Stochastic Generalized Complementarity Problems in Second-Order Cone: Box-Constrained Minimization Reformulation and Solving Methods

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Abstract—**In this paper, we reformulate the stochastic generalized second-order cone complementarity problems as box-constrained optimization problems. If satisfy the condition that the reformulation's objective value is zero, the solutions of box-constrained optimization problems are also solutions of stochastic generalized second-order cone complementarity problems. Since the box-constrained minimization problems contain an expectation function, we then employ sample average approximation method to give approximation problems of the box-constrained minimization reformulation. Meanwhile the convergence results of global solutions and stationary points for the corresponding approximation problems are considered.**

Index Terms—**box-constrained minimization problems, sample average approximation, convergence.**

I. Introduction

GENERALIZED second-order cone complementarity problems, denoted by GSOCCP(F,G,K) are to find a vector $x \in R^n$ satisfying

$$F(x) \in K^*, \quad G(x) \in K, \quad F(x)^T G(x) = 0,$$

where $F(x), G(x) : R^n \to R^n$ are continuously differentiable respect to $x \in R^n$. K is the convex cone

$$K = \{x \in R^n | x_1^2 \geq \sum_{j=2}^{n} a_j^2 x_j^2, x_1 \geq 0\}$$

and its polar cone K^* denoted by

$$K^* = \{x \in R^n | \forall\ y \in K, \langle x, y\rangle \geq 0\}.$$

Here, $a_j, j = 2, \ldots, n$ is the given parameter. The generalized second-order cone complementarity problems play a basic role and occupy an important position in the minimization theory, which have vital applications in many fields. In [2] the authors analyze the case where K is a polyhedral cone. In [3], [4], based on the well know Fischer-Burmeister NCP-function, smooth merit function is presented for the second-order cone product of Lorentz cones. In [9], the authors study the special case of GSOCCP(F,G,K), where $G(x) = x$ and K is Cartesian product of Lorentz cones. Moreover, in [1] R. Andreani etc. consider a special cone, which is denoted as $K = \{x \in R^n | x_1^2 \geq \|(x_2, \ldots, x_p)^T\|^2\}$. Since the origin is not a regular point, the authors make up for this lack by combining two functions which are mentioned in [2]. Then the authors construct two reformulations of GSOCCP(F,G,K) which are nonlinear minimization problems with box constraints. More basic theories, effective algorithms and important applications of GSOCCP(F,G,K) can be found in [7], [13].

However, in practice, stochastic elements are usually involved in several problems. Paying no attention to these stochastic elements will make mistake decision. Therefore, it is meaningful and interesting to study the stochastic generalized second-order cone complementarity problems.

Stochastic generalized second-order cone complementarity problems, denoted by SGSOCCP(F,G,K) are to find a vector $x \in R^n$ such that

$$\mathbf{E}[F(x,\omega)] \in K^*, \mathbf{E}[G(x,\omega)] \in K,$$
$$\mathbf{E}[F(x,\omega)]^T \mathbf{E}[G(x,\omega)] = 0,$$

where $\mathbf{E}$ denotes mathematical expectation, $\omega \in \Omega \subseteq R^m$ denotes stochastic variable, $F(x,\omega), G(x,\omega) : R^n \times \Omega \to R^n$ are continuously differentiable respect to $x \in R^n$, the convex cones K and K^* are defined as above.

Without loss of generality, set $A = diag(1, -a_2^2, -a_3^2, \ldots, -a_p^2, 0, \ldots, 0)$, where $a_i \neq 0, for\ 2 \leq i \leq p$, $\overline{A} = diag(1, \frac{-1}{a_2^2}, \ldots, \frac{-1}{a_p^2}, 0, \ldots, 0)$ and $M = diag(m_1, m_2, \ldots, m_n)$, where $m_i = 0$, for $1 \leq i \leq p$, and $m_i = 1$ for $i > p$, then the convex cone K can be expressed by matrix form, that is

$$K = \{x \in R^n | \frac{1}{2} x^T A x \geq 0,\ x_1 \geq 0\}$$

and its corresponding polar cone K^* is

$$K^* = \{x \in R^n | \frac{1}{2} x^T \overline{A} x \geq 0, x_1 \geq 0, Mx = 0\}.$$

In this paper, we suppose $a_i = 1$, for $i = 2, \ldots, p$. Then, we can rewrite SGSOCCP(F,G,K) as follows: Find $x \in R^n$ such that

$$\begin{aligned} \mathbf{E}[F(x,\omega)] \in K^*, \mathbf{E}[G(x,\omega)] \in K, \\ \mathbf{E}[F(x,\omega)]^T \mathbf{E}[G(x,\omega)] = 0, \end{aligned} \tag{1}$$

where

$$K = \{x \in R^n | x_1^2 \geq \|(x_2, \ldots, x_p)^T\|^2,\ x_1 \geq 0\}$$

and its polar cone

$$K^* = \{x \in R^n | x_1^2 \geq \|(x_2, \ldots, x_p)^T\|^2, x_1 \geq 0, Mx = 0\}.$$

Since p may be strictly less than n, the cone considered in this paper is more general. This implies, in particular, that K^* may be different from K.

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In fact, problem (1) is also a stochastic complementarity problem. About the classical stochastic complementarity problems(denoted by SCP), there are many good results have been published. For example, the expected value (EV) model suggested by Gürkan et al. [8]. The Expected residual minimization (ERM) model presented by Chen and Fukushima [5]. The Condition value-at-risk (CVaR) model presented by Chen and Lin [6]. The CVaR-constrained stochastic programming model presented by Xu and Yu [18]. More recently research of SCP can be seen in [11], [10], [19].

Based on [1], in this paper, we investigate SGSOCCP(F,G,K), where a special cone denoted as $K = \{x \in R^n | x_1^2 \geq \|(x_2, \ldots, x_p)^T\|^2\}$ is considered. The main difference between two papers is that SGSOCCP(F,G,K) contain expectation which is usually difficult to evaluate. Therefore, we employ sample average approximation (SAA) method to solve the corresponding box-constrained minimization reformulation. However, whether the solutions of SAA approximation problems are regarded as the solutions of SGSOCCP(F,G,K) is reasonable? To answer this question, we consider the convergence results of global solutions and stationary points for the corresponding approximation problems in theory.

The remainder of the paper is organized as follows: In Section 2, we introduce the box-constrained reformulation and the corresponding approximation problems of SGSOCCP(F, G, K). In Section 3, the convergence results of global solutions and stationary points for the approximation problems are considered. In Section 4, we give the conclusions.

II. Box-constrained reformulation of SGSOCCP(F,G,K)

In this section, we will study the equivalence problems of SGSOCCP(F,G,K) and the approximation problems of box-constrained minimization problems.

Inspired by [1], in order to reformulate SGSOCCP(F,G,K) via nonlinear programming, we employ the KKT conditions of minimization problems which are related to SGSOCCP(F,G,K) to construct two merit functions denoted by

$$\begin{aligned} f(x,t) &= \|\mathbf{E}[F(x,\omega)] - t_1 A\mathbf{E}[G(x,\omega)] - t_2 e_1\|^2 \\ &\quad + (\frac{1}{2}\mathbf{E}[G(x,\omega)]^T A\mathbf{E}[G(x,\omega)] - t_3)^2 \\ &\quad + (\mathbf{E}[G_1(x,\omega)] - t_4)^2 + (t_1 t_3)^2 + (t_2 t_4)^2, \\ g(x,s,\zeta) &= \|\mathbf{E}[G(x,\omega)] - s_1\overline{A}\mathbf{E}[F(x,\omega)] - s_2 e_1 - M\zeta\|^2 \\ &\quad + (\frac{1}{2}\mathbf{E}[F(x,\omega)]^T\overline{A}\mathbf{E}[F(x,\omega)] - s_3)^2 + (s_1 s_3)^2 \\ &\quad + (\mathbf{E}[F_1(x,\omega)] - s_4)^2 + \|M\mathbf{E}[F(x,\omega)]\|^2 + (s_2 s_4)^2, \end{aligned}$$

where $x, \zeta \in R^n$, setting $t = (t_1, t_2, t_3, t_4)^T \in R^4$, $s = (s_1, s_2, s_3, s_4)^T \in R^4$, and $e_1 = (1, 0, \ldots, 0)^T \in R^n$.

Then, SGSOCCP(F,G,K) is equivalent to the following box-constrained minimization problem:

$$\begin{aligned} \min_{(x,t,s,\zeta,r)} & \theta(x,t,s,\zeta,r) = rf(x,t) + (1-r)g(x,s,\zeta) \\ s.t. \quad & 0 \leq r \leq 1, \\ & t \geq 0, \\ & s \geq 0. \end{aligned} \tag{2}$$

If (x, t, s, ζ, r) is a solution of (2) with objective value zero, then x is a solution to (1). The detail proof see [1]. Noting that, (2) is a box-constrained minimization problem which is easily to solve and to deal.

Since the objective function of problem (2) contains an expectation which is generally difficult to evaluate, we use the sample average approximation (SAA) method to solve (2). The detail can be seen in [15]. We suppose that $\psi : \Omega \to R$ is integrable and $\Omega_k := \{\omega^1, \cdots, \omega^{N_k}\}$ represents sample space of random ω, we then use $\frac{1}{N_k}\sum_{\omega^j \in \Omega_k} \psi(\omega^j)$ to approximate $\mathbf{E}[\psi(\omega)]$. If $\{N_k\}$ is not a decreasing function respect to k, the strong law of large numbers guarantees that the following result holds with probability one (abbreviated by w.p.1). That is

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{\omega^j \in \Omega_k} \psi(\omega^j) = \mathbf{E}[\psi(\omega)].$$

For simplicity, we assume

$$\begin{aligned} l_1(x,t_1,t_2) &= \mathbf{E}[F(x,\omega)] - t_1 A\mathbf{E}[G(x,\omega)] - t_2 e_1, \\ l_2(x,t_3) &= \frac{1}{2}\mathbf{E}[G(x,\omega)]^T A\mathbf{E}[G(x,\omega)] - t_3, \\ l_3(x,t_4) &= \mathbf{E}[G_1(x,\omega)] - t_4, \\ l_4(x,s_1,s_2,\zeta) &= \mathbf{E}[G(x,\omega)] - s_1\overline{A}\mathbf{E}[F(x,\omega)] - s_2 e_1 - M\zeta, \\ l_5(x,s_3) &= \frac{1}{2}\mathbf{E}[F(x,\omega)]^T\overline{A}\mathbf{E}[F(x,\omega)] - s_3, \\ l_6(x,s_4) &= \mathbf{E}[F_1(x,\omega)] - s_4, \\ l_7(x) &= M\mathbf{E}[F(x,\omega)]. \end{aligned}$$

Corresponding, we let

$$\begin{aligned} l_1^k(x,t_1,t_2) &= \frac{1}{N_k}\sum_{\omega^j \in \Omega_k} F(x,\omega^j) - t_1 A\big(\frac{1}{N_k}\sum_{\omega^j \in \Omega_k} G(x,\omega^j)\big) - t_2 e_1, \\ l_2^k(x,t_3) &= \big(\frac{1}{2N_k}\sum_{\omega^j \in \Omega_k} G(x,\omega^j)\big)^T A\big(\frac{1}{N_k}\sum_{\omega^j \in \Omega_k} G(x,\omega^j)\big) - t_3, \\ l_3^k(x,t_4) &= \big(\frac{1}{N_k}\sum_{\omega^j \in \Omega_k} G_1(x,\omega^j)\big) - t_4, \\ l_4^k(x,s_1,s_2,\zeta) &= \big(\frac{1}{N_k}\sum_{\omega^j \in \Omega_k} G(x,\omega^j)\big) - s_1\overline{A}\big(\frac{1}{N_k}\sum_{\omega^j \in \Omega_k} F(x,\omega^j)\big) \\ &\quad - s_2 e_1 - M\zeta, \\ l_5^k(x,s_3) &= \big(\frac{1}{2N_k}\sum_{\omega^j \in \Omega_k} F(x,\omega^j)\big)^T\overline{A}\big(\frac{1}{N_k}\sum_{\omega^j \in \Omega_k} F(x,\omega^j)\big) - s_3, \\ l_6^k(x,s_4) &= \big(\frac{1}{N_k}\sum_{\omega^j \in \Omega_k} F_1(x,\omega^j)\big) - s_4, \\ l_7^k(x) &= M\big(\frac{1}{N_k}\sum_{\omega^j \in \Omega_k} F(x,\omega^j)\big). \end{aligned}$$

Thus, we have

$$\begin{aligned} f^k(x,t) &= \|l_1^k(x,t_1,t_2)\|^2 + (l_2^k(x,t_3))^2 + (l_3^k(x,t_4))^2 \\ &\quad + (t_1 t_3)^2 + (t_2 t_4)^2, \\ g^k(x,s,\zeta) &= \|l_4^k(x,s_1,s_2,\zeta)\|^2 + (l_5^k(x,s_3))^2 + (l_6^k(x's_4))^2 \\ &\quad + \|l_7^k(x)\|^2 + (s_1 s_3)^2 + (s_2 s_4)^2. \end{aligned}$$

Then, by SAA method one may construct, for each k, an approximation of problem (2) can be constructed as follows:

$$\begin{aligned} \min_{(x,t,s,\zeta,r)} & \theta^k(x,t,s,\zeta,r) = rf^k(x,t) + (1-r)g^k(x,s,\zeta) \\ s.t. \quad & 0 \leq r \leq 1, \\ & t \geq 0, \\ & s \geq 0. \end{aligned} \tag{3}$$

In this paper, we assume that, for each $x \in R^n$, $F(x,\omega)$ and $G(x,\omega)$ are integrable over $\omega \in \Omega$, and for any $\omega \in \Omega$, $F(x,\omega)$ and $G(x,\omega)$ are continuously differentiable respect to x. Furthermore, we suppose that Ω is a nonempty set and for all $x \in R^n$

$$\max\{\|\nabla_x F(x,\omega)\|^2, \|\nabla_x G(x,\omega)\|^2, \|F(x,\omega)\|^2, \|G(x,\omega)\|^2\} \le \sigma(\omega), \quad (4)$$

and

$$\mathbf{E}[\sigma(\omega)] < +\infty, \quad (5)$$

from (4),(5) and Cauchy-Schwarz inequality, for $x \in R^n$, we have

$$\max\{\|\nabla_x F(x,\omega)\|, \|\nabla_x G(x,\omega)\|, \|F(x,\omega)\|, \|G(x,\omega)\|\} \le \sqrt{\sigma(\omega)}, \quad (6)$$

and

$$\mathbf{E}[\sqrt{\sigma(\omega)}] < +\infty. \quad (7)$$

In addition, for an $n \times n$ matrix A, $\|A\|$ denotes spectrum norm and $\|A\|_{\mathcal{F}}$ denotes Frobenius norm, that is

$$\|A\|_{\mathcal{F}} := (\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2)^{\frac{1}{2}},$$

where a_{ij} be the factor of matrix A. By the definition, we have $\|A\| \le \|A\|_{\mathcal{F}}$. For a function $\eta(x)$: $R^n \to R$, $\nabla_x \eta(x)$ denotes gradient of $\eta(x)$ respect to x.

III. Convergence Analysis

Before considering the convergence results of (3), we give the following Definitions and Lemmas, which have been given in [12] firstly.

Definition 1: Let $h(x) : R^n \to R^n$, for each N, $h_N(x) : R^n \to R^n$ is a mapping. If for any $\epsilon > 0$, there exists $N(\epsilon)$, when $N > N(\epsilon)$, we have

$$\|h_N(x) - h(x)\| < \epsilon, \quad \forall x \in R^n.$$

Then we call $\{h_N(x)\}$ is uniformly convergent to $h(x)$ on R^n.

Definition 2: Let $h(x) : R^n \to R^n$, for each N, $h_N(x) : R^n \to R^n$ is a mapping. If for any sequence $\{x^N\} \to x$, when $N \to \infty$, we have

$$\lim_{N\to\infty} h_N(x^N) = h(x), \quad \forall x \in R^n.$$

Then we call $\{h_N(x^N)\}$ is continuously convergent to $h(x)$ on R^n.

Lemma 1: Let $h : R^n \to R^n$, for each N, $h_N : R^n \to R^n$ be a mapping, then h_N is continuously convergent to h on compact set $\mathcal{D}$ if and only if h_N is uniformly convergent to h on compact set $\mathcal{D}$.

Theorem 1: Suppose that for each k, $\{(x^k, t^k, s^k, \zeta^k, r^k)\}$ is a global optimal solution of problem (3) and $(x^*, t^*, s^*, \zeta^*, r^*)$ is an accumulation point of $\{(x^k, t^k, s^k, \zeta^k, r^k)\}$. Then, we have $(x^*, t^*, s^*, \zeta^*, r^*)$ is a global optimal solution of problem (2).

Proof: Without loss of generality, we may assume $\lim_{k\to\infty}(x^k, t^k, s^k, \zeta^k, r^k) = (x^*, t^*, s^*, \zeta^*, r^*)$. Let $B \subset S$ be a compact convex set containing the sequence $\{(x^k, t^k, s^k, \zeta^k, r^k)\}$, where $S := \{(x, t, s, \zeta, r)|x \in R^n, t \ge 0, s \ge 0, \zeta \in R^n, 0 \le r \le 1\}$ denotes the feasible region of (2). Let

$$\phi(x) = \mathbf{E}[F(x,\omega)], \phi^k(x) = \frac{1}{N_k}\sum_{\omega^j \in \Omega_k} F(x,\omega^j),$$

$$\varphi(x) = \mathbf{E}[G(x,\omega)], \varphi^k(x) = \frac{1}{N_k}\sum_{\omega^j \in \Omega_k} G(x,\omega^j).$$

From (6), (7) and Propositon 7 in Chapter 6 of [16], $\phi(x)$, $\varphi(x)$ are continuous respect to x and $\phi^k(x)$, $\varphi^k(x)$ are uniformly convergent to $\phi(x)$, $\varphi(x)$ on B, respectively. Moreover, from Lemma 1, we have $\phi^k(x^k)$, $\varphi^k(x^k)$ are continuously convergent to $\phi(x^*)$, $\varphi(x^*)$ on B, respectively. Taking a limit for $l_1^k(x^k, t_1^k, t_2^k)$, ..., $l_7^k(x^k)$, we obtain $l_1^k(x^k, t_1^k, t_2^k)$, ..., $l_7^k(x^k)$ are continuously convergent to $l_1(x^*, t_1^*, t_2^*)$, ..., $l_7(x^*)$ on B, respectively. From the definitions of $\theta(x, t, s, \zeta, r)$ and $\theta^k(x, t, s, \zeta, r)$, we have that $\theta^k(x^k, t^k, s^k, \zeta^k, r^k)$ is continuously convergent to $\theta(x^*, t^*, s^*, \zeta^*, r^*)$ on B. On the other hand, for each k, since $(x^k, t^k, s^k, \zeta^k, r^k)$ is a global solution of (3), for $\forall(x, t, s, \zeta, r) \in S$ there holds

$$\theta^k(x^k, t^k, s^k, \zeta^k, r^k) \le \theta^k(x, t, s, \zeta, r). \quad (8)$$

Letting $k \to \infty$ in (8) and by the fact that $\theta^k(x^k, t^k, s^k, \zeta^k, r^k)$ is continuously convergent to $\theta(x^*, t^*, s^*, \zeta^*, r^*)$ on B, we have that

$$\theta(x^*, t^*, s^*, \zeta^*, r^*) \le \theta(x, t, s, \zeta, r), \quad \forall(x, t, s, \zeta, r) \in S,$$

which indicates $(x^*, t^*, s^*, \zeta^*, r^*)$ is a global optimal solution of (2). ■

For simplicity, we set $\theta_\omega(x, t, s, \zeta, r) = rf_\omega(x, t) + (1 - r)g_\omega(x, s, \zeta)$, where

$$\begin{aligned} f_\omega(x,t) &= \|F(x,\omega) - t_1 AG(x,\omega) - t_2 e_1\|^2 \\ &\quad + (\frac{1}{2}G(x,\omega)^T AG(x,\omega) - t_3)^2 \\ &\quad + (G_1(x,\omega) - t_4)^2 + (t_1 t_3)^2 + (t_2 t_4)^2, \\ g_\omega(x,s,\zeta) &= \|G(x,\omega) - s_1\overline{A}F(x,\omega) - s_2 e_1 - M\zeta\|^2 \\ &\quad + (\frac{1}{2}F(x,\omega)^T\overline{A}F(x,\omega) - s_3)^2 + (s_1 s_3)^2 \\ &\quad + F_1(x,\omega - s_4)^2 + \|MF(x,\omega)\|^2 + (s_2 s_4)^2. \end{aligned}$$

We then show that with the increase of sample size, the optimal solutions of the approximation problem (3) converge exponentially to a solution of problem (2) with probability approaching one.

Theorem 2: Suppose that for each k, $\{(x^k, t^k, s^k, \zeta^k, r^k)\}$ is a global optimal solution of problem (3) and $\{(x^k, t^k, s^k, \zeta^k, r^k)\}$ itself converges to $(x^*, t^*, s^*, \zeta^*, r^*)$. Let B be a compact set that contains the whole sequence $\{(x^k, t^k, s^k, \zeta^k, r^k)\}$. Then, for any $\varepsilon > 0$ there exist positive constants $C(\varepsilon)$ and $\beta(\varepsilon)$, independent of N_k, such that

$$Prob\{sup_{(x,t,s,\zeta,r)\in B}|\theta^k(x, t, s, \zeta, r) - \theta(x, t, s, \zeta, r)| \ge \varepsilon\} \le C(\varepsilon)e^{-N_k\beta(\varepsilon)}.$$

Proof: For all $\omega \in \Omega$ and all $(x, t, s, \zeta, r) \in B$, noting that θ_ω and θ are all continuously differentiable with respect to (x, t, s, ζ, r), by the norm inequality or the absolute value inequality, following from (4)(5)(6)(7), after a simple calculation, we have that there exist $\kappa_1(\omega)$, $\kappa_2(\omega)$ satisfying

$$\begin{aligned}&|\theta_\omega(x,t,s,\zeta,r)-\theta(x,t,s,\zeta,r)|\le\kappa_1(\omega),\\&|\theta_\omega(x',t',s',\zeta',r')-\theta_\omega(x,t,s,\zeta,r)|\\\le&\kappa_2(\omega)\big(\|x'-x\|+\|t'-t\|+\|s'-s\|+\|\zeta'-\zeta\|+|r'-r|\big)\end{aligned}$$

and $\mathbf{E}[\kappa_1(\omega)]<+\infty$, $\mathbf{E}[\kappa_2(\omega)]<+\infty$. From above results, it is easy to obtain that the conditions of Theorem 5.1 in reference [17] are all hold, and hence completes the proof. ■

However, in general, approximation problems are non-convex programming. Then we probably get the stationary points rather than the global optimal solutions. Therefore, considering the stationary points are necessary. We next give some definitions associated with stationary points.

Definition 3: If there exist Lagrange multipliers $\beta^k=(\beta_1^k,\beta_2^k,\beta_3^k,\beta_4^k)^T\in R^4$, $\pi^k=(\pi_1^k,\pi_2^k,\pi_3^k,\pi_4^k)^T\in R^4$, $\gamma_1^k\in R$, $\gamma_2^k\in R$ satisfying

$$\nabla_x\theta^k(x^k,t^k,s^k,\zeta^k,r^k)=0, \tag{9}$$
$$\nabla_t\theta^k(x^k,t^k,s^k,\zeta^k,r^k)-\sum_{i=1}^4\beta_i^k e_i=0, \tag{10}$$
$$t_i^k\ge0,\beta_i^k\ge0,t_i^k\beta_i^k=0, \tag{11}$$
$$\nabla_s\theta^k(x^k,t^k,s^k,\zeta^k,r^k)-\sum_{j=1}^4\pi_j^k e_j=0, \tag{12}$$
$$s_j^k\ge0,\pi_j^k\ge0,s_j^k\pi_j^k=0, \tag{13}$$
$$\nabla_\zeta\theta^k(x^k,t^k,s^k,\zeta^k,r^k)=0, \tag{14}$$
$$\nabla_r\theta^k(x^k,t^k,s^k,\zeta^k,r^k)-\gamma_1^k+\gamma_2^k=0, \tag{15}$$
$$\gamma_1^k\ge0,\gamma_2^k\ge0,r^k\gamma_1^k=0,(1-r^k)\gamma_2^k=0,0\le r^k\le1, \tag{16}$$

where e_i denotes a unit vector with the ith component is 1, then we call $(x^k,t^k,s^k,\zeta^k,r^k)$ is a stationary point of (3).

Definition 4: If there exist lagrange multipliers $\beta^*=(\beta_1^*,\beta_2^*,\beta_3^*,\beta_4^*)^T\in R^4$, $\pi^*=(\pi_1^*,\pi_2^*,\pi_3^*,\pi_4^*)^T\in R^4$, $\gamma_1^*\in R$, $\gamma_2^*\in R$ satisfying

$$\nabla_x\theta(x^*,t^*,s^*,\zeta^*,r^*)=0, \tag{17}$$
$$\nabla_t\theta(x^*,t^*,s^*,\zeta^*,r^*)-\sum_{i=1}^4\beta_i^* e_i=0, \tag{18}$$
$$t^*\ge0,\beta_i^*\ge0,t_i^*\beta_i^*=0, \tag{19}$$
$$\nabla_s\theta(x^*,t^*,s^*,\zeta^*,r^*)-\sum_{j=1}^4\pi_j^* e_j=0, \tag{20}$$
$$s_j^*\ge0,\pi_j^*\ge0,s_j^*\pi^*=0, \tag{21}$$
$$\nabla_\zeta\theta(x^*,t^*,s^*,\zeta^*,r^*)=0, \tag{22}$$
$$\nabla_r\theta(x^*,t^*,s^*,\zeta^*,r^*)-\gamma_1^*+\gamma_2^*=0, \tag{23}$$
$$\gamma_1^*\ge0,\gamma_2^*\ge0,r^*\gamma_1^*=0,(1-r^*)\gamma_2^*=0,0\le r^*\le1, \tag{24}$$

then we call $(x^*,t^*,s^*,\zeta^*,r^*)$ is a stationary point of (2).

We now turn our attentions to Slater constraint qualification, which can ensure the boundedness of Lagrange multipliers in Definition 3.

Definition 5: If there exists a vector $y\in R^4$, for each $i\in I(t^*):=\{\ i\mid t_i^*=0,\ 1\le i\le4\}$ such that $y_i>0$. Then we call Slater constraint qualification holds.

Note that by the condtion (6) and Theory 16.8 of [14], we obtain that $f(x,t)$ is continuously differentiable with respect to x and

$$\begin{aligned}&\nabla_x f(x,t)\\=&2l_1(x,t_1,t_2)^T\mathbf{E}[\nabla_xF(x,\omega)-t_1A\nabla_xG(x,\omega)]\\&+2l_2(x,t_3)\mathbf{E}[\nabla_xG(x,\omega)]^TA\mathbf{E}[G(x,\omega)]\\&+2l_3(x,t_4)\mathbf{E}[\nabla_xG_1(x,\omega)].\end{aligned}$$

Similarly, $g(x,s,\zeta)$ is continuously differentiable over x and

$$\begin{aligned}&\nabla_x g(x,s,\zeta)\\=&2l_4(x,s_1,s_2)^T\mathbf{E}[\nabla_xG(x,\omega)-s_1\overline{A}\nabla_xF(x,\omega)]\\&+2l_5(x,s_3)\mathbf{E}[\nabla_xF(x,\omega)]^T\overline{A}\mathbf{E}[F(x,\omega)]\\&+2l_6(x,s_4)\mathbf{E}[\nabla_xF_1(x,\omega)]+2l_7^T(x)M\mathbf{E}[\nabla_xF(x,\omega)].\end{aligned}$$

Therefore, by the definition of $\theta(x,t,s,\zeta,r)$, it is easy to obtain that $\theta(x,t,s,\zeta,r)$ is continuously differentiable with respect to t, s, ζ, r, respectively. What means that $\theta(x,t,s,\zeta,r)$ is continuously differentiable on S.

Lemma 2: $\nabla\theta^k(x^k,t^k,s^k,\zeta^k,r^k)$ is continuously convergent to $\nabla\theta(x^*,t^*,s^*,\zeta^*,r^*)$ on $\mathcal{N}(x^*,t^*,s^*,\zeta^*,r^*)$.

Proof: We assume $\lim_{k\to\infty}(x^k,t^k,s^k,\zeta^k,r^k)=(x^*,t^*,s^*,\zeta^*,r^*)$, and $\mathcal{N}(x^*,t^*,s^*,\zeta^*,r^*)\in S$ is a neighborhood of $(x^*,t^*,s^*,\zeta^*,r^*)$ contains $\{(x^k,t^k,s^k,\zeta^k,r^k)\}$. For each $\omega\in\Omega$, combining (6), (7) and Proposition 7 in Chapter 6 of [16], we have $\nabla\theta^k(x^*,t^*,s^*,\zeta^*,r^*)$ is uniformly convergent to $\nabla\theta(x^*,t^*,s^*,\zeta^*,r^*)$. Combining with Lemma 1, we obtain the conclusion immediately. ■

Theorem 3: Suppose that for each k, $(x^k,t^k,s^k,\zeta^k,r^k)$ is a stationary point of (3), $(x^*,t^*,s^*,\zeta^*,r^*)$ is an accumulation point of $\{(x^k,t^k,s^k,\zeta^k,r^k)\}$. Then $(x^*,t^*,s^*,\zeta^*,r^*)$ is a stationary point of (2).

Proof: For simplicity, we assume $\lim_{k\to\infty}(x^k,t^k,s^k,\zeta^k,r^k)=(x^*,t^*,s^*,\zeta^*,r^*)$. Let β^k, π^k, γ_1^k, γ_2^k are the Lagrange multipliers of (9)–(16).

(i) We first show that the sequence of $\{\beta^k\}$ is bounded for all k. To this end, we set

$$\tau^k:=\sum_{i=1}^4\beta_i^k. \tag{25}$$

Suppose by contradiction that $\{\beta^k\}$ is not bounded, which means that there exists a sequence $\{\tau^k\}$ such that $\lim_{k\to\infty}\tau^k=+\infty$. We may further assume that the limit

$$\overline{\beta}_i:=\lim_{k\to\infty}\frac{\beta_i^k}{\tau^k},(i=1,\dots,4)$$

exists. By (11), we have for each $i\notin I(t^*)=\{\ i\mid t_i^*=0,1\le i\le4\}$, $\overline{\beta}_i=0$. Then following from (25), we obtain

$$\sum_{i\in I(t^*)}\overline{\beta}_i=\sum_{i=1}^4\overline{\beta}_i=1. \tag{26}$$

For (10), dividing by τ^k and taking a limit, following from Lemma 2 and formulation (26), we have

$$\sum_{i\in I(t^*)}\overline{\beta}_ie_i=\sum_{i=1}^4\overline{\beta}_ie_i=0. \tag{27}$$

Since Slater constraint qualification holds, then there exists a vector $y\in R^4$, for all $i\in I(t^*)$ satisfying $y_i>0$ and there holds:

$$(y-t^*)^Te_i=y_i-t_i^*=y_i>0,\quad\forall\ i\in I(t^*). \tag{28}$$

Let (27) multiplying $(y-t^*)^T$, by the fact that $\overline{\beta}_i \geq 0$ and (28) holds, we have $\overline{\beta}_i = 0$ for each $i \in I(t^*)$, which contradicts (26). Therefore $\{\beta^k\}$ is bounded. Similarly, we can obtain the boundedness of Lagrange multipliers $\{\pi^k\}$, $\{\gamma_1^k\}$, $\{\gamma_2^k\}$.

(ii) Since $\{\beta^k\}$, $\{\pi^k\}$, $\{\gamma_1^k\}$, $\{\gamma_2^k\}$ are bounded, we may assume that $\beta^* = \lim_{k\to\infty} \beta^k$, $\pi^* = \lim_{k\to\infty} \pi^k$, $\gamma_1^* = \lim_{k\to\infty} \gamma_1^k$, $\gamma_2^* = \lim_{k\to\infty} \gamma_2^k$ exist, which together with Lemma 2 and Definition 2, taking a limit in (9)$-$(16), we get (17)$-$(24), which mean $(x^*, t^*, s^*, \zeta^*, r^*)$ is a stationary point of problem (2). ■

IV. Conclusion

In this paper, we present an equivalent problem of SGSOCCP(F, G, K), which is constructed by the convex combination of two given merit functions. Since the box-constrained minimization problems contain an expectation function, we then use SAA method to give approximation problems. Furthermore the convergence results of global solutions and stationary points for the corresponding approximation problems are considered, the conclusions ensure that it is feasible to regard the solutions of approximation problems as the solutions of SGSOCCP(F, G, K) in theory.

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