Solvability of Quasilinear Euler-Lagrange Equations

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Abstract—In this paper we deal with the solvability of quasilinear Euler-Lagrange equation

\[-\text{div}((a(x)+ | u |^\gamma) | \nabla u |^{p-2} \nabla u) + \frac{\gamma}{p} | u |^{p-2} u | \nabla u |^p \]

\[= \lambda | u |^{\theta-2} u + | u |^{\varphi-2} u \text{ in } \Omega \]  

with zero Dirichlet boundary condition, under the assumption \(1 < \theta < p < q < \frac{\gamma}{\gamma-1} \) and \(\gamma > 1\). We concern with the existence of multiplicity solutions for the above equation in employing the critical point methods. Moreover, we obtain the trivial solution of such equation when \(\Omega\) is a smooth star-shaped domain in \(\mathbb{R}^N\).

Keywords: Euler-Lagrange equation; Weak solution; Truncated function; Nonsmooth critical point theory

1 Introduction

In this paper we study the following equation

\[-\text{div}((a(x)+ | u |^\gamma) | \nabla u |^{p-2} \nabla u) + \frac{\gamma}{p} | u |^{p-2} u | \nabla u |^p \]

\[= \lambda | u |^{\theta-2} u + | u |^{\varphi-2} u \text{ in } \Omega \]  

with zero Dirichlet boundary condition.

\[u = 0 \text{ on } \partial\Omega. \]

In this case, the corresponding functional to the quasilinear Euler-Lagrange equation is

\[J(u) = \int_\Omega (a(x)+ | u |^\gamma) | \nabla u |^p - \frac{\lambda}{\theta} \int_\Omega | u |^{\theta} - \frac{1}{q} \int_\Omega | u |^q \]

where \(\gamma > 1\), \(\Omega\) is a bounded, open subset of \(\mathbb{R}^N\) with \(N > 2\), \(1 < p < N\) and \(a(x)\) is a measurable function such that for two constants \(\alpha\) and \(\beta\)

\[0 < \alpha \leq a(x) \leq \beta \text{ a.e. } x \in \Omega. \]

We notice that the functional \(J\) is not Gâteau differentiable in \(W_0^{1,p}(\Omega)\) but is only differentiable through the direction of \(W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\).

The main difficulty of this work is due to the term \(| u |^\gamma\) in which the functional \(J\) is well defined in \(W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\), if we impose an additional condition on \(\gamma\), namely, \(\gamma + p < p^*\). We point out that our approach has been studied in [1], including \(L^\infty(\Omega)\) a priori estimates. We apply the Theorem 2.8 in [3] to establish the existence of multiplicity critical points under hypotheses \(0 < \lambda < \lambda_0\) and \(1 < \theta < p < q < \frac{\gamma}{\gamma-1} \) where \(\gamma > 1\). We notice that the multiplicity results for \(p\)-Laplacian with critical growth of concave-convex functions has been intensively studied. Recently, the existence of multiplicity of bounded weak solutions for the quasilinear singular Euler-Lagrange equation with natural growth with \(p = N\) has been investigated by Quincy Stevene Nkombo (see [10]). Finally, the novelty of this paper is that we study the existence of multiplicity bounded weak solutions for quasilinear Euler-Lagrange equation with \(1 < p < N\).

Notation: in the rest of this work we make use of the following notation. \(L^p(\Omega), 1 \leq p \leq \infty\), denote lebesgue spaces. The usual norm in \(L^p(\Omega)\) is denoted by \(| \cdot |_p\). \(W_0^{1,p}(\Omega)\) denote sobolev spaces; the norm in \(W_0^{1,p}(\Omega)\) is denoted by \(\| \cdot \|_p\). \(C_0, C_1, C_2, C_3, \ldots\) denote (possibly different) positive constants.

2. The case \(0 < \lambda < \lambda_0\)

Definition 2.1 A measurable function \(u\) is called a weak solution to the equation (1.1)-(1.2), if \(u \in W_0^{1,p}(\Omega)\) such that \(| u |^{\gamma-2} u | \nabla u |^p \in L^1(\Omega)\) and

\[\int_\Omega (a(x)+ | u |^\gamma) | \nabla u |^{p-2} \nabla u \nabla v + \frac{\gamma}{p} \int_\Omega | u |^{\gamma-2} u | \nabla u |^p v \]

\[= \lambda \int_\Omega | u |^{\theta-2} u v + \int_\Omega | u |^{\varphi-2} u v \]

holds for every \(v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\).

The main result of this paper is focus on the existence of multiplicity bounded weak solutions to the equation (1.1)-(1.2). For that the result is given by the following
Theorem 2.1 Suppose that $\gamma$ satisfies the condition $\gamma + p < p^*$. Moreover, there exists $\lambda_0$ such that

$$1 < \theta < p < q < \frac{2p}{p} (\gamma + p); \quad 0 < \lambda < \lambda_0.$$  

(2.2)

Then, there exist infinitely many weak solutions for the problem (1.1)-(1.2).

Proof. We use the theorem 2.8 in [3] in order to prove the existence of multiplicity weak solutions to the problem (1.1)-(1.2). So that we divide this proof into several steps.

• Step 1: A truncated function

If $m$ is a positive integer, we consider the truncated function $f_m(t)$ at level $m$, $T_m(t)$ is given by

$$T_m(t) = \begin{cases}
-m - \frac{1}{2} & \text{if } t \leq -m - 1
\end{cases}
\begin{cases}
(m + 1)t + \frac{t^2 + m^2}{2} & \text{if } -m - 1 \leq t \leq -m
\end{cases}
\begin{cases}
t & \text{if } -m \leq t \leq m
\end{cases}
\begin{cases}
(m + 1)t + \frac{t^2 + m^2}{2} & \text{if } m \leq t \leq m + 1
\end{cases}
\begin{cases}
m + \frac{1}{2} & \text{if } t \geq m + 1
\end{cases}
$$

(2.3)

which is introduced in [1].

Assuming that $q_0$ and $q_1$ are two numbers such that $1 < q_0 < \theta < p < q_1 < q$ and the truncated function $f_{n,\lambda}(t)$ is defined by

$$f_{n,\lambda}(t) = \lambda h_n(t) + g_n(t),$$

where

$$h_n(t) = \begin{cases}
\frac{|t|^\theta}{\theta} & \text{if } |t| < n
\end{cases}
\begin{cases}
\frac{|t|^\theta}{\theta - \frac{1}{q_0}} + n^{\theta - q_0} \frac{|t|^n}{q_0} & \text{if } |t| \geq n
\end{cases}
$$

(2.4)

and

$$g_n(t) = \begin{cases}
\frac{|t|^\theta}{\theta} & \text{if } |t| < n
\end{cases}
\begin{cases}
\frac{|t|^\theta}{\theta - \frac{1}{q_1}} + n^{\theta - q_1} \frac{|t|^n}{q_1} & \text{if } |t| \geq n
\end{cases}
$$

(2.5)

By observing the definition of $h_n(t)$ and $g_n(t)$, we deduce the following inequalities

$$0 \leq h_n(t) \leq \frac{n^{\theta - q_0}}{q_0} |t|^q_0 \quad \text{and} \quad 0 \leq h_n(t) \leq \frac{|t|^\theta}{\theta}.$$  

(2.6)

$$0 \leq g_n(t) \leq \frac{n^{\theta - q_1}}{q_1} |t|^q_1 \quad \text{and} \quad 0 \leq g_n(t) \leq \frac{|t|^\theta}{q}.$$  

(2.7)

Consequently, we infer that the estimate of the function $f_{n,\lambda}(t)$ as follows

$$0 \leq f_{n,\lambda}(t) \leq \frac{\lambda_0^{\theta - q_0}}{q_0} |t|^q_0 + \frac{n^{\theta - q_1}}{q_1} |t|^q_1.$$  

(2.8)

Let us consider the truncated functional,

$$J_m,n(u) = \frac{1}{p} \int_{\Omega} (a(x) + |T_m(u)|^\gamma) |\nabla u| p - \int_{\Omega} f_{n,\lambda}(u).$$  

(2.9)

for $u \in W_0^{1,p}$. Which is clearly well defined for $q_0 < q_1 < p^*$.

• Step 2: Geometry of truncated function

Let $r$ a positive real constant such that

$$B_r = \{u \in W_0^{1,p}(\Omega) / \|u\|_p \leq r\}.$$  

The fact that, $a(x) + |T_m(u)|^\gamma \geq \alpha$ and integrating inequality (2.8) on $\Omega$, we get

$$\int_{\Omega} f_{n,\lambda}(u) = \lambda C_0 n^{\theta - q_0} |u|_{p_0} |C_1 n^{q - q_1} |u|_{p_1}.$$  

(2.10)

where $C_0$ and $C_1$ are nonnegative constants.

Combining (2.10) with the hypothesis, $a(x) + |T_m(u)|^\gamma \geq \alpha$, we obtain the following result

$$J_{m,n}(u) \geq \frac{\alpha}{p} \|u\|_p - \lambda C_0 n^{\theta - q_0} \|u\|_{p_0} + C_1 n^{q - q_1} \|u\|_{p_1}.$$  

(2.11)

Thereby, there exist nonnegative constants $r_n, \lambda_n$ and $\lambda_0$ such that

$$J_{m,n}(u) > 0 \quad \text{in } B_{r_n,\lambda_n} \quad \text{and} \quad J_{m,n}(u) \geq \lambda_n \text{ in } \partial B_{r_n,\lambda_n}$$  

for all $0 < \lambda < \lambda_0$.

• Step 3: Compactness of the truncated function

Let $\{w_k\}$ be a sequence in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfying, for every $n \in \mathbb{N}$ the following conditions

$$J_{m,n}(w_k) \leq C_1.$$  

$$|w_k|_\infty \leq 2b_k.$$  

$$J_{m,n}(w_k) \leq \varepsilon_k \left( \frac{|w|_\infty}{b_k} + \|w\|_p \right).$$  

(2.4)

where $C_1$ is a nonnegative constant, $\{b_k\} \subset R^+ - \{0\}$ is a nonnegative sequence and $\{\varepsilon_k\} \subset R^+ - \{0\}$ is a sequence converging to zero.

Suppose that

$$g(\lambda, t) = \frac{1}{t + \gamma} + f_{n,\lambda}(t).$$

And

$$g_0(\lambda) = \max_{t \in R} g(\lambda, t).$$

Where $\lambda > 0$ and $t > 0$. Let $\varepsilon > 0$, be given and choose $t_0 > 0$ such that

$$\max_{t \in R} g(\lambda, t) \leq g(\lambda, t_0) + \varepsilon.$$  

(2.7)

Clearly $\tilde{g}(\lambda, t_0)$ is an increasing and continuous function with respect to $\lambda$ and there exists $\tilde{\lambda}_0$ a nonnegative number $0 < \tilde{\lambda}_0 < \infty$ and such that

$$\tilde{g}(\tilde{\lambda}_0, t_0) \leq \frac{1}{p + \gamma} \quad \text{for all } 0 < \lambda < \tilde{\lambda}_0.$$  

(2.8)

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Therefore\[ \max_{t \in \mathbb{R}} \tilde{g}(\lambda, t) \leq \tilde{g}(\lambda, t_0) + \varepsilon \leq \tilde{g}(\lambda_0, t_0) + \varepsilon. \]

Which leads to\[ \max_{t \in \mathbb{R}} \tilde{g}(\lambda, t) \leq \frac{1}{p + \gamma} + \varepsilon, \text{ for all } 0 < \lambda < \lambda_0. \]

Then it is easily verified using induction that\[ f_{n,\lambda}(t) < g_0(\lambda) < \frac{1}{p + \gamma}, \text{ for all } 0 < \lambda < \lambda_0. \]

After straightforward calculation of the term\[ J_{m,n}(w_k) = g_0(\lambda) \langle J_{m,n}'(w_k), w_k \rangle \]

yields\[ \left( \frac{1}{p} - g_0(\lambda) \right) \int_{\Omega} a(x) | \nabla w_k |^p \]
\[ + \int_{\Omega} \left( \frac{1}{p} - g_0(\lambda) - \frac{\gamma}{p^*} g_0(\lambda) \right) T_m(w_k) \]
\[ \times | T_m(w_k) |^\gamma | \nabla w_k |^p \]
\[ + \int_{\Omega} (g_0(\lambda) w_k J_{m,n}'(w_k) - f_{n,\lambda}(w_k)) \]
\[ \leq C_1 + \varepsilon_k \left( \frac{1}{b_k} \| w_k \|_{h,\lambda} + \| w_k \|_{p} \right). \]

Notice that all the left-hand side terms are positives. Indeed, the first one is nonnegative due to consequence of the definition of $g_0(\lambda)$, namely\[ f_{n,\lambda}(t) < g_0(\lambda) < \frac{1}{p + \gamma}. \]

For the second term, it is enough to use the assumption, $0 < \frac{\gamma}{p^*} g_0(\lambda) \leq 1$, and that $g_0(\lambda) < \frac{1}{p + \gamma}$. The positiveness of the third term is verified in using the definition of $g_0(\lambda)$. Therefore, we can conclude that the sequence $\{w_k\}$ is bounded in $W_0^{1,p}(\Omega)$ for every $p$ such that $1 < p < N$. So that the sequence $\{w_k\}$ admit a subsequence that we still denote $\{w_k\}$, which converges to a function $w$.

**Step 4:** Existence of critical points of the truncated function.

We point out that the main idea of this proof is in [4], for that we adapt the arguments of Theorem 2.8 in [3] in order to prove the existence of multiplicity critical points of $J_{m,n}$.

Let $H_k$ be a $k$-dimensional subspace of $W_0^{1,p}(\Omega)$ as we take $w_k \in H_k$, the norm of $w_k$, $\| w_k \|_p$ is finite.

We set\[ \Sigma = \{ C \subset W_0^{1,p}(\Omega) : \forall \in C, C = -C \}. \]

For $C \in \Sigma$ the $Z_2$-genus of $C$ is denoted by $\gamma(C)$. According to the step 2 and step 3, the assumptions $(I_1)$ and $(I_3)$ of Theorem 2.3 hold true (see [3]).

Moreover, letting\[ A_{m,n} = B_{r_n,\lambda} \cup \{ J_{m,n} \geq 0 \}. \]

We can clearly assert that $H_k \cap A_{m,n}$ is bounded for all $n \in N$, the assumption $(I_5)$ Theorem 2.3 is complete.

Next we set\[ \Gamma^* = \{ h \in C(W_0^{1,p}(\Omega), W_0^{1,p}(\Omega)) : h \text{ is an odd homeomorphism } h(0) = 0 \text{ and } h(B_1) \subset A_{m,n} \}. \]

And\[ \Gamma_k = \{ K \in \Sigma : \gamma(K \cap h(\partial B_1)) \geq k \forall h \in \Gamma^* \}. \]

And then\[ S_k = \inf_{K \in \Gamma_k} \max_{w \in K} J_{m,n}(w). \]

So that we can state that lemma 2.7 in [3] holds. We then choose $h(w) = r_{n,\lambda} w$

where $r_{n,\lambda}$ a nonnegative real which has been defined in the step 2 and $h$ belongs to $\Gamma^*$. Consequently, we infer that $K \cap B_{r_n,\lambda} \neq \emptyset$ for all $K \in \Gamma_k$. Since $J_{m,n}$ is bounded from below on $\partial B_{r_n,\lambda}$, then\[ S_k = \inf_{K \in \Gamma_k} \max_{w \in K} J_{m,n}(w) \geq a_{n,\lambda} > 0. \]

Since all assumptions of Theorem 2.8 in [3] are satisfied, thus there exist infinitely many critical points of $J_{m,n}$. Hence, the Dirichlet problem (2.11)-(1.2) possesses infinitely many nontrivial weak solutions $\square$.

**Step 5:** Uniformly $L^\infty$-estimates

Consider the following equation

\[ -\text{div} \{ (a(x) + |T_m(w_{m,n})|^\gamma | \nabla w_{m,n} |^{p-2} \nabla w_{m,n} \}
\]
\[ + \frac{\gamma}{p} T_m(w_{m,n}) T_m(w_{m,n}) | \nabla w_{m,n} |^p \]
\[ = f_{n,\lambda}^{\prime}(w_{m,n}). \quad (2.11) \]

Assuming that either $w_{m,n} = u_{m,n}$ or $w_{m,n} = u_0^{m,n}$ or \ldots or $w_{m,n} = u_{m,n}$ or \ldots \ldots solution of (2.11)-(1.2).

Setting that $T_m(w_{m,n}) = w_{m,n}$ and $v = |w_{m,n}|^b w_{m,n}$ as a test function, then we have

\[ (b + 1) \int_{\Omega} (a(x) + |w_{m,n}|^\gamma | \nabla w_{m,n} |^b | w_{m,n} |^b \]
\[ + \frac{\gamma}{p} \int_{\Omega} |w_{m,n}|^{b+\gamma} | \nabla w_{m,n} |^p \]
\[ \leq (\lambda + 1) n^b \theta_{q_0+q_1} \int_{\Omega} |w_{m,n}|^{b+q}. \quad (2.12) \]

Dropping the positive terms on the left hand side of (2.12), we get

\[ (b + 1) \int_{\Omega} a(x) |w_{m,n}|^b | \nabla w_{m,n} |^p \]
\[ \leq (\lambda + 1) n^b \theta_{q_0+q_1} \int_{\Omega} |w_{m,n}|^{b+q}. \quad (2.13) \]
On the other hand, we obtain the following result after using the Sobolev inequality
\[ C_p^p \left( \int_\Omega |w_{m,n}|^{\frac{p+p}{p}} \right)^{\frac{p}{p}} \leq \frac{(b+p)^{p}}{p^p} \]
\[ \times \int_\Omega |w_{m,n}|^{\frac{p}{p}} |\nabla w_{m,n}|^{p}. \]
Therefore
\[ C_p^p \left( \int_\Omega |w_{m,n}|^{\frac{p+p}{p}} \right)^{\frac{p}{p}} \leq \frac{(b+p)^{p}}{p^p} \frac{1}{\alpha} \int_\Omega \alpha(x) |w_{m,n}|^{\frac{p}{p}} |\nabla w_{m,n}|^{p}. \] (2.14)
Combining (2.13) with (2.14), we have
\[ \left( \int_\Omega |w_{m,n}|^{\frac{p+p}{p}} \right)^{\frac{p}{p}} \leq \frac{(b+p)^p (\alpha (pC_2)^p (b+1))}{\alpha (pC_2)^p (b+1)} \]
\[ \times n^{\theta-q_0+q-1} \int_\Omega |w_{m,n}|^{\frac{p}{p}} |\nabla w_{m,n}|^{p}. \]
It follows that
\[ \int w_{m,n}^{\frac{b+q}{p}} \leq \left( \frac{(b+p)^p (\alpha (pC_2)^p (b+1))}{\alpha (pC_2)^p (b+1)} \right)^{\frac{1}{\alpha(pC_2)^p (b+1)}} \]
\[ \times n^{\theta-q_0+q-1} \int_\Omega |w_{m,n}|^{\frac{p}{p}} |\nabla w_{m,n}|^{p}. \]
Let \( r = b + q \), then
\[ \int w_{m,n}^{\frac{b+q}{p}} \leq \left( \frac{(r-q+p)^p (\alpha (pC_2)^p (r+q-1))}{\alpha (pC_2)^p (r+q-1)} \right)^{\frac{1}{\alpha(pC_2)^p (r+q-1)}} \]
\[ \times n^{\theta-q_0+q-1} \int_\Omega |w_{m,n}|^{\frac{p}{p}} |\nabla w_{m,n}|^{p}. \]
Notice that \( w_{m,n} \) belongs to \( W^{1,p}(\Omega) \) and so to \( L^{p_0}(\Omega) \), we can choose \( r = r_0 = p^* - q \) to deduce that \( w_{m,n} \) belongs to \( L^{\frac{p^*+p}{p^*}}(\Omega) \), we can then choose \( r = r_1 = \frac{r_0}{p} \) to obtain \( w_{m,n} \) belongs to \( L^{\frac{p^*+p}{p^*}}(\Omega) \). Iterating this process and defining by induction \( r_k \) as
\[ \begin{aligned}
& r_0 = p^* - q \\
& r_k = r_{k-1} + \frac{p}{p} (p-q).
\end{aligned} \] (2.15)
We infer that \( w_{m,n} \) belongs to \( L^{p^*}(\Omega) \) with
\[ \int w_{m,n}^r \leq \left( \frac{(r_k-q+p)^p (\alpha (pC_2)^p (r_k-q-1))}{\alpha (pC_2)^p (r_k-q-1)} \right)^{\frac{1}{\alpha(pC_2)^p (r_k-q-1)}} \]
\[ \times n^{\frac{\theta-q_0+q-1}{r_k-1}} \int_\Omega |w_{m,n}|^{\frac{p}{p}} |\nabla w_{m,n}|^{p}. \]
Therefore
\[ \int w_{m,n}^r \leq \ldots \leq C_3 \int w_{m,n} \leq C_4. \]
Because of \( \int_\Omega \alpha(x) |\nabla w_{m,n}|^{p} \) is bounded with respect to \( m \) and \( n \).
Since \( \frac{p}{p} > 1 \), it is enough to show that \( r_k \) is increasing sequence which diverges to infinity, thus, if it is such that
\[ \frac{r_k+p-q}{p} \geq \frac{N}{q} \] an adaptation to the quasilinear case of the proof of a result of Stampacchia (see [5]) implies that there exists \( M_n > 0 \) such that
\[ |w_{m,n}| \leq M_n. \]
Let \( m_n \) be an integer such that \( m_n \geq \max \{ M_n + p, r \} \), if we define \( w_n = w_n^{m_n} \), namely, either \( w_n = u^m \) or \( w_n = u^k \) or \( u^k \) or \( w_n = u^k \).
Then \( T_{m_n}(w_n) = w_n \) and \( T_{m_n}(w_n) = 1 \), consequently, the equation which is satisfied by \( w_n \) is
\[ -\text{div} \left( (a(x) + |w_n|^{p}) |\nabla w_n|^{p-2} |\nabla w_n| \right) \]
\[ + \frac{p}{p} |w_n|^{\gamma^{-2}} |\nabla w_n|^{p} = f_n, \lambda(w_n) \] (2.16)
with zero Dirichlet boundary condition.
Notice that by the assumption \( q < \frac{p}{p} (p+\gamma) \), then \( w_n \) is bounded in \( L^6(\Omega) \) using this fact, we are going to show that \( w_n \) is uniformly bounded in \( L^\infty(\Omega) \).
Let \( b > 0 \) as before, and choose \( v = |w_n|^b \) as a test function in the equation (2.16)-(1.2) satisfied \( w_n \).
The fact that \( f_n, \lambda(t) \leq (\lambda + 1) \) \( |t|^{q+b-1} \) and we drop two nonnegative terms, and then we obtain
\[ (b+1) \int_\Omega \alpha(x) |w_n|^{b+q} |\nabla w_n|^{p} \leq (b+1) \int_\Omega |w_n|^{b+q} \]
However, we get another inequality when we apply the Sobolev inequality to \( w_n^{b+p+q} \), we then have
\[ C_p^p \left( \int_\Omega |w_n|^{\frac{p+p}{p}} \right)^{\frac{p}{p}} \leq \left( \frac{\gamma + b + p}{p} \right)^{\frac{p}{p}} \int_\Omega |w_n|^{b+q} |\nabla w_n|^{p}. \]
Thus
\[ \left( \int_\Omega |w_n|^{\frac{p+p}{p}} \right)^{\frac{p}{p}} \leq \left( \frac{\gamma + b + q}{p} \right)^{\frac{p}{p}} \int_\Omega |w_n|^{b+q}. \]
Where \( w_n \) belongs to \( L^{\frac{b+q+p}{b+q}}(\Omega) \) provided that \( w_n \) belongs to \( L^\infty(\Omega) \) with \( b = r - q \), yields
\[ |w_n|^{\frac{p+p}{p}} \leq \left( \frac{\gamma + b + q}{p} \right)^{\frac{p}{p}} |w_n|^{\frac{p+p}{p}}. \]
Because of \( \frac{p}{p} \geq p^* \). Arguing as before, if we consider the sequence \( r_k \) as follows
\[ \begin{aligned}
& r_0 = \frac{p}{p} (\gamma + p) \\
& r_k = r_{k-1} + \frac{p}{p} (\gamma + p - q).
\end{aligned} \] (2.17)
Thus \( w_{m,n} \) belongs to \( L^{p^*}(\Omega) \) for every \( k \) and so
\[ |w_n|^{r_k} \leq \left( \frac{\gamma + r_k - q + p}{p} \right)^{\frac{p}{p}} |w_n|^{\frac{p+p}{p}}. \]
It follows that
\[ | w_n | r_{2} \leq \ldots \leq C_{0} | w_n | \left( \frac{\phi_n}{| x^{+} |} \right)^{k+\frac{n}{p}} \leq C_{r}. \]

The fact that
\[ \int_{\Omega} | w_n | n \nabla w_n | P \] is bounded with respect to n.

And \( w_n \in L_{r}^{\theta}((\gamma + \rho))(\Omega) \), clearly the sequence \( \{ r_{n} \} \) is increasing and unbounded for \( \frac{\phi_{n}}{P} > 1 \). So that in a finite number of steps we conclude that
\[ \lambda \ | w_n | q-2 \ w_n + | w_n | q-2 \ w_n \]
is bounded in \( L_{r} \) with \( r > \frac{\phi_{n}}{2} \). Using again an adaptation of the proof theorem 2.1 in [5] yields that there exists a nonnegative constant \( C_{0}^{\prime} > 0 \), such that
\[ | w_n | \infty \leq C_{0}^{\prime}, \ \forall n \geq \max(\tau, \pi) \]

In other words, we obtain
\[ | u_{n}^{0} | \infty \leq C_{1}^{\prime}, \ | u_{n}^{1} | \infty \leq C_{2}^{\prime}, \ldots \ldots \ldots | u_{n}^{k} | \infty \leq C_{k}^{\prime} \]
\[ \forall n \geq \max(\tau, \pi). \]

• Step 6: Conclusion
Finally, if \( \forall n \geq \max(C_{0}^{\prime}, \tau, \pi) \) then
\[ f_{n, \lambda}(w_n) = \lambda | w_n | q-2 \ w_n + | w_n | q-2 \ w_n \]
and so \( w \equiv w_{\pi} \). In other words, either
\[ w \equiv u_{n}^{0} \equiv u_{n}^{0} \pi \text{ or } w \equiv u_{1}^{0} \equiv u_{n}^{0} \pi \text{ or \ldots \ldots \ldots \ldots or} \]
\[ w \equiv u_{k}^{0} \equiv u_{n}^{0} \pi \text{ or \ldots \ldots \ldots \ldots or} \]

Hence, we can conclude that the problem (1.1)-(1.2) has an infinitely many positive bounded weak solutions.

3. The case \( \lambda > \lambda_{0} \)
We complete the study of the equation (1.1)-(1.2) by showing that such equation does not have nontrivial solution. In order to prove this fact, we assume that \( \Omega \) is star-shaped, i.e., \( x, \nu > 0 \) on \( \partial \Omega \). Where \( \nu \) is outward normal to \( \partial \Omega \). For that we use the idea of [9] in the next proposition.

Proposition 3.1 If \( \Omega \) is a smooth star-shaped in \( R^{N} \)
containing \( 0 \), then \( u \equiv 0 \) is the unique \( H^{2}(\Omega) \cap H_{0}^{1}(\Omega) \)
nonnegative solution of (1.1)-(1.2).

Proof. Let \( u \) belongs to \( H^{2}(\Omega) \cap H_{0}^{1}(\Omega) \) be a nonnegative solution of (1.1). The divergence of the vector field \( (a(x)+ | u | \gamma) | \nabla u | p-2 \ \nabla u(x \nabla u) \) can be written as follows
\[ \text{div} \{(a(x)+ | u | \gamma) | \nabla u | p-2 \ \nabla u(x \nabla u)\} \]
\[ = (x \nabla u) \text{div} \{(a(x)+ | u | \gamma) | \nabla u | p-2 \ \nabla u\} \]
\[ + (a(x)+ | u | \gamma) | \nabla u | p-2 \ \nabla u \nabla(x \nabla u). \]

Since
\[ \nabla u \nabla(x \nabla u) = | \nabla u |^{2} + \frac{1}{2} \left( x \nabla(| \nabla u |^{2}) \right) . \]

And
\[ \frac{1}{2} (a(x)+ | u | \gamma) | \nabla u | p-2 \ \left( x \nabla(| \nabla u |^{2}) \right) \]
\[ = \frac{1}{p} (a(x)+ | u | \gamma) (x \nabla(| \nabla u |^{p})). \]

Consequently,
\[ \text{div} \{(a(x)+ | u | \gamma) | \nabla u | p-2 \ \nabla u(x \nabla u)\} \]
\[ = (x \nabla u) \text{div} \{(a(x)+ | u | \gamma) | \nabla u | p-2 \ \nabla u\} \]
\[ + (a(x)+ | u | \gamma) | \nabla u |^{p} \]
\[ + \frac{1}{p} (a(x)+ | u | \gamma) (x \nabla(| \nabla u |^{p})). \]

Multiplying the equation (1.1) by \( x \nabla u \), yields
\[ (x \nabla u) \text{div} \{(a(x)+ | u | \gamma) | \nabla u | p-2 \ \nabla u(x \nabla u)\} \]
\[ = \frac{\gamma}{p} | u |^{p-2} u(x \nabla u) + \frac{\lambda}{p} | u |^{p-2} \ \nabla u(x \nabla u), \]
\[ - \frac{\lambda}{p} u^{q-2} u(x \nabla u) - | u |^{q-2} u(x \nabla u) \]
\[ + (a(x)+ | u | \gamma) | \nabla u |^{p} \]
\[ + \frac{1}{p} (a(x)+ | u | \gamma) (x \nabla(| \nabla u |^{p})). \]

Replacing (3.3) into (3.2), we have
\[ \text{div} \{(a(x)+ | u | \gamma) | \nabla u | p-2 \ \nabla u(x \nabla u)\} \]
\[ = \frac{\gamma}{p} | u |^{p-2} u(x \nabla u) + \frac{\lambda}{p} | u |^{p-2} \ \nabla u(x \nabla u), \]
\[ - \frac{\lambda}{p} u^{q-2} u(x \nabla u) - | u |^{q-2} u(x \nabla u) \]
\[ + (a(x)+ | u | \gamma) | \nabla u |^{p} \]
\[ + \frac{1}{p} (a(x)+ | u | \gamma) (x \nabla(| \nabla u |^{p})). \]

On the other hand, applying Gauss formula to the vector field \( (a(x)+ | u | \gamma) | \nabla u | p-2 \ \nabla u(x \nabla u) \), we obtain
\[ \int_{\Omega} \text{div} \{(a(x)+ | u | \gamma) | \nabla u | p-2 \ \nabla u(x \nabla u)\} \]
\[ = \int_{\partial \Omega} (a(x)+ | u | \gamma) | \nabla u |^{p} (x.\nu) ds. \]

Combining (3.4) with (3.5), we get
\[ - \frac{\gamma N}{p} \int_{\Omega} | u |^{p} \ \nabla u |^{p} - \lambda \int_{\partial \Omega} | u |^{q} (x.\nu) ds \]
\[ - \int_{\partial \Omega} | u |^{q} (x.\nu) ds + \left( 1 - \frac{N}{p} \right) \int_{\partial \Omega} (a(x)+ | u | \gamma) | \nabla u |^{p} \]
\[ = \int_{\partial \Omega} (a(x)+ | u | \gamma) | \nabla u |^{p} (x.\nu) ds. \]

The fact that \( p < N \) and \( x.\nu > 0 \) on \( \partial \Omega \).
Therefore \( u \equiv 0 \). □
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References


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