The New Estimations of Diagonally Dominant Degree and Eigenvalues Distributions for the Schur Complements of Block Diagonally Dominant Matrices and Determinantal Bounds

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Abstract—In this paper, some new estimations of diagonally dominant degree on the Schur complement of I(II)-block diagonally dominant matrices are obtained by applying the properties of Schur complement and some inequality techniques, which improve some existing ones. Further, as an application, we present some new distribution theorems for eigenvalues of the Schur complement and some new upper and lower bounds for the determinant of I(II)-block diagonally dominant matrices. These results are proved to be sharper than some known ones. Finally, numerical examples are also presented to confirm the theoretical results studied in this paper.

Index Terms—block matrix, Schur complement, diagonally dominant degree, eigenvalue distribution, determinant.

I. INTRODUCTION

The Schur complement has been proved to be a useful tool in many fields such as control theory, statistics and computational mathematics, and many works have been done on it (see [1], [2], [3], [4], [5], [6]). Applying the Schur-based iteration method mentioned in [7], [8], we can solve large scale linear systems though reducing the order by the Schur complement. That is, for a non-homogeneous system of linear equation $Mx = b$ with a nonsingular leading principal submatrix. Partition $M$ as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A$ is supposed to be nonsingular. Partition $x = (x_1^T, x_2^T)^T$ and $b = (b_1^T, b_2^T)^T$ conformably with $M$. This linear equation can be formally regard as a special case of the saddle point problems [9] The linear system $Mx = b$ is equivalent to the pair of linear systems

$$Ax_1 + Bx_2 = b_1,$$
$$Cx_1 + Dx_2 = b_2.$$

If we multiply the first equation by $-CA^{-1}$ and add it to the second equation, the vector variable $x_1$ is eliminated and we obtain a linear system of smaller size

$$(D - CA^{-1}B)x_1 = b_2 - CA^{-1}b_1.$$

If the coefficient matrix $D - CA^{-1}B$ is a block diagonally dominant matrix or a block $H$-matrix, we can use some block or preconditioned iterative methods [10], [11] to continue resolving the linear system equation (1). In the meanwhile, when we solve linear equation system, the convergence rate of many iterate algorithms are closely related with spectral radius of coefficient matrix. Hu [12] obtained the following result which can be used to estimate the convergence rate:

Let $M = (M_{ij})_{m \times m}$ be a block strictly diagonally dominant matrix and $N = (N_{ij})_{m \times m}$ partitioned conformably with $M$. Then

$$\rho(M^{-1}N) \leq \max_i \frac{\sum_{j=1}^m \|N_{ij}\|}{\|M_{ii}^{-1}\| - \sum_{j \neq i} \|M_{ij}\|}.$$

Therefore, we know the estimate of block matrix’s spectral is closely related with the block diagonally dominant degree $\|M_{ii}^{-1}\| - \sum_{j \neq i} \|M_{ij}\|$ of each row when $M$ is a block strictly diagonally dominant matrix. Thus, after being reduced order, it is significant to study the block diagonally dominant degree of the coefficient matrix of the linear equation system (1). Additionally, as mentioned in [13], we see that the eigenvalues of Schur complement of diagonally dominant matrix are more concentrated than those of original matrix, and we predict that the Schur-based conjugate gradient method will compute faster than the ordinary conjugate gradient method. Hence, it is very important to estimate the eigenvalue distributions of (block) diagonally dominant matrix. Over the years, there has been a surge of interest in studying the locations of eigenvalues of the Schur complement of matrices in much literature, see [6], [7], [8], [13], [14], [15], [16], [17], [18], [19], [20], [21]. Moreover, the determinant of matrices has hitherto great influence on every branch of mathematics [22], [23], [24], [25], [26]. Zhang and Liu [17] proposed some upper and lower bounds for determinants of diagonally dominant matrices by making use of the results of the estimates of diagonally dominant degree for the Schur complement of the diagonally dominant matrices. On the other hand, the authors in [27], [28], [29], [30] extended the concept of diagonally dominant matrix and developed two kinds of block diagonally dominant matrices, which are referred to as the I-block [27] and II-block [31] diagonally dominant...
matrices, respectively. Later, two kinds of generalized block strictly diagonally dominant matrices (I-block [32] II-block [31]) \( H \)-matrices are established in [31], [32], [33]. In the sequel, Liu et al. [13] derived some estimations of diagonally dominant degree and eigenvalue inclusion sets for the Schur complement of \( I \)-block diagonally dominant matrices, and Wang [20], [21] put forward the new estimations of diagonally dominant degree and eigenvalue inclusion sets which are proved to be tighter than those of [13]. Zhu [34] obtained some upper and lower bounds for determinants of \( I \)-block diagonally dominant matrices, and Xu [35] arrived at some determinants bounds are sharper than the ones obtained by Zhu. In the current work, we first focus on investigating the following three aspects:

- Study the new estimates of \( I \)-block diagonally dominant degree for Schur complement of matrices.
- Derive the new distributions for the eigenvalues of \( I \)-block diagonally dominant matrices.
- Develop the new upper and lower bounds for determinants of \( II \)-block diagonally dominant matrices.

Afterward, we prove that the proposed results are superior to some known ones in theory. The numerical results are implemented to verify the theoretical results. Before presenting the main results of this paper, we give some definitions which are used throughout this paper as follows.

Let \( \mathbb{C}^{n \times n} \) denote the set of all \( n \times n \) complex matrices, \( N = \{1, 2, \ldots, n\} \) and \( A = (a_{ij}) \in \mathbb{C}^{n \times n} (n \geq 2) \). Denote

\[
A = (a_{ij}) \in \mathbb{C}^{n \times n} \quad \text{is a strictly diagonally dominant matrix (abbreviated to } SD_n \text{) if } |a_{ii}| > \sum_{j \neq i} |a_{ij}|, \quad i \in N.
\]

A matrix \( A \) is called a \( M \)-matrix if there exist a nonnegative matrix \( B \) and a real number \( s > \rho(B) \) such that \( A = sI - B \), where \( \rho(B) \) is the spectral radius of \( B \). It is well known that \( A \) is an \( H \)-matrix if and only if \( \parallel A \parallel \) is an \( M \)-matrix, then the Schur complement of \( A \) is also an \( M \)-matrix and \( \det A > 0 \) (see [14]).

For \( \alpha \subseteq N \), denote by \( |\alpha| \) the cardinality of \( \alpha \) and \( \alpha' = N - \alpha \). If \( \alpha, \beta \subseteq N \), then \( A(\alpha, \beta) \) is the submatrix of \( A \) lying in the rows indexed by \( \alpha \) and the columns indicated by \( \beta \). In particular, \( A(\alpha, \alpha) \) is abbreviated to \( A(\alpha) \). Assume that \( A(\alpha) \) is nonsingular. Then

\[
A(\alpha)A^{-1} = A/\alpha = A/\alpha' = (\alpha') - (\alpha', \alpha)[A(\alpha)]^{-1} A(\alpha, \alpha'),
\]

called the Schur complement of \( A \) with respect to \( A(\alpha) \).

Let \( A \in \mathbb{C}^{n \times n} \) be partitioned as the following form:

\[
A = \begin{pmatrix}
A(\alpha_1, \alpha_1) & A(\alpha_1, \alpha_2) & \cdots & A(\alpha_1, \alpha_s) \\
A(\alpha_2, \alpha_1) & A(\alpha_2, \alpha_2) & \cdots & A(\alpha_2, \alpha_s) \\
\vdots & \vdots & \ddots & \vdots \\
A(\alpha_s, \alpha_1) & A(\alpha_s, \alpha_2) & \cdots & A(\alpha_s, \alpha_s)
\end{pmatrix},
\]

where \( 1 \leq s \leq n \), \( \alpha_0 = 0 \),

\[
\alpha_i = \left\{ \begin{array}{l}
\sum_{t=0}^{i-1} |\alpha_t| + 1, \ldots, \sum_{t=0}^{i} |\alpha_t| \\
(1 \leq i \leq s), \quad \sum_{t=0}^{i} |\alpha_t| = n
\end{array} \right.
\]

and \( A(\alpha_1, \alpha_1) \) is a \( |\alpha| \times |\alpha| \) nonsingular principal submatrix of \( A \), \( t = 1, 2, \ldots, s \).

Without loss of generality, we assume that \( \mathbb{C}_n^{r \times r} \) denote the set of all \( n \times n \) block matrices in \( \mathbb{C}^{n \times n} \) partitioned as (1), \( A = (A(\alpha_i, \alpha_m))_{r \times r} \in \mathbb{C}_n^{r \times r} \) and \( N(A) = (\|A(\alpha_i, \alpha_m)\|)_{s \times s} \), denote the norm matrix of block matrix \( A \).

In this paper, the matrix norm \( \|A\| \) of \( A \in \mathbb{C}^{n \times n} \) is defined as

\[
\|A\| = \sup_{x \in \mathbb{C}^m, x \neq 0} \|Ax\|/\|x\|.
\]

Thus if \( A \in \mathbb{C}^{n \times n} \) is nonsingular, then it holds that

\[
\|A^{-1}\|^{-1} = \left\{ \begin{array}{l}
\sup_{x \in \mathbb{C}^m, x \neq 0} \|A^{-1}x\|^{-1} \quad \text{if } x \in \mathbb{C}^m, x \neq 0
\end{array} \right.
\]

The remainder of this paper is organized as follows. In Section II, we recollect some useful lemmas which are utilized in the next sections. Several new estimates for the \( I \)-block diagonally dominant degree of the Schur complement of matrices are established in Section III. As applications, some new distribution theorems for eigenvalues of the Schur complement and the new bounds for the determinant of \( II \)-block diagonally dominant matrices are obtained in Section IV and Section V, respectively. Section VI is devoted to performing some numerical experiments to confirm the advantages and the validity of the established results. Finally, the paper is ended with some conclusions in Section VII.
In this section, we start with some lemmas. They will be useful in the following proofs.

**Lemma 2.1** [13] If $A \in SD_n$, then $\mu(A)$ is $M$-matrix, i.e., $A$ is $H$-matrix.

**Lemma 2.2** [2] If $A$ is a $H$-matrix, then $[\mu(A)]^{-1} \geq [A^{-1}]$.

**Lemma 2.3** [30] If $A \in I - BSDs$, then $[\mu_I(A)]^{-1} \geq N(A^{-1})$.

**Lemma 2.4** [30] If $A \in I - BSDs$, then $[\mu_{II}(A)]^{-1} \geq N(A^{-1}D)$, where $D = \text{diag}(A_{11}, A_{12}, \ldots, A_{ss})$.

Let $A \in C_s^{n \times n}$, $\alpha = \sum_{i=1}^{k} \alpha_i \in N$, $\alpha' = N - \alpha = \sum_{i=1}^{l} \alpha_j \in N$, and $k + l = s$. For any $\alpha_j \in \alpha'$, we denote:

$$B_j = \left( \begin{array}{cc} x & -G_j \mu(A(\alpha)) \\ -H^T & \eta \end{array} \right).$$

If $A \in I - BSD_s$, we take $\tilde{\mu}(A) = \mu_I[A(\alpha)]$,

$$C_t = \left\{ \|A(\alpha_j, \alpha_i)\|, \ldots, \|A(\alpha_j, \alpha_i)\| \right\},$$

$$H = \left\{ \sum_{u=1}^{l} \|A(\alpha_i, \alpha_j)\|, \ldots, \sum_{u=1}^{l} \|A(\alpha_i, \alpha_j)\| \right\}.$$

If

$$x \geq f \sum_{\alpha_j \in \alpha'} \|A(\alpha_j, \alpha_i)\| \sum_{t=1}^{l} \|A(\alpha_j, \alpha_i)\|,$$

then $\eta = \max_{1 \leq w \leq k} \frac{\sum_{t=1}^{l} \|A(\alpha_i, \alpha_j)\|}{1 - \sum_{t=1, t \neq w}^{l} \|A(\alpha_i, \alpha_j)\|}$.

Thus

$$\eta = \sum_{t=1}^{l} \|A(\alpha_i, \alpha_j)\| \sum_{t=1}^{l} \|A(\alpha_i, \alpha_j)\|.$$

We construct a positive diagonal matrix $D = \text{diag}(d_1, d_2, \ldots, d_{k+1})$, where

$$d_v = \begin{cases} 1, & v = 1, \\ hA_{i1}(1)^{-1} \|A(\alpha_i, \alpha_j)\| + \varepsilon, & 2 \leq v \leq k + 1. \end{cases}$$

Denote $C_t = B_j D = (c_{sv})_{(k+1) \times (k+1)}$. If $s = 1$, then

$$x = \sum_{v=1}^{k+1} \|A(\alpha_j, \alpha_i)\| \left[ \frac{P_{i1}(A)}{\|A(\alpha_i, \alpha_j)\|} \right] + \varepsilon > 0;$$

If $s = 2, 3, \ldots, k + 1$, then it has

$$x = \sum_{v=1}^{k+1} \|A(\alpha_j, \alpha_i)\| \left[ \frac{P_{i1}(A)}{\|A(\alpha_i, \alpha_j)\|} \right] + \varepsilon > 0.$$

Since $A \in I - BSD_s$, it holds that $0 \leq r < 1$. Moreover, for $1 \leq u \leq k$, we have

$$r \geq \frac{\sum_{v=1}^{k} \|A(\alpha_i, \alpha_j)\|}{1 - \sum_{t=1, t \neq w}^{l} \|A(\alpha_i, \alpha_j)\|}.$$
From the above inequality, for $1 \leq u \leq k$, we obtain
$$0 \leq P_{iu}(A) \| A(\alpha_{iu}, \alpha_{iu}) \| + r \sum_{t=1,t\neq u}^k \| A(\alpha_{it}, \alpha_{it}) \| = P_{iu}(A).$$

From the above inequality, for $1 \leq u \leq k$, we obtain
$$0 \leq P_{iu}(A) \| A(\alpha_{iu}, \alpha_{iu}) \| \leq r < 1.$$

By the definition of $P_{iu}(A)$, for $1 \leq u \leq k$, we have
$$P_{iu}(A) = \sum_{v=1}^l \| A(\alpha_{iv}, \alpha_{iv}) \|,$$
$$P_{iu}(A) = \sum_{v=1}^l \| A(\alpha_{iv}, \alpha_{iv}) \| \leq 1,$$
which leads to $0 \leq h \leq 1$. Furthermore, for $1 \leq u \leq k$,
$$h \geq \sum_{v=1}^l \| A(\alpha_{iv}, \alpha_{iv}) \|,$$
which can be rewritten as
$$hP_{iu}(A) \geq \sum_{v=1}^l \| A(\alpha_{iv}, \alpha_{iv}) \| + h \sum_{t=1,t \neq u}^k \| A(\alpha_{it}, \alpha_{it}) \|.$$

Thus, it follows from Equality (7) that for $s = 2, 3, \cdots, k+1$,
$$\langle c_s \rangle = \sum_{u=2}^{k+1} \langle c_s \rangle =$$
$$hP_{iu}(A) + \sum_{v=1}^l \| A(\alpha_{iv}, \alpha_{iv}) \| + h \sum_{t=1,t \neq u}^k \| A(\alpha_{it}, \alpha_{it}) \|$$
$$\geq \sum_{v=1}^l \| A(\alpha_{iv}, \alpha_{iv}) \| + h \sum_{t=1,t \neq u}^k \| A(\alpha_{it}, \alpha_{it}) \|$$
$$\times \frac{P_{iu}(A)}{\| A(\alpha_{iu}, \alpha_{iu}) \|} + \| A(\alpha_{iu}, \alpha_{iu}) \|$$
$$- \sum_{t=1,t \neq u}^k \| A(\alpha_{it}, \alpha_{it}) \|$$
$$- \sum_{v=1}^l \| A(\alpha_{iv}, \alpha_{iv}) \|$$
$$= hP_{iu}(A) + \sum_{v=1}^l \| A(\alpha_{iv}, \alpha_{iv}) \| + h \sum_{t=1,t \neq u}^k \| A(\alpha_{it}, \alpha_{it}) \|$$
$$\geq \sum_{v=1}^l \| A(\alpha_{iv}, \alpha_{iv}) \| + h \sum_{t=1,t \neq u}^k \| A(\alpha_{it}, \alpha_{it}) \|$$
$$\times \frac{P_{iu}(A)}{\| A(\alpha_{iu}, \alpha_{iu}) \|} + \| A(\alpha_{iu}, \alpha_{iu}) \|$$
$$- \sum_{t=1,t \neq u}^k \| A(\alpha_{it}, \alpha_{it}) \|$$
$$- \sum_{v=1}^l \| A(\alpha_{iv}, \alpha_{iv}) \|$$
$$= \varepsilon h(\| A(\alpha_{iu}, \alpha_{iu}) \| - \sum_{w=1,w \neq u}^l \| A(\alpha_{iw}, \alpha_{iw}) \|) > 0,$$
which means that $C_i$ is a $SD_{k+1}$. By Lemma 2.1, $\mu(B_{j_l})$ is a M-matrix. Note that $\mu(B_{j_l}) = B_{j_l}$, then det $B_{j_l} > 0$.

When the equality holds in (5), for any $\varepsilon > 0$, denote $B_\varepsilon = B + \text{diag}(\varepsilon, 0, \cdots, 0)$. In a similar way to the above proof, we have $B_\varepsilon \in SD_{k+1}$ and hence $\det B_{j_l} > 0$. Let $\varepsilon \to 0^+$, we get det $B_{j_l} \geq 0$ immediately.

For the case of $A \in II - BSD_s$, the proof is similar. ■

Lemma 2.6 [30] Let $A \in I - (II) - BSD_s$, $\alpha = \bigcup_{u=1}^k \alpha_{iu} \subset N$, $\alpha' = N - \alpha = \bigcup_{v=1}^l \alpha_{iv}$, and $k + l = s$.

For any $t = 1, 2, \cdots, l$,
$$\Psi_t = 1 - \sum_{v=1}^l \| A(\alpha_{iv}, \alpha_{iv}) \| \left( A(\alpha_{iv}, \alpha_{iv}) \right)^{-1} \left( A(\alpha_{iv}, \alpha_{iv}) \right) > 0.$$
By the definition of Schur complement, denote by $J_t = |\alpha_t|$ and $I_m$ the identity matrix. According to Lemma 2.5, we obtain $\|A(\alpha_t, \alpha_t)\|^{-1} < 1$. It follows that
\begin{align*}
\|A(\alpha_t, \alpha_t)\|^{-1} - \sum_{r=1}^t \|A(\alpha_t, \alpha_r)\| &
\geq \|A(\alpha_t, \alpha_t)\|^{-1} - \sum_{r=1}^t \|A(\alpha_t, \alpha_r)\| - \|\Psi_t\| \\
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\geq \|A(\alpha_t, \alpha_t)\|^{-1} - \sum_{r=1}^t \|A(\alpha_t, \alpha_r)\| - \|\Psi_t\| \\
&\]
This means that
\[
\begin{align*}
  w_{j_1} &= \sum_{v=1}^{k} \| A(\alpha_{j_v}, \alpha_u) \| \frac{\| A(\alpha_{j_v}, \alpha_u) \| -1}{\| A(\alpha_{j_v}, \alpha_u) \|^2} - hP_v(A) \\
  \geq & \sum_{v=1}^{k} \| A(\alpha_{j_v}, \alpha_u) \| \frac{\| A(\alpha_{j_v}, \alpha_u) \| -1}{\| A(\alpha_{j_v}, \alpha_u) \|^2} - P_v(A) \\
  \geq & (1 - r) \sum_{v=1}^{k} \| A(\alpha_{j_v}, \alpha_u) \|
  \\
  &= \min_{1 \leq n \leq k} \frac{\| A(\alpha_{j_v}, \alpha_u) \| - 1 - R_n(A)}{\| A(\alpha_{j_v}, \alpha_u) \|^2} - \sum_{\ell=1, \ell \neq n}^{k} \| A(\alpha_{j_v}, \alpha_u) \|
  \\
  \geq & \min_{1 \leq n \leq k} \frac{\| A(\alpha_{j_v}, \alpha_u) \| - 1 - R_n(A)}{\| A(\alpha_{j_v}, \alpha_u) \|^2} - \sum_{v=1}^{k} \| A(\alpha_{j_v}, \alpha_u) \|.
\end{align*}
\]

From Inequality (11), it’s obvious that Theorem 1 improves the results of Theorem 3.1 in [13], Theorem 2.10 in [21] and Theorem 2.1.1 in [35].

Based on Theorem 3.1, the following corollary can be obtained immediately.

**Corollary 3.1** Let $A \in I - BSD_s$, and take $\alpha = \sum_{u=1}^{s-1} \alpha_u < N$. Then
\[
\begin{align*}
  \| (A/\alpha)^{-1} \| &\leq \| (A, \alpha_s)^{-1} \| - 1 \\
  - h \sum_{v=1}^{s-1} \| A(\alpha_{j_v}, \alpha_u) \| \| P_v(A) \| \| A(\alpha_{j_v}, \alpha_u) \|^2 - 1 \\
  + h \sum_{v=1}^{s-1} \| A(\alpha_{j_v}, \alpha_u) \| \| P_v(A) \| \| A(\alpha_{j_v}, \alpha_u) \|^2 - 1.
\end{align*}
\]

**Proof.** Notice that $\alpha' = \alpha_s$. Thus, $A/\alpha = (\hat{A}(\alpha_s, \alpha_s))$, and $R_s(A/\alpha) = 0$, so by the definition of $w_{j_1}$, we have
\[
\begin{align*}
  w_{j_1} &= w_s \\
  &= \sum_{v=1}^{s-1} \| A(\alpha_{j_v}, \alpha_u) \| \| A(\alpha_{j_v}, \alpha_u) \|^2 - 1 - hP_v(A) \\
  &= \sum_{v=1}^{s-1} \| A(\alpha_{j_v}, \alpha_u) \| - h \sum_{v=1}^{s-1} \| A(\alpha_{j_v}, \alpha_u) \| \| P_v(A) \| \| A(\alpha_{j_v}, \alpha_u) \|^2 - 1.
\end{align*}
\]

Substituting Equation (12) into Inequality (8) and in a manner similar to that done for Theorem 3.1, the results are derived.

**Theorem 3.2** Let $A \in I - BSD_s, \alpha = \sum_{u=1}^{k} \alpha_u < N, \alpha' = N - \alpha = \sum_{v=1}^{l} \alpha_{j_v}$, and $k + l = s$. Denote $A/\alpha = (\hat{A}(\alpha_t, \alpha_j))$. Then
\[
\begin{align*}
  1 - \bar{R}_t(A/\alpha) \geq 1 - \bar{R}_{j_1}(A) + \bar{w}_j \geq 1 - \bar{R}_{j_1}(A) > 0 \Rightarrow (13) \text{ and}
  1 - \bar{R}_t(A/\alpha) \leq 1 - \bar{R}_{j_1}(A) - \bar{w}_j \leq 1 + \bar{R}_{j_1}(A),
\end{align*}
\]

where
\[
\begin{align*}
  \hat{R}_{j_1}(A) &= \sum_{m=1, m \neq j_1}^{s} \| A(\alpha_{j_1}, \alpha_m) \| - \| A(\alpha_{j_1}, \alpha_m) \| A(\alpha_{j_1}, \alpha_m) \|
  \\
  \hat{w}_j &= \sum_{v=1}^{k} \| A(\alpha_{j_1}, \alpha_v) \| - \| A(\alpha_{j_1}, \alpha_v) \| A(\alpha_{j_1}, \alpha_v) \| (1 - f \bar{P}_v(A)),
\end{align*}
\]

and $f$ and $\bar{P}_v(A)$ ($v = 1, 2, \ldots, k$) are defined as in Lemma 2.4.

**Proof.** For $t, r = 1, 2, \ldots, l$, denote $J_t = |\alpha_{j_r}|$, let
\[
\begin{align*}
  D &= \text{diag}(\alpha_{j_1}, \alpha_{j_1}, \ldots, \alpha_{j_k}, \alpha_{j_k}), \\
  \Psi_{\alpha_r} &= (A(\alpha_{j_1}, \alpha_{j_1}), \ldots, A(\alpha_{j_k}, \alpha_{j_k})) \\
  \Upsilon_r &= \{ A(\alpha_{j_1}, \alpha_{j_1})^{-1} A(\alpha_{j_1}, \alpha_{j_1}), \ldots, A(\alpha_{j_k}, \alpha_{j_k})^{-1} A(\alpha_{j_k}, \alpha_{j_k}) \}, \\
  \Gamma_r &= \{ (A(\alpha_{j_1}, \alpha_{j_1})^{-1} A(\alpha_{j_1}, \alpha_{j_1}), \ldots, (A(\alpha_{j_k}, \alpha_{j_k})^{-1} A(\alpha_{j_k}, \alpha_{j_k})) \}^T, \\
  L_t &= \{ (A(\alpha_{j_1}, \alpha_{j_1})^{-1} A(\alpha_{j_1}, \alpha_{j_1}), \ldots, (A(\alpha_{j_k}, \alpha_{j_k})^{-1} A(\alpha_{j_k}, \alpha_{j_k})) \}^T, \\
  H_t &= \left( \sum_{r=1}^{l} (A(\alpha_{j_1}, \alpha_{j_1})^{-1} A(\alpha_{j_1}, \alpha_{j_1})) \right)^T.
\end{align*}
\]

It follows from the definition of $\Psi_{\alpha}$ in Lemma 2.6 that
\[
\begin{align*}
  \Psi_{\alpha} &= 1 - \| (A(\alpha_{j_1}, \alpha_{j_1}))^{-1} \Psi_{\alpha} \| = 1 - \| \Upsilon_r [A(\alpha)]^{-1} D \Gamma_t \|,
\end{align*}
\]

which is equivalent to
\[
\begin{align*}
  \frac{1}{\Psi_{\alpha}} [1 - \| \Upsilon_r [A(\alpha)]^{-1} D \Gamma_t \|] &= 1. \quad (15)
\end{align*}
\]

According to lemma 2.7, we obtain
\[
\begin{align*}
  \| \{ I_{j_1} - [A(\alpha_{j_1}, \alpha_{j_1})]^{-1} \Psi_{\alpha} \}^{-1} \| \\
  &\leq \frac{1}{1 - \| (A(\alpha_{j_1}, \alpha_{j_1})^{-1} \Psi_{\alpha} \|} = \frac{1}{\Psi_{\alpha}}. \quad (16)
\end{align*}
\]

By making use of the definition of the Schur complement, we deduce that
\[
\begin{align*}
  1 - \bar{R}_t(A/\alpha) &= 1 - \sum_{r=1, r \neq t}^{l} \| (A(\alpha_{j_1}, \alpha_{j_1}) - \Psi_{\alpha} \|^2 [A(\alpha_{j_1}, \alpha_{j_1}) - \Psi_{\alpha} \|^2 \\
  &= 1 - \sum_{r=1, r \neq t}^{l} \| [I_{j_1} - [A(\alpha_{j_1}, \alpha_{j_1})]^{-1} \Psi_{\alpha} \|^{-1} \\
  \times \| [A(\alpha_{j_1}, \alpha_{j_1}) - [A(\alpha_{j_1}, \alpha_{j_1})]^{-1} \Psi_{\alpha} \|^{-1} \| \{ I_{j_1} - [A(\alpha_{j_1}, \alpha_{j_1})]^{-1} \Psi_{\alpha} \}^{-1} \| \\
  \| [A(\alpha_{j_1}, \alpha_{j_1})^{-1} A(\alpha_{j_1}, \alpha_{j_1}) - \Upsilon_r [A(\alpha)]^{-1} D \Gamma_t \| + \| \Upsilon_r [A(\alpha)]^{-1} D \Gamma_t \| \text{ (by (16))} \\
  &= \frac{1}{\Psi_{\alpha}} \left( 1 - \| \Upsilon_r [A(\alpha)]^{-1} D \Gamma_t \| \right)
\end{align*}
\]

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\[ \sum_{r=1}^{t} \|A(\alpha_{j}, \alpha_{i})^{-1}A(\alpha_{j}, \alpha_{i})\| \]

\[ \sum_{r=1}^{t} \|\Pi_{r}\|\|A(\alpha)\|^{-1}D\|\Pi_{r}\| \] (by (15))

\[ \geq \frac{1}{\Psi_{t}} \left( 1 - \sum_{r=1, r \neq t}^{t} \|A(\alpha_{j}, \alpha_{i})^{-1}A(\alpha_{j}, \alpha_{i})\| \right) \]

\[ - \sum_{r=1}^{t} \|\Pi_{r}\|N(\|A(\alpha)\|^{-1}D\|\Pi_{r}\|) \]

\[ \geq 1 - \sum_{r=1, r \neq t}^{t} \|A(\alpha_{j}, \alpha_{i})^{-1}A(\alpha_{j}, \alpha_{i})\| \]

\[ - L_{T}^{T}\{\mu_{T}(A(\alpha))\}^{-1}H' \text{(by Lemma 2.4)} \]

\[ = 1 - R_{j_{i}}(A) + \tilde{\omega}_{j_{i}} + \sum_{r=1}^{k} \|A(\alpha_{j}, \alpha_{i})^{-1}A(\alpha_{j}, \alpha_{i})\| \]

\[ - \tilde{\omega}_{j_{i}} - L_{T}^{T}\{\mu_{T}(A(\alpha))\}^{-1}H' \]

\[ = 1 - R_{j_{i}}(A) + \tilde{\omega}_{j_{i}} - \varepsilon + \frac{1}{\det[\mu_{T}(A(\alpha))]} \]

\[ = 1 - R_{j_{i}}(A) + \tilde{\omega}_{j_{i}} - \varepsilon + \frac{\det B_{2}}{\det[\mu_{T}(A(\alpha))]} \] (17)

Since \( A \in II - BSD_{s} \), it holds that

\[ \sum_{r=1}^{k} \|A(\alpha_{j}, \alpha_{i})^{-1}A(\alpha_{j}, \alpha_{i})\| - \tilde{\omega}_{j_{i}} + \varepsilon \leq f \sum_{r=1}^{k} \|A(\alpha_{j}, \alpha_{i})^{-1}A(\alpha_{j}, \alpha_{i})\| \tilde{P}_{t_{r}}(A) + \varepsilon \]

\[ > f \sum_{r=1}^{k} \|A(\alpha_{j}, \alpha_{i})^{-1}A(\alpha_{j}, \alpha_{i})\| \tilde{P}_{t_{r}}(A). \]

By Lemma 2.3, it is easy to see that \( \det B_{2} > 0 \). By Lemma 2.1, we deduce that \( \mu_{T}(A(\alpha)) \) is nonsingular \( M \)-matrix, thus \( \det[\mu_{T}(A(\alpha))] > 0 \), which yields that

\[ \lambda_{r}(A(\alpha))/1 - R_{j_{i}}(A) > 1 - R_{j_{i}}(A) + \tilde{\omega}_{j_{i}} - \varepsilon \geq 1 - R_{j_{i}}(A) - \varepsilon. \]

Let \( \varepsilon \to 0 \), thus we can get

\[ 1 - R_{t}(A(\alpha)) \geq 1 - R_{j_{i}}(A) + \tilde{\omega}_{j_{i}} \geq 1 - R_{j_{i}}(A) > 0, \]

which proves the desired Inequality (13). We can prove Inequality (14) with a quite similar strategy utilized in this theorem.

Remark 3.2 Note that

\[ f \tilde{P}_{t_{r}}(A) \leq \tilde{P}_{t_{r}}(A) \leq \eta \leq \max_{1 \leq u \leq k} \tilde{R}_{u}(A), 1 \leq u \leq k, \]

which leads to

\[ \tilde{w}_{j_{i}} = \sum_{u=1}^{k} \|A(\alpha_{j}, \alpha_{i})^{-1}A(\alpha_{j}, \alpha_{i})\| (1 - f \tilde{P}_{t_{r}}(A)) \]

\[ \geq \sum_{u=1}^{k} \|A(\alpha_{j}, \alpha_{i})^{-1}A(\alpha_{j}, \alpha_{i})\| (1 - \tilde{P}_{t_{r}}(A)) \]

\[ = \min_{1 \leq u \leq k} \left( 1 - \tilde{R}_{u}(A) \right) \]

\[ \times \sum_{v=1}^{k} \|A(\alpha_{j}, \alpha_{i})^{-1}A(\alpha_{j}, \alpha_{i})\| \]

\[ \geq \min_{1 \leq u \leq k} (1 - \tilde{R}_{u}(A)) \sum_{v=1}^{k} \|A(\alpha_{j}, \alpha_{i})^{-1}A(\alpha_{j}, \alpha_{i})\|. \] (18)

Evidently, from Inequality (18) we see that Theorem 2 improves the results of Theorem 3.2 in [13], Theorem 2.1.2 in [35] and Theorem 2.13 in [21].

IV. DISTRIBUTION FOR EIGENVALUES OF THE SCHUR COMPLEMENT OF I-(II)-BSD_{s}

In this section, as an application of our results in Section II and Section III, we establish some new locations for the eigenvalues of the Schur complements of \( I - (II - BSD_{s}) \) by the elements of the original matrix. Without loss of generality, we assume that \( \alpha = \bigcup_{u=1}^{k} \alpha_{u} \subset N, \alpha' = N - \alpha = \bigcup_{u=1}^{l} \alpha_{u} \subset N, \) and \( k + l = s \). Let \( A(\alpha') = (A(\alpha_{r}, \alpha_{r}'), \alpha_{r} \in \alpha') \) be the identity matrix. Denote by \( \lambda(\alpha'/\alpha) \) and \( \lambda(\alpha) \) the set of eigenvalues of \( A(\alpha'/\alpha) \) and \( A \) respectively.

Lemma 4.1 [13] Let \( A \in I - BSD_{s} \) and \( \lambda(\alpha) \) denote the set of eigenvalues of \( A \). Then

\[ \lambda(\alpha) \subset G = \bigcup_{i=1}^{s} [G_{i} \cup \lambda(A(\alpha_{i}, \alpha_{i})), \]

where

\[ G_{i} = \left\{ \lambda : \lambda \not\in \lambda(A(\alpha_{i}, \alpha_{i})) \right\} \]

Theorem 4.1 Let \( A \in I - BSD_{s} \) and \( w_{j_{i}} \) be defined as in Theorem 3.1. Then

\[ \lambda(\alpha) \subset G = \bigcup_{t=1}^{s} [G_{t} \cup \lambda(A(\alpha_{j}, \alpha_{j})), \]

where

\[ G_{t} = \left\{ \lambda : \lambda \not\in \lambda(A(\alpha_{j}, \alpha_{j})) \right\} \]

Proof. Let \( \Psi_{t} \) be such as in Theorem 3.1. If \( \lambda \not\in \lambda(\tilde{A}(\alpha_{i}, \alpha_{i})) \) and \( \lambda \not\in \lambda(A(\alpha_{j}, \alpha_{j})) \), then combining In-
equality (4) with Lemma 4.1 results in
\[ \| [\lambda I_t - \tilde{A}(\alpha_t, \alpha_t)]^{-1} x \|^{-1} \]
\[ = \left\{ \sup_{x \in \mathbb{C}^n, x \neq 0} \frac{\| [\lambda I_t - \tilde{A}(\alpha_t, \alpha_t)]^{-1} x \|}{\| x \|} \right\} \]
\[ = \inf_{x \in \mathbb{C}^n, x \neq 0} \frac{\| [\lambda I_t - \tilde{A}(\alpha_t, \alpha_t)] x \|}{\| x \|} \text{ (by (4))} \]
\[ = \inf_{x \in \mathbb{C}^n, x \neq 0} \frac{\| [\lambda I_t - [A(\alpha_j, \alpha_j) - \Psi_t]] x \|}{\| x \|} \]
\[ \geq \inf_{x \in \mathbb{C}^n, x \neq 0} \frac{\| [\lambda I_j - [A(\alpha_j, \alpha_j)] x \|}{\| x \|} - \sup_{x \in \mathbb{C}^n, x \neq 0} \| \Psi_t x \|} \]
\[ = \| [\lambda I_j - [A(\alpha_j, \alpha_j)]^{-1} x \|^{-1} - \| \Psi_t x \|. \]

Moreover,
\[ \| [\lambda I_j - [A(\alpha_j, \alpha_j)]^{-1} \|^{-1} \]
\[ \leq \| [\lambda I - [A(\alpha_t, \alpha_t)]^{-1} \|^{-1} + \| \Psi_t x \| \]
\[ = \sum_{r=1}^{l} \| A(\alpha_j, \alpha_j) - \Psi_{tr} \| + \| \Psi_{tr} \| \]
\[ \leq \sum_{r=1}^{l} \| A(\alpha_j, \alpha_j) \| + \| \Psi_{tr} \| \]
\[ \leq R_{j, t}(A) - w_{j, t} - \sum_{r=1}^{k} \| A(\alpha_j, \alpha_j) \| \]
\[ + w_{j, t} + G_t^T [\mu_t(A)]^{-1} H^t \]
\[ = R_{j, t}(A) - w_{j, t} + \varepsilon - \frac{1}{\det [\mu_t(A)]} \]
\[ \times \det \left( \sum_{r=1}^{k} A(\alpha_j, \alpha_j) - w_{j, t} + \varepsilon - G_t^T \right) \]
\[ = R_{j, t}(A) - w_{j, t} + \varepsilon - \frac{1}{\det B_t} \]
\[ \text{where } H^t, B_t, \text{ and } \mu_t(A) \text{ are defined as in the proof of Theorem 3.1. Thus } \det B_t > 0 \text{ and } \det j_t(A) \| > 0. \]
\[ \| [\lambda I_j - [A(\alpha_j, \alpha_j)]^{-1} \|^{-1} < R_{j, t}(A) - w_{j, t} + \varepsilon, \]

Letting \( \varepsilon > 0 \) yields
\[ \| [\lambda I_j - [A(\alpha_j, \alpha_j)]^{-1} \|^{-1} \leq R_{j, t}(A) - w_{j, t}. \]

If \( \lambda \in \lambda[\tilde{A}(\alpha_t, \alpha_t)] \) and \( \lambda \notin \lambda[A(\alpha_j, \alpha_j)] \), we assume that \( \tilde{x} \neq 0 \) is the eigenvector of \( A \) corresponding to \( \lambda \). Then
\[ 0 = \| [\lambda I_t - \tilde{A}(\alpha_t, \alpha_t)] x \| \]
\[ \geq \inf_{x \in \mathbb{C}^n, x \neq 0} \frac{\| [\lambda I_t - \tilde{A}(\alpha_t, \alpha_t)] x \|}{\| x \|} \]
\[ = \inf_{x \in \mathbb{C}^n, x \neq 0} \frac{\| [\lambda I_t - A(\alpha_j, \alpha_j)] x \|}{\| x \|} \]
\[ \geq \inf_{x \in \mathbb{C}^n, x \neq 0} \frac{\| [\lambda I_j - A(\alpha_j, \alpha_j)] x \|}{\| x \|} - \| \Psi_t x \| \]
\[ \geq \inf_{x \in \mathbb{C}^n, x \neq 0} \frac{\| [\lambda I_j - A(\alpha_j, \alpha_j)] x \|}{\| x \|} - \| \Psi_t x \| \]
\[ = \| [\lambda I_j - A(\alpha_j, \alpha_j)]^{-1} \|^{-1} - \| \Psi_t x \|. \]

Therefore,
\[ \| [\lambda I_j - [A(\alpha_j, \alpha_j)]^{-1} x \|^{-1} \]
\[ \leq \| \Psi_t x \| \leq R_t(A/\alpha) + \| \Psi_t x \| \leq R_{j, t}(A) - w_{j, t}, \]

which proves this theorem.

**Remark 4.1** By Remark 3.1, it is obvious that Theorem 4.1 improves the results of Theorem 4.1 in [13], Theorem 3.1.1 in [35] and Theorem 3.5 in [21].

By Theorem 3.2, similar to the proof of Theorem 4.1, the following theorem can be derived.

**Theorem 4.2** Let \( A \in II - BSD_s \) and \( w_j \) be defined as in Theorem 3.2. Then
\[ \lambda(A) \subset G = \bigcup_{t=1}^{s} [G_t \cup \lambda(A(\alpha_j, \alpha_j))], \]

where
\[ G_t = \left\{ \lambda : \lambda \notin \lambda[A(\alpha_j, \alpha_j)] \text{ and } \| [\lambda I_j - A(\alpha_j, \alpha_j)]^{-1} \|^{-1} \leq \Upsilon_t \right\} \]

and \( \Upsilon_t = \| A(\alpha_j, \alpha_j) \| [R_{j, t}(A) - w_{j, t}] \).**

**Remark 4.2** Similar to the discussions in Remark 3.2, it can be seen that the results of Theorem 4.2 improve those in Theorem 4.2 of [13], Theorem 3.12 of [35] and Theorem 3.6 of [21].

V. SOME NEW BOUNDS FOR DETERMINANTS OF I-(II-)BSD_s

In this section, we make use of the results in Sections II-IV to exhibit some new upper and lower bounds for the determinants of \( I - (I - II) - BSD_s \).

**Lemma 5.1** [36] Let \( A = (a_{ij})_{n \times n}, \emptyset \neq \alpha \subseteq N, \) assume that \( A(\alpha) \) is nonsingular. Then
\[ \det A = \det [A(\alpha)] \det A/\alpha. \]

**Lemma 5.2** [29] Let \( A \in II - BSD_s \), then \( D^{t-1}A \in I - BSD_s \), where \( D \) is defined as in Lemma 2.4.

Let \( \{ j_1, j_2, \ldots, j_s \} \) be a rearrangement of the elements in \{1, 2, \ldots, s\}. Denote \( \beta_1 = \alpha_{j_1}, \beta_2 = \alpha_{j_2}, \ldots, \beta_s = \alpha_{j_s}, \cup \alpha_{j_s+1}, \ldots, \cup \alpha_{j_s} = N. \) Then \( \beta_s - t + 1 - S = \alpha_{j_s}, \beta_1 = \alpha_{j_1} = \alpha_t, \).

\[ R_t(A(\beta_s-t+1)) = \sum_{\alpha_t \subseteq \beta_s-t} \| A(\alpha_{j_1}, \alpha_s) \|, \]

\[ R_{j, t}(A(\beta_s-t+1)) = \sum_{\alpha_t \subseteq \beta_s-t} \| A(\alpha_{j_1}, \alpha_s) \| - 1 \cdot A(\alpha_{j_1}, \alpha_s) \| \]

Let \( \varphi \) represent any rearrangement \( \{ j_1, j_2, \ldots, j_s \} \) of the elements in \{1, 2, \ldots, s\} with \( \beta_1, \beta_2, \ldots, \beta_s \) defined as above. Next, we establish some bounds for determinants of I-(II-)BSD_s in the following theorems.

**Theorem 5.1** Let \( A \in I - BSD_s \) and be partitioned as in (1). Then
\[ | \det A | \geq \max_{\varphi} \prod_{t=1}^{s} \{ [A(\alpha_{j_1}, \alpha_{j_s})]^{-1} - 1 - \Theta_{j, t}]^{[\alpha_{j_1}]}, \]

and
\[ | \det A | \leq \min_{\varphi} \prod_{t=1}^{s} \{ [A(\alpha_{j_1}, \alpha_{j_s})] + 1 - \Theta_{j, t}]^{[\alpha_{j_1}]}, \]

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where
\[
\Theta_{\beta_t} = h[A(\beta_{t-1})] \sum_{\alpha \in \beta_{t-1}} \|A(\alpha_j, \alpha_v)\| \frac{P_r[A(\beta_{t-1})]}{\|A(\alpha_v, \alpha_v)\|^{-1}} - 1.
\]

By Lemma 5.1, it follows that
\[
r[A(\beta_{t-1})] = \max_{t \leq s \leq n} \frac{\|A(\alpha_{j_1}, \alpha_{j_v})\|}{K_t},
\]
\[
K_t = \|A(\alpha_{j_1}, \alpha_{j_v})\|^{-1} - \sum_{v = t+1, v \neq \alpha} \|A(\alpha_{j_1}, \alpha_{j_v})\|,
\]
\[
P_r[A(\beta_{t-1})] = r[A(\beta_{t-1})] \sum_{t = 1}^{r[A(\beta_{t-1})]} \|A(\alpha_{j_1}, \alpha_{j_v})\| + \|A(\alpha_{j_1}, \alpha_{j_v})\|, \quad \alpha_v \in \beta_{t-1},
\]
\[
h[A(\beta_{t-1})] = \max_{t \leq s \leq n} \|A(\alpha_{j_1}, \alpha_{j_v})\| L_t = P_r[A(\beta_{t-1})]
\]
\[
- \sum_{v = t+1, v \neq \alpha} \|A(\alpha_{j_1}, \alpha_{j_v})\| \frac{P_r[A(\beta_{t-1})]}{\|A(\alpha_v, \alpha_v)\|^{-1}} - 1.
\]

**Proof.** Inasmuch as \(\beta_{t-1}\) is contained in \(\beta_{t-1} - \beta_{t-1} = \alpha_j\), by Corollary 3.1, we have
\[
\|A(\beta_{t-1})/\beta_{t-2}\|^{-1} - 1 \geq \|A(\alpha_j, \alpha_{j_v})\|^{-1} - \Theta_{\beta_t} > 0,
\]
\[
\|A(\beta_{t-1})/\beta_{t-2}\| \leq \|A(\alpha_j, \alpha_{j_v})\| + \Theta_{\beta_t}.
\]

By Lemma 5.1, it follows that
\[
|\det A| = \prod_{t \leq s \leq n} \frac{\det A[\beta_{s-t}]}{\det A[\beta_{s-t}]}|\det A[\beta_{s-t}]/\beta_{s-t}]| |\det A[\beta_{s-t}]/\beta_{s-t}| \det A[\beta_{s-t}]/\beta_{s-t}|
\]
\[
|\det A[\beta_{s-t}]/\beta_{s-t}]| = \prod_{s \geq t} \lambda_i(A(\beta_{s-t}/\beta_{s-t})) \prod_{s \geq t} \lambda_i(A(\beta_{s-t}/\beta_{s-t}))
\]
\[
\geq \prod_{s \geq t} \lambda_i(A(\beta_{s-t}/\beta_{s-t})) \prod_{s \geq t} \lambda_i(A(\beta_{s-t}/\beta_{s-t}))
\]
\[
\geq \prod_{t \leq s \leq n} \|A(\alpha_{j_1}, \alpha_{j_v})\|^{-1} - \Theta_{\beta_t}^{\alpha_j}.
\]

which proves the desired bound (19). The bound (20) can be similarly proved.

**Remark 5.1** Similar to the discussions in Remark 3.1, for \(\alpha_v \in \beta_{s-1}\), we have
\[
h[A(\beta_{s-1})] \frac{P_r[A(\beta_{s-1})]}{\|A(\alpha_v, \alpha_v)\|^{-1}} - 1 \leq r[A(\beta_{s-1})] \leq \max_{\alpha_v \in \beta_{s-1}} \frac{R_r[A(\beta_{s-1})]}{\|A(\alpha_v, \alpha_v)\|^{-1}} - 1,
\]

which results in
\[
\min_{r} \prod_{t \leq s \leq n} \|A(\alpha_{j_1}, \alpha_{j_v})\|^{-1} - \Theta_{\beta_t}^{\alpha_j}.
\]

\[
\geq \max_{r} \prod_{t \leq s \leq n} \|A(\alpha_{j_1}, \alpha_{j_v})\|^{-1} - r[A(\beta_{s-1})] \frac{R_r[A(\beta_{s-1})]}{\|A(\alpha_v, \alpha_v)\|^{-1}} - 1,
\]

\[
\geq \max_{r} \prod_{t \leq s \leq n} \|A(\alpha_{j_1}, \alpha_{j_v})\|^{-1} - \max_{\alpha_v \in \beta_{s-1}} \frac{R_r[A(\beta_{s-1})]}{\|A(\alpha_v, \alpha_v)\|^{-1}} - 1.
\]

The above discussions verify that Theorem 5.1 improves Theorem 2.3.1 in [35] and Theorem 3.6.1 in [34].

**Theorem 5.2** Let \(A \in II - BSD\) and be partitioned as in (1). Then
\[
|\det A| \geq \min_{r} \prod_{t \leq s \leq n} \|A(\alpha_{j_1}, \alpha_{j_v})\|^{-1} - (1 - \Delta_{\beta_t})^{\alpha_j} \geq \prod_{t \leq s \leq n} \|A(\alpha_{j_1}, \alpha_{j_v})\|^{-1} + \alpha_j
\]

where
\[
D = diag(A(\alpha_1, \alpha_1), \cdots, A(\alpha_s, \alpha_s)),
\]
\[
\Delta_{\beta_t} = h[D^{-1}A(\beta_{s-1})] = \prod_{t \leq s \leq n} \|A(\alpha_{j_1}, \alpha_{j_v})\|^{-1} A(\alpha_j, \alpha_v)\|P_r[D^{-1}A(\beta_{s-1})].
\]

**Proof.** Combining Lemma 5.2 and Theorem 5.1 yields that
\[
\min_{r} \prod_{t \leq s \leq n} \|A(\alpha_{j_1}, \alpha_{j_v})\|^{-1} - (1 - \Delta_{\beta_t})^{\alpha_j} \geq \prod_{t \leq s \leq n} \|A(\alpha_{j_1}, \alpha_{j_v})\|^{-1} + \alpha_j
\]

i.e.,
\[
|\det A| \leq |\det D| \min_{r} \prod_{t \leq s \leq n} \|A(\alpha_{j_1}, \alpha_{j_v})\|^{-1} \alpha_j
\]

So Inequality (22) is obtained, similarly, we can prove the Inequality (21).

**VI. NUMERICAL EXAMPLES**

In this section, we present some numerical examples to illustrate the theory results in this paper and show the advantages of our derived results.

**Example 6.1** Let
\[
A = \begin{pmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\
A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\
A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\
A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\
A_{51} & A_{52} & A_{53} & A_{54} & A_{55}
\end{pmatrix}
\]
\[
A_{11} = diag(16, \cdots, 16)_{20 \times 20}, A_{22} = diag(15, \cdots, 15)_{20 \times 20},
\]
\[
A_{33} = diag(18, \cdots, 18)_{20 \times 20}, A_{44} = diag(8, \cdots, 8)_{15 \times 15},
\]
\[
A_{55} = diag(9, \cdots, 9)_{15 \times 15}, A_{12} = diag(-1, \cdots, -1)_{20 \times 20},
\]
\[
A_{14} = A_{15} = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
& \ddots & \cdots & \ddots & \cdots \\
& & -2 & -2 & \cdots \\
& & & 20 & 15 \\
& & & & 0
\end{pmatrix}
\]
\[
A_{31} = A_{32} = \begin{pmatrix}
0 & -2 & \cdots & \cdots & \cdots \\
& 0 & \cdots & \cdots & \cdots \\
& & 0 & \cdots & \cdots \\
& & & \ddots & \cdots \\
& & & & 0
\end{pmatrix}
\]
By Theorem 4.1 in [13], any eigenvalues \( \lambda \) of \( A/\alpha \) satisfies
\[
\lambda \in \{ \lambda : |\lambda - 15| \leq 7.9186 \} \cup \{ \lambda : |\lambda - 8| \leq 6.8310 \}
\cup \{ \lambda : |\lambda - 9| \leq 6.2910 \} = \Gamma_1.
\]

By Theorem 3.13 in [20] and Theorem 3.5 in [21], any eigenvalues \( \lambda \) of \( A/\alpha \) satisfies
\[
\lambda \in \{ \lambda : |\lambda - 15| \leq 9.2727 \} \cup \{ \lambda : |\lambda - 8| \leq 9.5455 \}
\cup \{ \lambda : |\lambda - 9| \leq 8.1818 \} = \Gamma_2.\]

By Theorem 3.1.1 in [35], any eigenvalues \( \lambda \) of \( A/\alpha \) satisfies
\[
\lambda \in \{ \lambda : |\lambda - 15| \leq 9.2424 \} \cup \{ \lambda : |\lambda - 8| \leq 9.4697 \}
\cup \{ \lambda : |\lambda - 9| \leq 8.1515 \} = \Gamma_3.
\]

By Theorem 4.1 in [13], any eigenvalues \( \lambda \) of \( A/\alpha \) satisfies
\[
\lambda \in \{ \lambda : |\lambda - 15| \leq 10.1250 \} \cup \{ \lambda : |\lambda - 8| \leq 11.2500 \}
\cup \{ \lambda : |\lambda - 9| \leq 9.3750 \} = \Gamma_4.
\]

To further confirm the facts in the above results, Figures 1-3 depict the eigenvalue distributions of the Schur complement. From these numerical results and figures, we have the following observations:

- As observed in the comparison results, the Theorem 3.13 in [20], Theorem 3.5 in [21], Theorem 3.1.1 in [35], Theorem 4.1 in [13] and Theorem 4.1 can succeed in computing and determining the the eigenvalue distributions of the Schur complement by using the elements of the original matrix, whereas the eigenvalue distributions derived by Theorem 4.1 are sharper than the ones computed by Theorem 3.13 in [20], Theorem 3.5 in [21], Theorem 3.1.1 in [35] and Theorem 4.1 in [13], that is, \( \Gamma_1 \subset \Gamma_2 \), \( \Gamma_1 \subset \Gamma_3 \) and \( \Gamma_1 \subset \Gamma_4 \).
- From Figures 1-3, we clearly find that \( \Gamma_1 \) is the tightest among all eigenvalue distributions, which demonstrates the validity of the conclusion given in Remark 4.1.

Example 6.2 Let
\[
A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},
\]
where
\[
A_{11} = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad A_{13} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]
\[
A_{21} = \begin{pmatrix} 0 & 2 \end{pmatrix}, \quad A_{22} = 10, \quad A_{23} = \begin{pmatrix} 3 & 0 \end{pmatrix},
\]
\[
A_{31} = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}, \quad A_{32} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A_{33} = \begin{pmatrix} 0 & 9 \\ 0 & 9 \end{pmatrix}.
\]

It is easy to see that \( A \in I - BSD_3 \). We compare the bounds in Theorem 5.1 with those in Theorem 3.6.1 of [34]
and Theorem 3.2.3 of [35]. By utilizing Theorem 3.61 in [34], we have
\[35403 \leq |\det A| \leq 70596.\]
By making use of Theorem 3.2.3 in [35], we have
\[43643 \leq |\det A| \leq 60938.\]
Now, by applying Theorem 5.1, we derive
\[43841 \leq |\det A| \leq 60703,\]
which is an improvement on the bounds in Theorem 3.61 of [34] and Theorem 3.2.3 of [35]. This example shows that the upper and lower bounds in Theorem 5 are better than those in Theorem 3.61 of [34] and Theorem 3.2.3 of [35]. In fact, \(\det A = 47448\).

VII. CONCLUSIONS

To estimates diagonally dominant degree on the Schur complement of matrices, we first exhibit some new estimations of diagonally dominant degree on the Schur complement of I(II)-block diagonally dominant matrices in this paper, which are proved to be sharper than the ones in [13], [35], [21]. As applications, some new distributions for the eigenvalues of the Schur complement of matrices as well as the new upper and lower bounds for determinants of the I(II)-block diagonally dominant matrices are derived, these results are better compared with those of [13], [35], [21], [34]. Numerical examples are also given to illustrate these facts.

It would be nice if we can find more precise estimates of I(II)-block diagonally dominant degree for Schur complement of matrices, distributions for the eigenvalues of the Schur complement of matrices and upper and lower bounds for determinants of the I(II)-block diagonally dominant matrices compared those proposed in this paper. We will continue to research this topic in our further work.

REFERENCES


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