

$$B^2_\mu(a+b-1+p,p) \cong B^2_\theta(a+b-1+p,p),$$

$$B^2_\mu(a+b+p-2,p-1) \cong B^2_\theta(a+b+p-2,p-1).$$

By Theorem 3, we can obtain that

$$B^2_\mu(a+b-1+p,p) \prec B^2(a+b-1+p,p),$$

and

$$B^2_\mu(a+b+p-2,p-1) \prec B^2(a+b+p-2,p-1),$$

so we have $B^2_\mu(a+b+p,p) \prec D_{13}1-v'$, that is $B^2_\mu(a+b+p,p) \prec B^2(a+b+p,p)$. Thus, for $n=a+b+p+1$, we have $B^2_\mu(a+b+p,p) \prec B^2(a+b+p,p)$, i. e.,

$$B^2_\mu(n-1,p) \prec B^2(n-1,p).$$

On the basis above, to add one vertex to graph $D_{13}1$ but the number of pendent vertices will not change, there exist two methods: lengthening the pendent path which contains v' and u' on $D_{13}1$ or other pendent path. Without loss of generality, assume that we add one vertex such that the graph has the same pendent vertices, then we obtain $G \cong D_{13}2$ or $G \cong D_{13}2'$, where $D_{13}2, D_{13}2' \in B^2(a+b+p+2,p)$.

(2) If $n-(a+b-1)-p=3$, then $n=a+b+p+2$, so it is not difficult to see that

$$V(D_{13}2-v')=a+b+p+1,$$

$$D_{13}2-v' \in B^2(a+b+p+1,p),$$

$$V(D_{13}2'-v')=a+b+p+1,$$

and

$$D_{13}2'-v' \in B^2(a+b+p+1,p).$$

Comparing $B^2_\mu(a+b+p+1,p)$ and $D_{13}2-v'$ as well as $B^2_\mu(a+b+p+1,p)$ and $D_{13}2'-v'$, by Lemma 1, we get

$$\begin{aligned} &b_i(B^2_\mu(a+b+p+1,p)) = b_i(B^2_\mu(a+b+p,p)) \\ &+b_{i-2}(B^2_\mu(a+b+p-1,p)), \end{aligned}$$

$$\begin{aligned} &b_i(D_{13}2-v') = b_i(B^2(a+b+p,p)) \\ &+b_{i-2}(B^2(a+b+p-1,p), \end{aligned}$$

and

$$\begin{aligned} &b_i(D_{13}2'-v') = b_i(B^2(a+b+p,p)) \\ &+b_{i-2}(B^2(a+b+p-1,p-1)). \end{aligned}$$

Comparing three equations above, combining with the proof of (1), the result of the comparison between the first term of the right side of the equal is $B^2_\mu(a+b+p,p) \prec B^2(a+b+p,p)$.

For the second term of the right side of the equal, since

$$B^2_\mu(a+b-1+p,p) \cong B^2_\theta(a+b-1+p,p),$$

by Lemma 1, we get $B^2_\mu(a+b-1+p,p) \prec B^2(a+b+p-1,p)$.

For $B^2_\mu(a+b+p-1,p)$ and $B^2(a+b+p-1,p-1)$, by Theorem 1, we have

$$B^2_\mu(a+b+p-1,p-1) \prec B^2(a+b+p-1,p-1).$$

Now, we need only to prove that

$$B^2_\mu(a+b+p-1,p) \prec B^2_\mu(a+b+p-1,p-1).$$

From Lemma 1,we can get

$$\begin{aligned} &b_i(B^2_\mu(a+b+p-1,p)) \\ &= b_i(B^2_\theta(a+b+p-2,p-1)) + b_{i-2}(P_{a-1} \cup P_{b-1}) \end{aligned}$$

and

$$\begin{aligned} &b_i(B^2(a+b+p-1,p-1)) \\ &= b_i(B^2_\theta(a+b+p-2,p-1)) \\ &+b_{i-2}(B^2(a+b+p-3,p-2)). \end{aligned}$$

Since $P_{a-1} \cup P_{b-1}$ is a proper subgraph of $B^2(a+b+p-3,p-2)$, the relationship on the second term of the right side of the equal is

$$P_{a-1} \cup P_{b-1} \prec B^2(a+b+p-3,p-2),$$

so

$$B^2_\mu(a+b+p-1,p) \prec B^2(a+b+p-1,p-1).$$

Thus, for $n=a+b+p+2$, we obtain that

$$B^2_\mu(a+b+p+1,p) \prec B^2(a+b+p+1,p),$$

i. e.,

$$B^2_\mu(n-1,p) \prec B^2(n-1,p).$$

(3) If $n-(a+b-1)-p=N-1$, then $n=a+b+p+N-2$. Set $n_1=n$, assume that the theorem holds for n_1, i. e., we have

$$B^2_\mu(n_1-2,p) \prec B^2(n_1-2,p),$$

and

$$B^2_\mu(n_1-1,p) \prec B^2(n_1-1,p).$$

Suppose that $n-(a+b-1)-p=N$, then $n=a+b+p+N-1$. Set $n_2=n$. Since $V(G-v)=n_2-1$, without loss of generality, assume that $G \cong D_{13}$ (the pendent pathes are not shown all in the Fig.31). Obviously, $D_{13}-v \in B^2(n_2-1,p)$. Comparing $B^2_\mu(n_2-1,p)$ with $D_{13}-v$, by Lemma 1, we have

$$\begin{aligned} &b_i(B^2_\mu(n_2-1,p)) \\ &= b_i(B^2_\mu(n_2-2,p)) + b_{i-2}(B^2_\mu(n_2-3,p)), \end{aligned}$$

and

$$\begin{aligned} &b_i(D_{13}-v) \\ &= b_i(B^2(n_2-2,p)) + b_{i-2}(B^2(n_2-3,p)). \end{aligned}$$

Due to $n_2=n_1+1$, we can get

$$\begin{aligned} B^2_\mu(n_2-2,p) &\cong B^2_\mu(n_1-1,p), \\ B^2(n_2-2,p) &\cong B^2(n_1-1,p), \\ B^2_\mu(n_2-3,p) &\cong B^2_\mu(n_1-2,p), \end{aligned}$$

and

$$B^2(n_2-3,p) \cong B^2(n_1-2,p).$$

By induction hypothesis, we have

$$B^2_\mu(n_1-1,p) \prec B^2(n_1-1,p),$$

and

$$B^2_\mu(n_1-2,p) \prec B^2(n_1-2,p).$$

Therefore,

$$B_\mu^2(n_2-2,p) \prec B^2(n_2-2,p),$$

and

$$B_\mu^2(n_2-3,p) \prec B^2(n_2-3,p)),$$

that is to say,

$$B_\mu^2(n-2,p) \prec B^2(n-2,p),$$

and

$$B_\mu^2(n-1,p) \prec B^2(n-1,p).$$

So

$$B_\mu^2(n-1,p) \prec D_{13}-v.$$

According to the induction above, we have

$$B_\mu^2(n-1,p) \prec D_{13}-v,$$

and

$$B_\mu^2(n-2,p) \prec D_{13}-v-u,$$

that is,

$$b_i(B_\mu^2(n-1,p)) < b_i(D_{13}-v),$$

and

$$b_i(B_\mu^2(n-2,p) < \ b_i(D_{13}-v-u).$$

Thus, $B_\mu^2(n,p) \prec D_{13}$ and then the result is obtained.

Subcase 2.1.1.2. $d(u_1) \geq 3$.

Suppose that $G \cong D_{14}$, where D_{14} is the graph with n vertices and p pendent vertices and a part of pendent vertices and a pendent path are shown in Fig.35. For $B_\mu^2(n,p)$ and D_{14}, we have

$$b_i(B_\mu^2(n,p)) = b_i(B_\mu^2(n-1,p)) + b_{i-2}(B_\mu^2(n-2,p)),$$
$$b_i(D_{14}) = b_i(D_{14}-v) + b_{i-2}(D_{14}-v-u).$$

Obviously, $D_{14}-v \in B^2(n-1,p)$ and $D_{14}-v-u \in B^2(n-2,p-1)$. Similar as the subcase 2.1.1.1 and subcase 1.2, we have $B_\mu^2(n-1,p) \prec B^2(n-1,p)$, and $B_\mu^2(n-2,p-1) \prec B^2(n-2,p-1)$. So $B_\mu^2(n-1,p) \prec D_{14}-v$, $B_\mu^2(n-2,p-1) \prec D_{14}-v-u$. Hence the problem is turned into showing $B_\mu^2(n-2,p) \prec B_\mu^2(n-2,p-1)$.

(i)For $n-2-(a+b-1+p)=0$, then $B_\mu^2(n-2,p) \cong B_\theta^2(n-2,p)$.

By Lemma 1, we get

$$B_\theta^2(n-2,p) = b_i(B_\theta^2(n-3,p-1)) + b_{i-2}(P_{a-1} \cup P_{b-1}),$$

$$B_\mu^2(n-2,p-1) = b_i(B_\theta^2(n-3,p-1)) + b_{i-2}(B_\theta^2(n-4,p-2)).$$

It is not difficult to find that $P_{a-1} \cup P_{b-1}$ is a proper subgraph of $B_\theta^2(n-4,p-2)$, so

$$B_\mu^2(n-2,p) \prec B_\mu^2(n-2,p-1).$$

(ii)For $n-2-(a+b-1+p) \geq 1$. By Lemma 1, we get

$$b_i(B_\mu^2(n-2,p)) = b_i(B_\mu^2(n-2,p)-v_1) + b_{i-2}(B_\mu^2(n-2,p)-v_1-u_1),$$

$$b_i(B_\mu^2(n-2,p-1)) = b_i(B_\mu^2(n-2,p-1)-v_2) + b_{i-2}(B_\mu^2(n-2,p-1)-v_2-u_2).$$

Note that $B_\mu^2(n-2,p)-v_1 \cong B_\mu^2(n-2,p-1)-v_2$. We have

$$b_i(B_\mu^2(n-2,p)-v_1) = b_i(B_\mu^2(n-2,p-1)-v_2).$$

Since

$$B_\mu^2(n-2,p)-v_1-u_1 = P_{a-1} \cup P_{b-1} \cup P_{n-a-b-p},$$

and $P_{a-1} \cup P_{b-1} \cup P_{n-a-b-p}$ is a proper subgraph of $B_\mu^2(n-2,p-1)-v_2-u_2$, we obtain that

$$B_\mu^2(n-2,p)-v_1-u_1 \prec B_\mu^2(n-2,p-1)-v_2-u_2,$$

thus

$$B_\mu^2(n-2,p) \prec B_\mu^2(n-2,p-1).$$

In summary, we get $B_\mu^2(n,p) \prec D_{14}$.

Subcase 2.1.2. $d(u) \geq 3$.

Subcase 2.1.2.1. $d(u_1) = 2$.

Suppose that $G \cong D_{15}$ where D_{15} is the graph with n vertices and p pendent vertices that are adjacent to u_1, and a part of pendent vertices and pendent path are shown in Fig.38.

For $B_\mu^2(n,a,b,p)$ and D_{15}, by Lemma 1, we have

$$b_i(B_\mu^2(n,p)) = b_i(B_\mu^2(n,p)-v_1) + b_{i-2}(B_\mu^2(n,p)-v_1-u_0)$$

$$= b_i(B_\mu^2(n-1,p-1)) + b_{i-2}(P_{a-1} \cup P_{b-1} \cup P_{n-a-b-p+2}),$$

$$b_i(D_{15}) = b_i(D_{15}-v) + b_{i-2}(D_{15}-v-u).$$

Obviously, $D_{15}-v \in B^2(n-1,p-1)$, and all vertices are pendent vertices except one point u_1 in $N(u)$.

Suppose that there are m pendent vertices adjacent to u, then $|N(u)| = m+1$ (please see D'_{15} in Fig.39). Let $D'_{15}-v-u = G' \cup (m-1)P_1$. Then $G' \in B^2(n-m-1,p-m+1)$. Similar as subcase 1.2, we have

$$B_\mu^2(n-1,p-1) \prec B^2(n-1,p-1),$$

and

$$B_\mu^2(n-m-1,p-m+1) \prec B^2(n-m-1,p-m+1),$$

so we obtain that $B_\mu^2(n-1,p-1) \prec D'_{15}-v$ and $B_\mu^2(n-m-1,p-m+1) \prec D'_{15}-v-u$. What remains is to prove that $P_{a-1} \cup P_{b-1} \cup P_{n-a-b-p+2} \prec B_\mu^2(n-m-1,p-m+1)$. Since $G \ncong Q_n^{a,b,p}, p-m \geq 1$.

(i) For $p = m+1$, $B_\mu^2(n-m-1,p-m+1) \cong B_\mu^2(n-p,2)$, then by Lemma 1, we can get

$$\begin{aligned} & b_i(B_\mu^2(n-m-1,p-m+1)) \\ & = b_i(B_\mu^2(n-p,2)) \\ & = b_i(B_\mu^2(n-p-1,1)) \\ & +b_{i-2}(P_{a-1} \cup P_{b-1} \cup P_{n-p-a-b}), \end{aligned}$$

and

$$\begin{aligned} & b_i(P_{a-1} \cup p_{b-1} \cup P_{n-a-b-p+2}) \\ & = b_i(P_{a-1} \cup P_{b-1} \cup P_{n-a-b-p+1}) \\ & +b_{i-2}(P_{a-1} \cup P_{b-1} \cup P_{n-a-b-p}). \end{aligned}$$

Comparing the first term of the right side of the equal, it is not difficult to find that $P_{a-1} \cup P_{b-1} \cup P_{n-a-b-p+1}$ is a proper subgraph of $B^2_\mu(n-p-1,1)$, so we have

$$P_{a-1} \cup P_{b-1} \cup P_{n-a-b-p+1} \prec B^2_\mu(n-p-1,1).$$

Therefore,

$$P_{a-1} \cup P_{b-1} \cup P_{n-a-b-p+2} \prec B^2_\mu(n-m-1,p-m+1)$$

holds.

(ii) For $p \geq m+2$, by Lemma 1, we get

$$\begin{aligned} & b_i(B^2_\mu(n-m-1,p-m+1)) \\ & = b_i(B^2_\mu(n-m-1,p-m+1)-v) \\ & +b_{i-2}(B^2_\mu(n-m-1,p-m+1)-v-u_0) \\ & = b_i(B^2_\mu(n-m-2,p-m)) \\ & +b_{i-2}(P_{a-1} \cup P_{b-1} \cup P_{n-a-b-p}), \end{aligned}$$

and

$$\begin{aligned} & b_i(P_{a-1} \cup p_{b-1} \cup P_{n-a-b-p+2}) \\ & = b_i(P_{a-1} \cup P_{b-1} \cup P_{n-a-b-p+1}) \\ & + b_{i-2}(P_{a-1} \cup P_{b-1} \cup P_{n-a-b-p}) \end{aligned}$$

holds for all $i \geq 0$. Therefore,

$$\begin{aligned} & P_{a-1} \cup P_{b-1} \cup P_{n-a-b-p+2} \\ & \prec B^2_\mu(n-m-1,p-m+1). \end{aligned}$$

Subcase 2.1.2.2. $d(u_1) \geq 3$.

Without loss of generality, assume that $G \cong D_{16}$ where D_{16} is the graph with n vertices and p pendent vertices, and a part of pendent vertices and pendent path are shown in Fig.41. Since $d(u_1) \geq 3$, $p \geq m+1$. Furthermore, $D_{16} - v \in B^2(n-1,p-1)$ and $D_{16} - v - u \in B^2(n-m-1,p-m)$.The problem is transformed into the similar situation above, so the corresponding conclusion is still established.

In summary, the result in Subcase 2.1 is proved completely.

Subcase 2.2. $d_G(v, C_{a,b}) = 2$.

Subcase 2.2.1. $d(u) = 2$.

Without loss of generality, assume that $G \cong D_{17}$ where D_{17} is the graph with n vertices and p pendent vertices, and a part of pendent vertices and pendent path are shown in Fig.42.

Obviously, $D_{17} - v \in B^2(n-1,p)$ and $D_{17} - v - u \in B^2(n-2,p-1)$. The problem is transformed into the above similar situation. Similar as Subcase 2.1.1.2, we have the corresponding conclusion.

Subcase 2.2.2. $d(u) \geq 3$.

Without loss of generality, assume that $G \cong D_{18}$ where D_{18} is the graph with n vertices and p pendent vertices, and a part of pendent vertices and pendent paths are shown in Fig.43. Assume that G' is discussed as above. Due to $G \ncong Q^{a,b,p}_n$, $p \geq m+1$. Obviously, we have

$$D_{18} - v \in B^2(n-1,p-1),$$

and

$$D_{18} - v - u \in B^2(n-m-1,p-m),$$

the problem is transformed into the subcases above. Similar as Subcase 1.2, we can get the corresponding conclusion.

The proof is completed.

References

[1] L. Chen, J. Meng, Y. Tian, F. Xia, Restricted connectivity of Cartesian product graphs, IAENG International Journal of Applied Mathematics, 46:1, pp.58-63, 2016.

[2] G. V. Ghodasra, S. G. Sonchhatra, Further results on 3-equitable labeling, IAENG International Journal of Applied Mathematics, 45:1, pp.1-15, 2015.

[3] I. Gutman, Acyclic conjugated molecules, trees and their energies, J. Math. Chen. 2 (1987) 123-143.

[4] I. Gutman, Total -electron energy of benzenoid hydrocarbon, Topic. Curr. C-hem. 162 (1992) 29-63.

[5] H. Hua, M. Wang, Unicycle graphs with given number of pendent vertices and minimal energy, Linear Algebra Appl. 426 (2007) 478-489.

[6] H. Hua, On minimal energy of unicycle graphs with prescribed girth and pendent vertices, MATCH Commun. Math. Comput. Chem. 57 (2007) 351-361.

[7] Y. Hou, Unicycle graphs with minimal energy, J. Math. Chem. 3 (2001) 163-168.

[8] Y. Hou, Bicyclic graphs with minimal energy, Linear and Multilinear A lgebra. 49 (2001) 347-354.

[9] Y. Hou, I. Gutman, C. Woo, Unicyclic graphs with maximal energy, Linear Algebra Appl. 356 (2002) 27-36.

[10] X. Li, Y. Shi and I. Gutman, Graph energy, Springer New York Heidelberg Dordrec-t London, 2012.

[11] X. Li, J. Zhang, On bicyclic graphs with maximal energy, Linear Algebra Appl. 427 (2007) 87-98.

[12] J. Liu and X. Zhang, Cube-connected complete graphs, IAENG International Journal of Applied Mathematics, 44:3, pp.134-136, 2014.

[13] Y. Yang, B. Zhou, Minimal energy of bicyclic graphs of a given diameter, MATCH Commun. Math. Comput. Chem. 59 (2008) 321-342.

[14] F. Zhang, H. Li, On acyclic conjugated molecules with minimal energies, Discr. Appl. Math. 92 (1999) 71-84.

[15] F. Zhang, Z. Li, L. Wang, Hexagonal chuains with maximal total π-electr on energy, Chem. Phys. Lett. 337 (2001) 125-130.