Comparison of Methods for Solving Time-Fractional Drinfeld-Sokolov-Wilson System

Xumei Chen, Wei Shen, Linjun Wang, Fang Wang

***Abstract*—In this paper, Coupled Fractional Reduced Differential Transform method (CFRDTM) is applied to solve time-fractional Drinfeld-Sokolov-Wilson system. In the procedure of the method, we can obtain the solutions which take the form of convergent series. This paper will present a comparison between the solutions obtained by CFRDTM and the previously known results using the residual power series method. The comparison demonstrates that both of the methods are efficient and easily implemented for getting approximate solutions of time-fractional Drinfeld-Sokolov-Wilson system. Furthermore, it also indicates that the results in previous literature contain errors.**

***Index Terms*—Coupled fractional reduced differential transform, Generalized Taylor series, Residual power series method, Time-fractional Drinfeld-Sokolov-Wilson system.**

I. Introduction

THE fractional differential equations as mathematical tools are powerful for describing and modeling various phenomena in many fields of engineering, physics and other fields of applied sciences [1], [2], [3]. Thanks to their wide application, fractional differential equations have become a hot research domain. It is interesting to note that the theory and application of fractional differential equations have been discussed in detail in the literature [4], [5], [6], [7] and the references therein.

Consider a generalization of time-fractional Drinfeld-Sokolov-Wilson (DSW) system [8]

$$\begin{aligned} D_t^{\alpha}u &= -avv_x, \\ D_t^{\beta}v &= -bv_{xxx} - \gamma uv_x - \epsilon u_x v, \end{aligned} \tag{1}$$

where a, b, γ and ϵ are nonzero parameters and α, β are the order of fractional derivatives with $0 \leq \alpha, \beta \leq 1$. The choice $\alpha = \beta$ leads system (1) to the following time-fractional DSW system, which was introduced in [9]

$$\begin{aligned} D_t^{\alpha}u &= -avv_x, \\ D_t^{\alpha}v &= -bv_{xxx} - \gamma uv_x - \epsilon u_x v. \end{aligned} \tag{2}$$

Note that for $\alpha = \beta = 1$, system (1) represents the standard DSW system, which was proposed as a model of water waves by Drinfeld and Sokolov [10] and Wilson [11] when $a = 1$, $b = \gamma = 2$ and $\epsilon = 1$.

A great deal of effort has focused on the exact or approximate solutions for the standard DSW system. Inc [12] applied the Adomian decomposition method to obtain approximate doubly periodic wave solutions. Zha and zhi [13] used the improved F-Expansion method to construct new exact doubly periodic solutions. Zhang [14] implemented the variational approach to get the solitary solutions. Ullah et al. [15] made use of optimal homotopy asymptotic method to acquire the approximate doubly periodic wave solutions.

Compared to the standard DSW system, it is more difficult to construct and develop approximate or analytical methods to solve time-fractional DSW system. Therefore, a large number of approximate methods concerning the solutions of time-fractional system have aroused wide attention. These methods include homotopy perturbation transform method [8], residual power series method (RPSM) [9], [16], coupled fractional reduced differential transform method (CFRDTM) [17], collocation method [18], [19] and so on.

The RPSM was initially developed to solve the first-order and the second-order fuzzy differential equations. It provides power series solutions with rapid convergence. For more details, see [20], [21], [22].

The CFRDTM has been presented and developed in [17], [23]. This method was based upon generalized Taylor's formula [24]. Using the CFRDTM, we can get the power series approximate solutions with fast convergence like the RPSM. The CFRDTM is proved to construct approximate solutions effectively and accurately. It has been successfully implemented to solve time fractional coupled modified KdV equations [17], time fractional Whitham-Broer-Kaup equations [23] and et al..

The main purpose of this paper is to compare the CFRDTM with the RPSM for solving time-fractional DSW system. In the paper, the CFRDTM will be applied to obtain the approximate analytical solutions with high accuracy. For the purpose of comparison, our second method is the RPSM as presented in [9] where approximate solutions are obtained. The comparison demonstrates that both of the methods are efficient and easily implemented for getting approximate solutions of time-fractional Drinfeld-Sokolov-Wilson system. On the other aspect as well, it also indicates that the results in previous literature contain errors. The correct results are also given in this paper.

This paper is organized as follows. In the next section, some basic definitions and theorems concerning fractional calculus are introduced. In Section 3, the procedure of CFRDTM is described in detail. In Section 4, using the CFRDTM, we derive the approximate solutions for time-fractional Drinfeld-Sokolov-Wilson system. In the mean time, we also compare the results with the solutions obtained in the previous literature. By comparing, we find the previous results contain errors. The last section contains some remarks and conclusions.

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II. Basic definitions and theorems

In this section, we present some definitions and theorems of fractional calculus [25]. There are different definitions of fractional integration and differentiation involving Grunwald-Letnikov's definition, Riemann-Liouville's definition, Caputo's definition and Riesz derivative. For the purpose of this paper, the Caputo's type of fractional derivative will be used.

Definition 1. For m to be the smallest integer that exceeds α, the Caputo fractional derivatives of order $\alpha > 0$ is defined as

$$D_t^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau, \\ m-1 < \alpha < m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, \quad \alpha = m \in N. \end{cases}$$

For the Caputo's derivative, we have some properties:

$$D^\alpha C = 0, \text{ (C is a constant)},$$
$$D^\alpha(\gamma f(t) + \delta g(t)) = \gamma D^\alpha f(t) + \delta D^\alpha g(t),$$

where γ and δ are constants, and

$$D^\alpha t^\beta = \begin{cases} 0, & \beta \leq \alpha - 1, \\ \frac{\Gamma(\beta+1) t^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}, & \beta \geq \alpha - 1. \end{cases}$$

The Caputo's derivative satisfies the following Leibnitz's rule

$$D^\alpha(g(t)f(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} g^{(k)}(t) D^{\alpha-k} f(t),$$

if $f(\tau)$ is continuous in $[0,t]$ and $g(\tau)$ has $n+1$ continuous derivatives in $[0,t]$.

Theorem 1. ((Generalized Taylor's formula) [24]) Suppose that $D_a^{k\alpha} f(t) \in C(a,b]$ for $k = 0, 1, ..., n+1$, where $0 < \alpha \leq 1$, we have

$$f(t) = \sum_{i=0}^{n} \frac{(t-a)^{i\alpha}}{\Gamma(i\alpha+1)} [D_a^{i\alpha} f(t)]_{t=a} + \mathcal{R}_n^\alpha(t;a) \qquad (3)$$

with $\mathcal{R}_n^\alpha(t;a) = \frac{(t-a)^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} [D_a^{(n+1)\alpha} f(t)]_{t=\xi}$, $a \leq \xi \leq t$, $\forall\ t \in (a,b]$, where $D_a^{k\alpha} = D_a^\alpha \cdot D_a^\alpha \cdot D_a^\alpha \cdots D_a^\alpha$ $(k\ times)$.

III. Couple fractional reduce differential transform method (CFRDTM)

For a better introduction of the couple fractional reduce differential transform method, we use $U(h, k-h)$ to denote the coupled fractional reduced differential transform of $u(x,t)$. If $u(x,t)$ is analytic and differentiated continuously with respect to time t, and then the fractional coupled reduced differential transform of $u(x,t)$ is defined as

$$\begin{aligned} &U(h, k-h) \\ &= \frac{1}{\Gamma(h\alpha+(k-h)\beta+1)} [D_t^{(h\alpha+(k-h)\beta)} u(x,t)]_{t=0}, \end{aligned} \qquad (4)$$

whereas the inverse transform of $U(h, k-h)$ is

$$u(x,t) = \sum_{k=0}^{\infty} \sum_{h=0}^{k} U(h, k-h) t^{h\alpha+(k-h)\beta}. \qquad (5)$$

This is one of the solutions of coupled fractional differential equations.

Next, we will give the theorem of the fractional couple reduced differential transform [23], which is the key of the method.

Theorem 2. Suppose that $U(h, k-h)$ and $V(h, k-h)$ are the coupled fractional reduced differential transform of the functions $u(x,t)$ and $v(x,t)$, respectively.

(i) If $u(x,t) = f(x,t) \pm g(x,t)$, then $U(h, k-h) = F(h, k-h) \pm G(h, k-h)$.

(ii) If $u(x,t) = af(x,t)$, where $a \in R$, then $U(h, k-h) = aF(h, k-h)$.

(iii) If $f(x,t) = u(x,t)v(x,t)$, then $F(h, k-h) = \sum_{l=0}^{h} \sum_{s=0}^{k-h} U(h-l, s) V(l, k-h-s)$.

(iv) If $f(x,t) = D_t^\alpha u(x,t)$, then

$$F(h, k-h) = \frac{\Gamma((h+1)\alpha + (k-h)\beta + 1)}{\Gamma(h\alpha + (k-h)\beta + 1)} U(h+1, k-h).$$

(v) If $f(x,t) = D_t^\beta v(x,t)$, then

$$F(h, k-h) = \frac{\Gamma(h\alpha + (k-h+1)\beta + 1)}{\Gamma(h\alpha + (k-h)\beta + 1)} V(h, k-h+1).$$

IV. Comparison of numerical application

In this section, we will demonstrate the effectiveness of the CFRDTM and compare the results with RPSM.

Consider time-fractional DSW system [9]:

$$\begin{aligned} D_t^\alpha u &= -3vv_x, \\ D_t^\alpha v &= -2v_{xxx} - 2uv_x - u_x v, \end{aligned} \qquad (6)$$

subject to the initial conditions:

$$\begin{aligned} u(x,0) &= 3\text{sech}^2(x), \\ v(x,0) &= 2\text{sech}(x). \end{aligned} \qquad (7)$$

For the special case, where $\alpha = 1$, the exact solutions of (6) and (7) are

$$\begin{aligned} u(x,t) &= 3\text{sech}^2(x-2t), \\ v(x,t) &= 2\text{sech}(x-2t). \end{aligned} \qquad (8)$$

In order to use the CFRDTM for solving (6) and (7), we derive the recursive formulas from Eq. (6). We assume that $U(h, k-h)$ and $V(h, k-h)$ are the coupled fractional reduced differential transform of $u(x,t)$ and $v(x,t)$, respectively. Here, $u(x,t)$ and $v(x,t)$ are the solutions of Eq. (6) with the initial data

$$U(0,0) = u(x,0),\ V(0,0) = v(x,0).$$

Without loss of generality, we assume that

$$U(0,j) = 0,\ j = 1, 2, 3, \cdots$$

and

$$V(i,0) = 0,\ i = 1, 2, 3 \cdots$$

Next, we will deduce the recursive formulas. By taking CFRDTM to Eq. (6), we get

$$\begin{aligned} &\frac{\Gamma((k+1)\alpha+1)}{\Gamma(k\alpha+1)} U(h+1, k-h) \\ &= -3\Big(\sum_{i=0}^{h} \sum_{s=0}^{k-h} V(h-i, s) \frac{\partial}{\partial x} V(i, k-h-s)\Big), \end{aligned} \qquad (9)$$

and

$$\begin{aligned} &\frac{\Gamma((k+1)\alpha+1)}{\Gamma(k\alpha+1)} V(h, k-h+1) \\ &= -2\Big(\frac{\partial^3}{\partial x^3} V(h, k-h)\Big) \\ &\quad -2\Big(\sum_{i=0}^{h} \sum_{s=0}^{k-h} U(i, k-h-s) \frac{\partial}{\partial x} V(h-i, s)\Big) \\ &\quad -\Big(\sum_{i=0}^{h} \sum_{s=0}^{k-h} \frac{\partial}{\partial x} U(i, k-h-s) V(h-i, s)\Big). \end{aligned} \qquad (10)$$

Thanks to the initial condition (7), we have

$$\begin{array}{l} U(0,0) = u(x,0) = 3\text{sech}^2(x), \\ V(0,0) = v(x,0) = 2\text{sech}(x). \end{array} \quad (11)$$

According to the procedure of the CFRDTM, using recursive formulas (9), (10) together with initial conditions (11), we can easily obtain

$$\begin{aligned} U(1,0) &= \frac{12\text{sech}^2(x)\tanh(x)}{\Gamma(1+\alpha)}, \\ V(0,1) &= \frac{4\text{sech}(x)\tanh(x)}{\Gamma(1+\alpha)}, \\ U(1,1) &= \frac{24(-3\text{sech}^4(x)+2\text{sech}^2(x))}{\Gamma(1+2\alpha)}, \\ V(0,2) &= \frac{8(15\text{sech}^5(x)-14\text{sech}^3(x)+\text{sech}(x))}{\Gamma(1+2\alpha)}, \\ U(2,0) &= 0, \\ V(1,1) &= \frac{24(-5\text{sech}^5(x)+4\text{sech}^3(x))}{\Gamma(1+2\alpha)}. \end{aligned}$$

Similarly, if we repeat the process of recursive formulas, we will get the other recursive expressions.

Thus, if we collect all of the recursive expressions and substitute them into (5), then the approximate solutions of Eqs.(6) and (7) are obtained. The approximate solutions are constructed as follows:

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{k} U(h,k-h)t^{k\alpha} \\ &= U(0,0) + \sum_{k=1}^{\infty}\sum_{h=1}^{k} U(h,k-h)t^{k\alpha} \\ &= 3\text{sech}^2(x) + \frac{12\text{sech}^2(x)\tanh(x)}{\Gamma(1+\alpha)}t^{\alpha} \\ &+ \frac{24(-3\text{sech}^4(x)+2\text{sech}^2(x))}{\Gamma(1+2\alpha)}t^{2\alpha} + \cdots (12) \end{aligned}$$

and

$$\begin{aligned} v(x,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{k} V(h,k-h)t^{k\alpha} \\ &= V(0,0) + \sum_{k=1}^{\infty}\sum_{h=0}^{k} V(h,k-h)t^{k\alpha} \\ &= 2\text{sech}(x) + \frac{4\text{sech}(x)\tanh(x)}{\Gamma(1+\alpha)}t^{\alpha} \\ &+ \frac{8(-2\text{sech}^3(x)+\text{sech}(x))}{\Gamma(1+2\alpha)}t^{2\alpha} + \cdots \quad (13) \end{aligned}$$

For the special case of $\alpha = 1$, the solutions are given by

$$\begin{aligned} u(x,t) &= 3\text{sech}^2(x) + 12\text{sech}^2(x)\tanh(x)t \\ &+ 12(-3\text{sech}^4(x)+2\text{sech}^2(x))t^2 + \cdots (14) \end{aligned}$$

and

$$\begin{aligned} v(x,t) &= 2\text{sech}(x) + 4\text{sech}(x)\tanh(x)t \\ &+ 4(-2\text{sech}^3(x)+\text{sech}(x))t^2 + \cdots \quad (15) \end{aligned}$$

The results for the special case of $\alpha = 1$ show that the solutions (14) and (15) are exactly same as the Taylor series expansions of the exact solutions

$$\begin{aligned} u(x,t) &= 3\text{sech}^2(x-2t) \\ &= 3\text{sech}^2(x) + 12\text{sech}^2(x)\tanh(x)t \\ &+ 12(-3\text{sech}^4(x)+2\text{sech}^2(x))t^2 + \cdots (16) \end{aligned}$$

and

$$\begin{aligned} v(x,t) &= 2\text{sech}(x-2t) \\ &= 2\text{sech}(x) + 4\text{sech}(x)\tanh(x)t \\ &+ 4(-2\text{sech}^3(x)+\text{sech}(x))t^2 + \cdots \quad (17) \end{aligned}$$

In order to compare the results with the residual power series solutions in [9], we construct *n*-th truncated series solutions of $u(x,t)$, $v(x,t)$, which are exactly same as the *n*-th truncated Taylor series of the exact solutions. Since both of the solutions are calculated in the form of generalized Taylor series, it is possible to compare. It is notable that one can achieve better approximation by adding more new terms of the truncated series. For convenience, $u_n(x,t)$ and $v_n(x,t)$ are denoted as the *n*-th truncated series solutions:

$$\begin{aligned} u_n(x,t) &= \sum_{k=0}^{n}\sum_{h=0}^{k} U(h,k-h)t^{k\alpha}, \\ v_n(x,t) &= \sum_{k=0}^{n}\sum_{h=0}^{k} V(h,k-h)t^{k\alpha}. \end{aligned}$$

It is easy to get the 1-st truncated series solutions by setting $n = 1$

$$\begin{aligned} u_1(x,t) &= 3\text{sech}^2(x) + \frac{12\text{sech}^2(x)\tanh(x)}{\Gamma(1+\alpha)}t^{\alpha}, \\ v_1(x,t) &= 2\text{sech}(x) + \frac{4\text{sech}(x)\tanh(x)}{\Gamma(1+\alpha)}t^{\alpha}. \quad (18) \end{aligned}$$

The 2-nd truncated series solutions are represented in the following forms

$$\begin{aligned} u_2(x,t) &= u_1(x,t) + \frac{24(-3\text{sech}^4(x)+2\text{sech}^2(x))}{\Gamma(1+2\alpha)}t^{2\alpha}, \\ v_2(x,t) &= v_1(x,t) + \frac{8(-2\text{sech}^3(x)+\text{sech}(x))}{\Gamma(1+2\alpha)}t^{2\alpha}. (19) \end{aligned}$$

The 3-rd truncated series solutions take the forms

$$\begin{aligned} u_3(x,t) &= u_2(x,t) \\ &+ \Big(96\tanh(x)(-4\text{sech}^4(x)+\text{sech}^2(x)) \\ &\times \frac{1}{\Gamma(1+3\alpha)} \\ &+ 48\tanh(x)(-2\text{sech}^4(x)+\text{sech}^2(x)) \\ &\times \frac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2\Gamma(1+3\alpha)}\Big)t^{3\alpha}, \end{aligned}$$

$$\begin{aligned} v_3(x,t) &= v_2(x,t) \\ &+ \Big(16\tanh(x)(42\text{sech}^5(x)-30\text{sech}^3(x) \\ &+ \text{sech}(x)) \times \frac{1}{\Gamma(1+3\alpha)} \\ &+ 48\tanh(x)(-7\text{sech}^5(x)+4\text{sech}^3(x)) \\ &\times \frac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2\Gamma(1+3\alpha)}\Big)t^{3\alpha}. \quad (20) \end{aligned}$$

The 4-th and 5-th truncated series solutions are

$$\begin{aligned}
u_4(x,t) &= u_3(x,t) \\
&+ \Big(96(-294\text{sech}^8(x) + 402\text{sech}^6(x) \\
&- 123\text{sech}^4(x) + 2\text{sech}^2(x)) \times \frac{1}{\Gamma(1+4\alpha)} \\
&+ 288(49\text{sech}^8(x) - 62\text{sech}^6(x) \\
&+ 16\text{sech}^4(x)) \times \frac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2\Gamma(1+4\alpha)} \\
&+ 96(10\text{sech}^6(x) - 11\text{sech}^4(x) + 2\text{sech}^2(x)) \\
&\times \frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)\Gamma(1+4\alpha)}\Big)t^{4\alpha},
\end{aligned}$$

$$\begin{aligned}
v_4(x,t) &= v_3(x,t) \\
&+ \Big(32(13230\text{sech}^9(x) - 20874\text{sech}^7(x) \\
&+ 8811\text{sech}^5(x) - 800\text{sech}^3(x) + \text{sech}(x)) \\
&\times \frac{1}{\Gamma(1+4\alpha)} \\
&+ 96(-2205\text{sech}^9(x) + 3416\text{sech}^7(x) \\
&- 1388\text{sech}^5(x) + 112\text{sech}^3(x)) \\
&\times \frac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2\Gamma(1+4\alpha)} \\
&+ 288(14\text{sech}^7(x) - 17\text{sech}^5(x) + 4\text{sech}^3(x)) \\
&\times \frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)\Gamma(1+4\alpha)}\Big)t^{4\alpha}, \qquad (21)
\end{aligned}$$

and

$$\begin{aligned}
u_5(x,t) &= u_4(x,t) + \Big(384\tanh(x)(66150\text{sech}^{10}(x) \\
&- 83496\text{sech}^8(x) + 26433\text{sech}^6(x) \\
&- 1600\text{sech}^4(x) + \text{sech}^2(x))\frac{1}{\Gamma(1+5\alpha)} \\
&+ 1152\tanh(x)(-11025\text{sech}^{10}(x) \\
&+ 13664\text{sech}^8(x) - 4164\text{sech}^6(x) \\
&+ 224\text{sech}^4(x))\frac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2\Gamma(1+5\alpha)} \\
&+ 3456\tanh(x)(56\text{sech}^8(x) - 51\text{sech}^6(x) \\
&+ 8\text{sech}^4(x))\frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)\Gamma(1+5\alpha)} \\
&+ 384\tanh(x)(-168\text{sech}^8(x) + 216\text{sech}^6(x) \\
&- 62\text{sech}^4(x) + \text{sech}^2(x)) \\
&\times \frac{\Gamma(1+4\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)\Gamma(1+5\alpha)} \\
&+ 1152\tanh(x)(28\text{sech}^8(x) - 33\text{sech}^6(x) \\
&+ 8\text{sech}^4(x)) \\
&\times \frac{\Gamma(1+2\alpha)\Gamma(1+4\alpha)}{(\Gamma(1+\alpha))^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)} \\
&+ 192\tanh(x)(12\text{sech}^6(x) - 8\text{sech}^4(x) \\
&+ \text{sech}^2(x))\frac{\Gamma(1+4\alpha)}{(\Gamma(1+2\alpha))^2\Gamma(1+5\alpha)}\Big)t^{5\alpha},
\end{aligned}$$

$$\begin{aligned}
v_5(x,t) &= v_4(x,t) \\
&+ \Big(64\tanh(x)(-12700800\text{sech}^{11}(x) \\
&+ 19655370\text{sech}^9(x) - 8841846\text{sech}^7(x) \\
&+ 1137561\text{sech}^5(x) - 21576\text{sech}^3(x) \\
&+ \text{sech}(x)) \times \frac{1}{\Gamma(1+5\alpha)} \\
&+ 192\tanh(x)(2116800\text{sech}^{11}(x) \\
&- 3245655\text{sech}^9(x) + 1436696\text{sech}^7(x) \\
&- 178588\text{sech}^5(x) + 3024\text{sech}^3(x)) \\
&\times \frac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2\Gamma(1+5\alpha)} \\
&+ 192\tanh(x)(-20160\text{sech}^9(x) \\
&+ 24278\text{sech}^7(x) - 7017\text{sech}^5(x) \\
&+ 332\text{sech}^3(x))\frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)\Gamma(1+5\alpha)} \\
&+ 576\tanh(x)(-210\text{sech}^9(x) + 302\text{sech}^7(x) \\
&- 103\text{sech}^5(x) + 4\text{sech}^3(x)) \\
&\times \frac{\Gamma(1+4\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)\Gamma(1+5\alpha)} \\
&+ 192\tanh(x)(315\text{sech}^9(x) - 366\text{sech}^7(x) \\
&+ 77\text{sech}^5(x) + 4\text{sech}^3(x)) \\
&\times \frac{\Gamma(1+2\alpha)\Gamma(1+4\alpha)}{(\Gamma(1+\alpha))^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)} \\
&+ 384\tanh(x)(30\text{sech}^7(x) - 25\text{sech}^5(x) \\
&+ 4\text{sech}^3(x))\frac{\Gamma(1+4\alpha)}{(\Gamma(1+2\alpha))^2\Gamma(1+5\alpha)}\Big)t^{5\alpha}. \qquad (22)
\end{aligned}$$

Now, let us compare our results with the solutions obtained by the RPSM in the previous literature. It is obvious that the 1-st and 2-nd residual power series (RPS) approximate solutions obtained in [9] are identical to (18) and (19), respectively. However, we find that the 3-rd, 4-th and 5-th RPS approximate solutions presented in [9] are not in agreement with (20), (21) and (22). Notice that there are errors on p.66 in [9]: when the authors applied D_t^α on both sides of Eqs.(16) and (17), the terms in (18) and (19) are incorrect, which leads to the wrong expressions of $f_3(x)$, $g_3(x)$, $f_4(x)$, $g_4(x)$, $f_5(x)$ and $g_5(x)$ in (21), (22) and (23) in [9]. The main reason for the errors is that the authors misused the derivation rule of fractional derivative. According to the properties of the Caputo fractional derivative, when $\beta > \alpha - 1$, we have

$$D_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}. \qquad (23)$$

Unfortunately, the author in [9] treated it as integer-order derivative and used

$$D_t^\alpha t^\beta = \beta t^{\beta-1}D_t^\alpha t = \frac{\beta}{\Gamma(2-\alpha)}t^{\beta-\alpha}. \qquad (24)$$

Obviously, (24) only holds when $\alpha = 1$. The correct expressions of $f_3(x)$, $g_3(x)$, $f_4(x)$, $g_4(x)$, $f_5(x)$ and $g_5(x)$

in [9] should be

$$
\begin{aligned}
f_3(x) &= -a\Big(g_2(x)g'(x) + g_1(x)g_1'(x)\frac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2} \\
&+ g(x)g_2'(x)\Big), \\
g_3(x) &= -bg_2'''(x) - \gamma\Big(f_2(x)g'(x) \\
&+ f_1(x)g_1'(x)\frac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2} + f(x)g_2'(x)\Big) \\
&- \epsilon\Big(f'(x)g_2(x) + f_1'(x)g_1(x)\frac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2} \\
&+ f_2'(x)g(x)\Big),
\end{aligned}
$$

$$
\begin{aligned}
f_4(x) &= -a\Big(g_3(x)g'(x) \\
&+ g_2(x)g_1'(x)\frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} \\
&+ g_1(x)g_2'(x)\frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} + g(x)g_3'(x)\Big), \\
g_4(x) &= -bg_3'''(x) - \gamma\Big(f_3(x)g'(x) \\
&+ f_2(x)g_1'(x)\frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} \\
&+ f_1(x)g_2'(x)\frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} \\
&+ f(x)g_3'(x)\Big) - \epsilon\Big(f'(x)g_3(x) \\
&+ f_1'(x)g_2(x)\frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} \\
&+ f_2'(x)g_1(x)\frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} \\
&+ f_3'(x)g(x)\Big),
\end{aligned}
$$

$$
\begin{aligned}
f_5(x) &= -a\Big(g_4(x)g'(x) \\
&+ g_3(x)g_1'(x)\frac{\Gamma(1+4\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)} \\
&+ g_2(x)g_2'(x)\frac{\Gamma(1+4\alpha)}{(\Gamma(1+2\alpha))^2} \\
&+ g_1(x)g_3'(x)\frac{\Gamma(1+4\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)} + g(x)g_4'(x)\Big), \\
g_5(x) &= -bg_4'''(x) - \gamma\Big(f_4(x)g'(x), \\
&+ f_3(x)g_1'(x)\frac{\Gamma(1+4\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)} \\
&+ f_2(x)g_2'(x)\frac{\Gamma(1+4\alpha)}{(\Gamma(1+2\alpha))^2} \\
&+ f_1(x)g_3'(x)\frac{\Gamma(1+4\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)} + f(x)g_4'(x)\Big) \\
&- \epsilon\Big(f'(x)g_4(x) + f_1'(x)g_3(x)\frac{\Gamma(1+4\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)} \\
&+ f_2'(x)g_2(x)\frac{\Gamma(1+4\alpha)}{(\Gamma(1+2\alpha))^2} \\
&+ f_3'(x)g_1(x)\frac{\Gamma(1+4\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)} \\
&+ f_4'(x)g(x)\Big).
\end{aligned}
$$

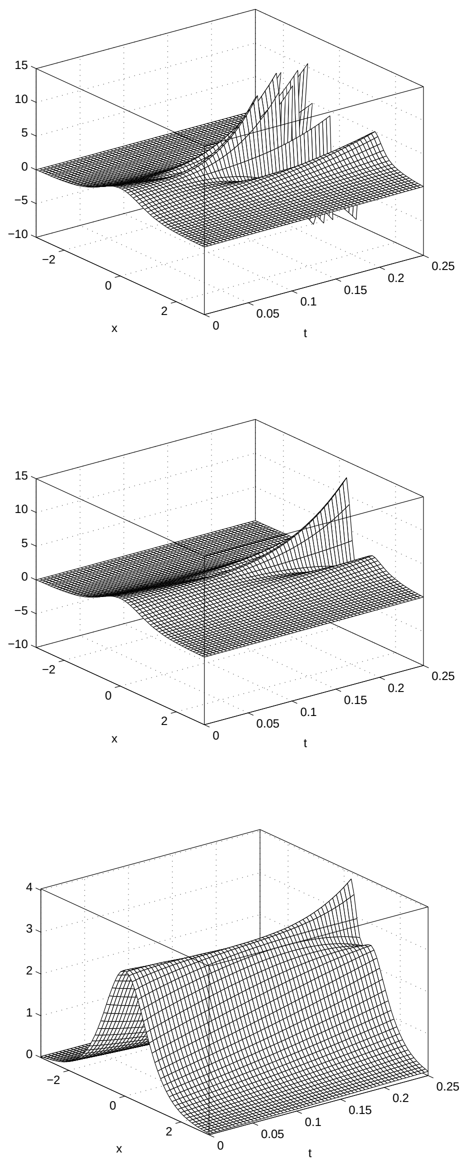

Fig. 1. The 5-th truncated series solution of $u(x,t)$ when $\alpha = 0.7$, $\alpha = 0.8$, $\alpha = 0.9$, respectively.

By replacing the expressions of $f_3(x)$, $g_3(x)$, $f_4(x)$, $g_4(x)$, $f_5(x)$ and $g_5(x)$ in [9], we obtain the correct 3-rd, 4-th and 5-th RPS approximate solutions, which appear to coincide with (20), (21) and (22).

After acquiring the correct 3-rd, 4-th and 5-th RPS approximate solutions, we can redraw the figures of reference [9]. The 5-th approximate solutions of $u(x,t)$ for different values of the fractional derivative α are in Figure 1. Figure 2 explores the 5-th approximate solution of $u(x,t)$ for $\alpha = 1$ and the corresponding absolute error. The graphical results of $v(x,t)$ are shown in Figure 3 and 4.

There is another error of the initial conditions (25) on p.67 in [9]. The initial conditions should be replaced by

$$
\begin{aligned}
u(x,0) &= \frac{3c}{2}\operatorname{sech}^2(\sqrt{\frac{c}{2}}x), \\
v(x,0) &= c\,\operatorname{sech}(\sqrt{\frac{c}{2}}x).
\end{aligned}
$$

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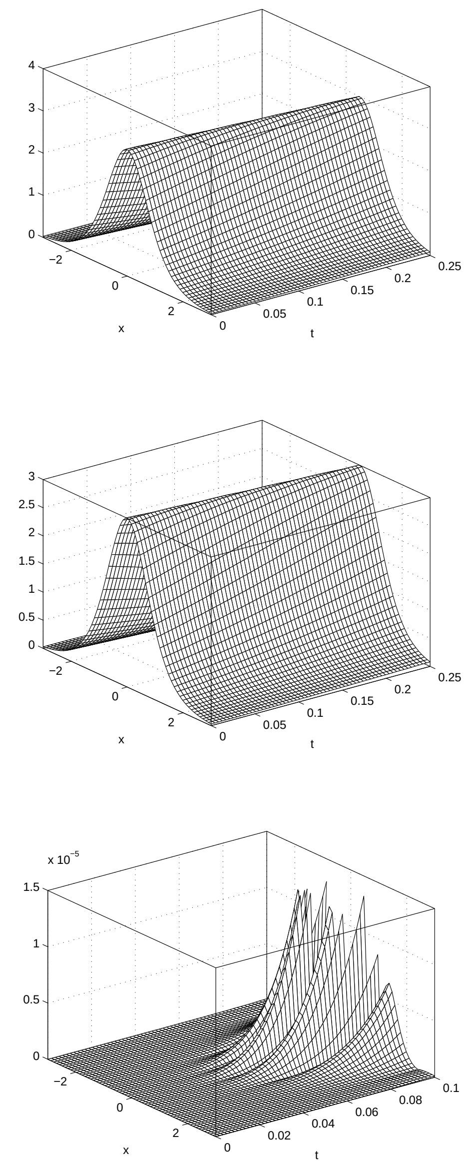
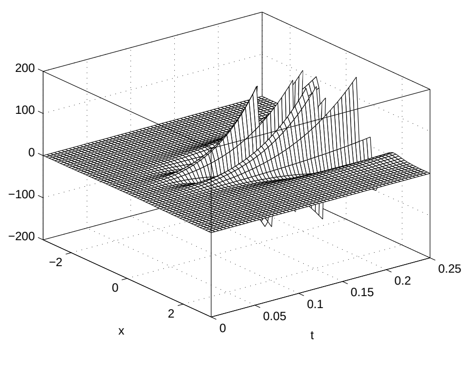


Fig. 2. The 5-th truncated series solution $u_5(x,t)(x,t,\alpha=1)$, the exact solution $u(x,t,\alpha=1)$ and the absolute error.



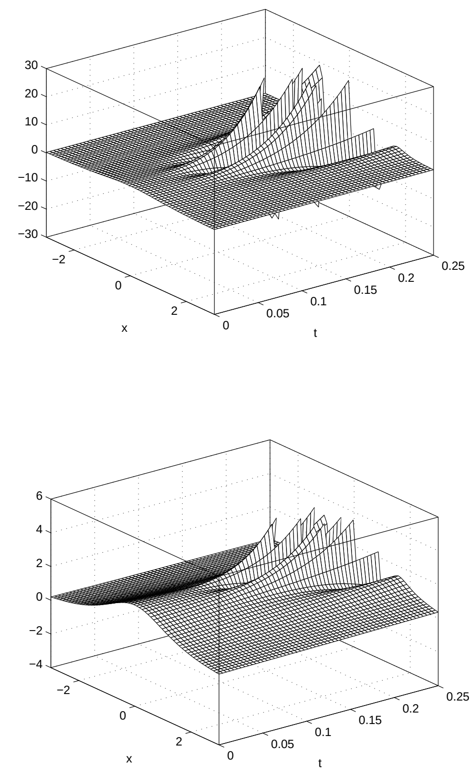
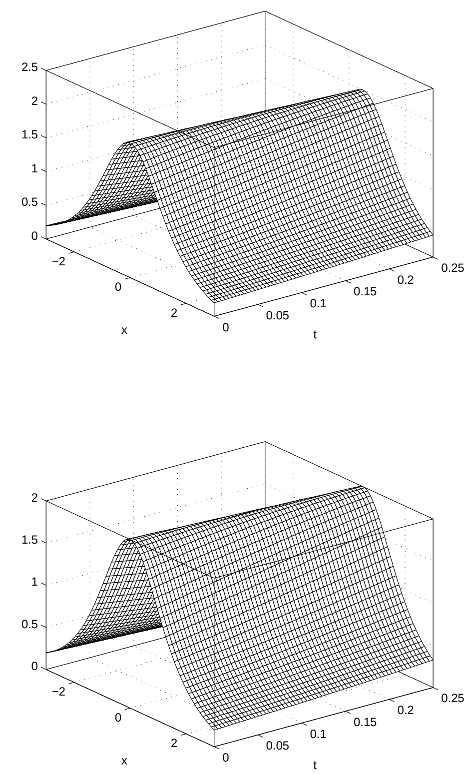


Fig. 3. The 5-th truncated series solution of $v(x,t)$ when $\alpha = 0.7$, $\alpha = 0.8$, $\alpha = 0.9$, respectively.



V. Conclusion

In this paper, the CFRDTM is extended to solve time-fractional DSW system. By using this method, the solutions in the form of generalized Taylor series are obtained. Comparing the results with the other ones obtained by the RPSM, it may be concluded that both of the methods are efficient and the CFRDTM can be taken as an alternative to handle time-fractional nonlinear differential equations. Moreover, we correct the errors in the expressions of $f_3(x)$, $g_3(x)$, $f_4(x)$, $g_4(x)$, $f_5(x)$ and $g_5(x)$ in the previous literature.

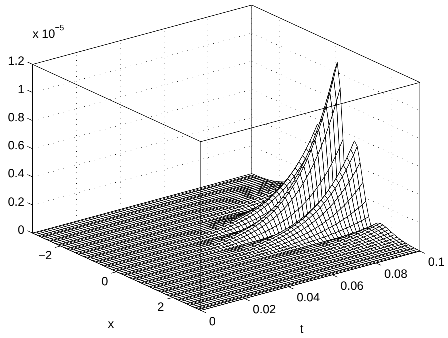


Fig. 4. The 5-th truncated series solution $v_5(x,t)(x,t,\alpha=1)$, the exact solution $v(x,t,\alpha=1)$ and the absolute error.

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