The New Estimations of Diagonally Dominant Degree and Eigenvalues Distributions for the Schur Complements of Block Diagonally Dominant Matrices and Determinantal Bounds

Zhengge Huang, Ligong Wang*, Zhong Xu, Jingjing Cui

***Abstract*—In this paper, some new estimations of diagonally dominant degree on the Schur complement of I(II)-block diagonally dominant matrices are obtained by applying the properties of Schur complement and some inequality techniques, which improve some existing ones. Further, as an application, we present some new distribution theorems for eigenvalues of the Schur complement and some new upper and lower bounds for the determinant of I(II)-block diagonally dominant matrices. These results are proved to be sharper than some known ones. Finally, numerical examples are also presented to confirm the theoretical results studied in this paper.**

***Index Terms*—block matrix, Schur complement, diagonally dominant degree, eigenvalue distribution, determinant.**

I. Introduction

THE Schur complement has been proved to be a useful tool in many fields such as control theory, statistics and computational mathematics, and many works have been done on it (see [1], [2], [3], [4], [5], [6]). Applying the Schur-based iteration method mentioned in [7], [8], we can solve large scale linear systems though reducing the order by the Schur complement. That is, for a non-homogeneous system of linear equation $Mx = b$ with a nonsingular leading principal submatrix. Partition M as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A is supposed to be nonsingular. Partition $x = (x_1^T, x_2^T)^T$ and $b = (b_1^T, b_2^T)^T$ conformably with M. This linear equation can be formally regard as a special case of the saddle point problems [9] The linear system $Mx = b$ is equivalent to the pair of linear systems

$$\begin{cases} Ax_1 + Bx_2 = b_1, \\ Cx_1 + Dx_2 = b_2. \end{cases}$$

If we multiply the first equation by $-CA^{-1}$ and add it to the second equation, the vector variable x_1 is eliminated and we obtain a linear system of smaller size

$$(D - CA^{-1}B)x_1 = b_2 - CA^{-1}b_1.$$

If the coefficient matrix $D - CA^{-1}B$ is a block diagonally dominant matrix or a block H-matrix, we can use some block or preconditioned iterative methods [10], [11] to continue resolving the linear system equation (1). In the meanwhile, when we solve linear equation system, the convergence rate of many iterate algorithms are closely related with spectral radius of coefficient matrix. Hu [12] obtained the following result which can be used to estimate the convergence rate:

Let $M = (M_{ij})_{m\times m}$ be a block strictly diagonally dominant matrix and $N = (N_{ij})_{m\times m}$ partitioned conformably with M. Then

$$\rho(M^{-1}N) \le \max_i \frac{\sum\limits_{j=1}^{m} \|N_{ij}\|}{\|M_{ii}^{-1}\|^{-1} - \sum\limits_{j\neq i} \|M_{ij}\|}.$$

Therefore, we know the estimate of block matrix's spectral is closely related with the block diagonally dominant degree $\|M_{ii}^{-1}\|^{-1} - \sum\limits_{j\neq i} \|M_{ij}\|$ of each row when M is a block strictly diagonally dominant matrix. Thus, after being reduced order, it is significant to study the block diagonally dominant degree of the coefficient matrix of the linear equation system (1). Additionally, as mentioned in [13], we see that the eigenvalues of Schur complement of diagonally dominant matrix are more concentrated than those of original matrix, and we predict that the Schur-based conjugate gradient method will compute faster than the ordinary conjugate gradient method. Hence, it is very important to estimate the eigenvalue distributions of (block) diagonally dominant matrix. Over the years, there has been a surge of interest in studying the locations of eigenvalues of the Schur complement of matrices in much literature, see [6], [7], [8], [13], [14], [15], [16], [17], [18], [19], [20], [21]. Moreover, the determinant of matrices has hitherto great influence on every branch of mathematics [22], [23], [24], [25], [26]. Zhang and Liu [17] proposed some upper and lower bounds for determinants of diagonally dominant matrices by making use of the results of the estimates of diagonally dominant degree for the Schur complement of the diagonally dominant matrices. On the other hand, the authors in [27], [28], [29], [30] extended the concept of diagonally dominant matrix and developed two kinds of block diagonally dominant matrices, which are referred to as the I-block [27] and II-block [31] diagonally dominant

Manuscript received September 01, 2016; revised December 09, 2016. This work was supported by the National Natural Science Foundations of China (No. 11171273) and Innovation Foundation for Doctor Dissertation of Northwestern Polytechnical University (No. CX201628).

*Ligong Wang, Corresponding Author, is with the Department of Applied Mathematics, School of Science, Northwestern Polytechnical University, Xi'an, Shaanxi, 710072, PR China. e-mail: lgwang@nwpu.edu.cn.

Zhengge Huang (e-mail: ZhenggeHuang@mail.nwpu.edu.cn), Zhong Xu (e-mail: zhongxu@nwpu.edu.cn) and Jingjing Cui (e-mail: JingjingCui@mail.nwpu.edu.cn) are with the Department of Applied Mathematics, School of Science, Northwestern Polytechnical University, Xi'an, Shaanxi, 710072, PR China.

matrices, respectively. Later, two kinds of generalized block strictly diagonally dominant matrices (I-block [32](II-block [31]) H-matrices) are established in [31], [32], [33]. In the sequel, Liu et al. [13] derived some estimations of diagonally dominant degree and eigenvalue inclusion sets for the Schur complement of I(II)-block diagonally dominant matrices, and Wang [20], [21] put forward the new estimations of diagonally dominant degree and eigenvalue inclusion sets which are proved to be tighter than those of [13]. Zhu [34] obtained some upper and lower bounds for determinants of I(II)-block diagonally dominant matrices, and Xu [35] arrived at some determinants bounds are sharper than the ones obtained by Zhu. In the current work, we first focus on investigating the following three aspects:

- Study the new estimates of I(II)-block diagonally dominant degree for Schur complement of matrices.
- Derive the new distributions for the eigenvalues of the Schur complement of matrices.
- Develop the new upper and lower bounds for determinants of the I(II)-block diagonally dominant matrices.

Afterward, we prove that the proposed results are superior to some known ones in theory. The numerical results are implemented to verify the theoretical results. Before presenting the our main results of this paper, we give some definitions which are used throughout this paper as follows.

Let $\mathbb{C}^{n\times n}$ denote the set of all $n\times n$ complex matrices, $N=\{1,2,\cdots,n\}$ and $A=(a_{ij})\in\mathbb{C}^{n\times n}(n\geq 2)$. Denote

$$\tau_i(A)=\sum_{j\neq i}|a_{ij}|,\quad i\in N.$$

$A=(a_{ij})\in\mathbb{C}^{n\times n}$ is a strictly diagonally dominant matrix (abbreviated to SD_n) if $|a_{ii}|>\tau_i(A)$, for $i\in N$.

The comparison matrix of A, denoted by $\mu(A)=(t_{ij})_{n\times n}$, is defined to be

$$t_{ij}=\begin{cases}|a_{ij}|, & \text{if } i=j,\\ -|a_{ij}|, & \text{if } i\neq j.\end{cases}$$

A matrix A is called an M-matrix if there exist a nonnegative matrix B and a real number $s>\rho(B)$ such that $A=sI-B$, where $\rho(B)$ is the spectral radius of B. It is well known that A is an H-matrix if and only if $\mu(A)$ is an M-matrix, then the Schur complement of A is also an M-matrix and $\det A>0$ (see [14]).

For $\alpha\subseteq N$, denote by $|\alpha|$ the cardinality of α and $\alpha'=N-\alpha$. If $\alpha,\beta\subseteq N$, then $A(\alpha,\beta)$ is the submatrix of A lying in the rows indicated by α and the columns indicated by β. In particular, $A(\alpha,\alpha)$ is abbreviated to $A(\alpha)$. Assume that $A(\alpha)$ is nonsingular. Then

$$A/\alpha=A/A(\alpha)=A(\alpha')-A(\alpha',\alpha)[A(\alpha)]^{-1}A(\alpha,\alpha'),$$

is called the Schur complement of A respect to $A(\alpha)$.

Let $A\in\mathbb{C}^{n\times n}$ be partitioned as the following form:

$$A=\begin{pmatrix} A(\alpha_1,\alpha_1) & A(\alpha_1,\alpha_2) & \cdots & A(\alpha_1,\alpha_s)\\ A(\alpha_2,\alpha_1) & A(\alpha_2,\alpha_2) & \cdots & A(\alpha_2,\alpha_s)\\ \vdots & \vdots & \ddots & \vdots\\ A(\alpha_s,\alpha_1) & A(\alpha_s,\alpha_2) & \cdots & A(\alpha_s,\alpha_s)\end{pmatrix}, \quad (1)$$

where $1\leq s\leq n$, $\alpha_0=0$,

$$\alpha_i=\left\{\sum_{t=0}^{i-1}|\alpha_t|+1,\cdots,\sum_{t=0}^{i}|\alpha_t|\right\}(1\leq i\leq s),\ \sum_{t=0}^{i}|\alpha_t|=n$$

and $A(\alpha_t,\alpha_t)$ is a $|\alpha_t|\times|\alpha_t|$ nonsingular principal submatrix of A, $t=1,2,\cdots,s$.

Without loss of generality, we assume that $\mathbb{C}_s^{n\times n}$ denote the set of all $s\times s$ block matrices in $\mathbb{C}^{n\times n}$ partitioned as (1), $A=(A(\alpha_l,\alpha_m))_s^{n\times n}\in\mathbb{C}_s^{n\times n}$ and $N(A)=(\|A(\alpha_l,\alpha_m)\|)_s$ denote the norm matrix of block matrix A.

In this paper, the matrix norm $\|.\|$ of $A\in\mathbb{C}^{n\times n}$ is defined as

$$\|A\|=\sup_{x\in\mathbb{C}^m,x\neq 0}\frac{\|Ax\|}{\|x\|}.$$

Thus if $A\in\mathbb{C}^{n\times n}$ is nonsingular, then it holds that

$$\|A^{-1}\|^{-1}=\left\{\sup_{x\in\mathbb{C}^m,x\neq 0}\frac{\|A^{-1}x\|}{\|x\|}\right\}^{-1}=\inf_{x\in\mathbb{C}^m,x\neq 0}\frac{\|Ax\|}{\|x\|}. \quad (2)$$

Definition 1.1 A is called an I-block strictly diagonally dominant matrix ($I-BSD_s$) [27] if for all $1\leq l\leq s$,

$$\|[A(\alpha_l,\alpha_l)]^{-1}\|^{-1}>\sum_{m=1,m\neq l}^{s}\|A(\alpha_l,\alpha_m)\|. \quad (3)$$

Denote by $\|[A(\alpha_l,\alpha_l)]^{-1}\|^{-1}-\sum\limits_{m=1,m\neq l}^{s}\|A(\alpha_l,\alpha_m)\|$ the I-block diagonally dominant degree for $1\leq l\leq s$ of A.

Definition 1.2 A is called an II-block strictly diagonally dominant matrix ($II-BSD_s$) [28] if for all $1\leq l\leq s$,

$$\sum_{m=1,m\neq l}^{s}\|[A(\alpha_l,\alpha_l)]^{-1}A(\alpha_l,\alpha_m)\|<1. \quad (4)$$

$1-\sum\limits_{m=1,m\neq l}^{s}\|[A(\alpha_l,\alpha_l)]^{-1}A(\alpha_l,\alpha_m)\|$ represents the II-block diagonally dominant degree for $1\leq l\leq s$ of A. It is noteworthy that if $A\in I-BSD_s$, then it follows from (3), (4) and the inequality

$$\|A(\alpha_l,\alpha_l)A(\alpha_l,\alpha_m)\|\leq\|A(\alpha_l,\alpha_l)\|\|A(\alpha_l,\alpha_m)\|$$

that $A\in II-BSD_s$.

Definition 1.3 A is called an I-block H-matrix and II-block H-matrix, respectively, if the comparison matrices of block matrix A which are defined by $\mu_I(A)=(\omega_{l,m})\in\mathbb{R}^{s\times s}$ and $\mu_{II}(A)=(\bar{\omega}_{l,m})\in\mathbb{R}^{s\times s}$ are M-matrix, where

$$\omega_{l,m}=\begin{cases}\|[A(\alpha_l,\alpha_l)]^{-1}\|^{-1}, & \text{if } l=m,\\ -\|A(\alpha_l,\alpha_m)\|, & \text{if } l\neq m,\end{cases}$$

$$\bar{\omega}_{l,m}=\begin{cases}1, & \text{if } l=m,\\ -\|[A(\alpha_l,\alpha_l)]^{-1}A(\alpha_l,\alpha_m)\|, & \text{if } l\neq m.\end{cases}$$

The remainder of this paper is organized as follows. In Section II, we recollect some useful lemmas which are utilized in the next sections. Several new estimates for the I(II)-block diagonally dominant degree of the Schur complement of matrices are established in Section III. As applications, some new distribution theorems for eigenvalues of the Schur complement and the new bounds for the determinant of I(II)-block diagonally dominant matrices are obtained in Section IV and Section V, respectively. Section VI is devoted to performing some numerical experiments to confirm the advantages and the validity of the established results. Finally, the paper is ended with some conclusions in Section VII.

II. Preliminaries

In this section, we start with some lemmas. They will be useful in the following proofs.

Lemma 2.1 [13] If $A \in SD_n$, then $\mu(A)$ is M-matrix, i.e., A is H-matrix.

Lemma 2.2 [2] If A is a H-matrix, then $[\mu(A)]^{-1} \geq |A^{-1}|$.

Lemma 2.3 [30] If $A \in I - BSD_s$, then $[\mu_I(A)]^{-1} \geq N(A^{-1})$.

Lemma 2.4 [30] If $A \in II - BSD_s$, then $[\mu_{II}(A)]^{-1} \geq N(A^{-1}D)$, where $D = \mathrm{diag}(A(\alpha_1, \alpha_1), A(\alpha_2, \alpha_2), \cdots, A(\alpha_s, \alpha_s))$.

Lemma 2.5 Let $A \in \mathbb{C}_s^{n\times n}$, $\alpha = \bigcup\limits_{u=1}^{k} \alpha_{i_u} \subset N$, $\alpha' = N - \alpha = \bigcup\limits_{v=1}^{l} \alpha_{j_v} \subset N$, and $k + l = s$. For any $\alpha_{j_t} \subset \alpha'$, we denote:

$$B_{j_t} = \begin{pmatrix} x & -G_t \\ -H^T & \tilde{\mu}[A(\alpha)] \end{pmatrix}.$$

If $A \in I - BSD_s$, we take $\tilde{\mu}[A(\alpha)] = \mu_I[A(\alpha)]$,

$$G_t = \{\|A(\alpha_{j_t}, \alpha_{i_1})\|, \cdots, \|A(\alpha_{j_t}, \alpha_{i_k})\|\},$$
$$H = \left\{\sum_{u=1}^{l} \|A(\alpha_{i_1}, \alpha_{j_u})\|, \cdots, \sum_{u=1}^{l} \|A(\alpha_{i_k}, \alpha_{j_u})\|\right\}.$$

If

$$x \geq h \sum_{v=1}^{k} \|A(\alpha_{j_t}, \alpha_{i_v})\| \frac{P_{i_v}(A)}{\|[A(\alpha_{i_v}, \alpha_{i_v})]^{-1}\|^{-1}}, \tag{5}$$

where

$$r = \max_{1\leq w\leq k} \frac{\sum\limits_{v=1}^{l} \|A(\alpha_{i_w}, \alpha_{j_v})\|}{\|[A(\alpha_{i_w}, \alpha_{i_w})]^{-1}\|^{-1} - \sum\limits_{t=1, t\neq w}^{k} \|A(\alpha_{i_w}, \alpha_{i_t})\|},$$

$$P_{i_w}(A) = r \sum_{t=1, t\neq w}^{k} \|A(\alpha_{i_w}, \alpha_{i_t})\| + \sum_{v=1}^{l} \|A(\alpha_{i_w}, \alpha_{j_v})\|,$$

$$h = \max_{1\leq w\leq k} \frac{\sum\limits_{v=1}^{l} \|A(\alpha_{i_w}, \alpha_{j_v})\|}{H_i},$$

$$H_i = P_{i_w}(A) - \sum_{t=1, t\neq w}^{k} \|A(\alpha_{i_w}, \alpha_{i_t})\| \frac{P_{i_t}(A)}{\|[A(\alpha_{i_t}, \alpha_{i_t})]^{-1}\|^{-1}},$$

then $\det B_{j_t} > 0$. If $A \in II - BSD_s$, we take $\tilde{\mu}[A(\alpha)] = \mu_{II}[A(\alpha)]$,

$$\begin{aligned} G_t &= \{\|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1} A(\alpha_{j_t}, \alpha_{i_1})\|, \\ &\quad \cdots, \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1} A(\alpha_{j_t}, \alpha_{i_k})\|\}, \\ H &= \left\{\sum_{u=1}^{l} \|[A(\alpha_{i_1}, \alpha_{i_1})]^{-1} A(\alpha_{i_1}, \alpha_{j_u})\|, \right. \\ &\quad \left. \cdots, \sum_{u=1}^{l} \|[A(\alpha_{i_k}, \alpha_{i_k})]^{-1} A(\alpha_{i_k}, \alpha_{j_u})\| \right\}. \end{aligned}$$

If

$$x \geq f \sum_{v=1}^{k} \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1} A(\alpha_{j_t}, \alpha_{i_v})\| \tilde{P}_{i_v}(A), \tag{6}$$

where

$$\eta = \max_{1\leq w\leq k} \frac{\sum\limits_{v=1}^{l} \|[A(\alpha_{i_w}, \alpha_{i_w})]^{-1} A(\alpha_{i_w}, \alpha_{j_v})\|}{1 - \sum\limits_{t=1, t\neq w}^{k} \|[A(\alpha_{i_w}, \alpha_{i_w})]^{-1} A(\alpha_{i_w}, \alpha_{i_t})\|},$$

$$\tilde{P}_{i_w}(A) = \eta \sum_{t=1, t\neq w}^{k} \|[A(\alpha_{i_w}, \alpha_{i_w})]^{-1} A(\alpha_{i_w}, \alpha_{i_t})\|$$
$$+ \sum_{v=1}^{l} \|[A(\alpha_{i_w}, \alpha_{i_w})]^{-1} A(\alpha_{i_w}, \alpha_{j_v})\|,$$

$$f = \max_{1\leq w\leq k} \frac{\sum\limits_{v=1}^{l} \|[A(\alpha_{i_w}, \alpha_{i_w})]^{-1} A(\alpha_{i_w}, \alpha_{j_v})\|}{G_i},$$

$$G_i = \tilde{P}_{i_w}(A) - \sum_{t=1, t\neq w}^{k} \|[A(\alpha_{i_w}, \alpha_{i_w})]^{-1} A(\alpha_{i_w}, \alpha_{i_t})\| \tilde{P}_{i_t}(A),$$

then $\det B_{j_t} > 0$.

Proof. If strict inequality in (5) holds, we take $\varepsilon > 0$, sufficiently small such that

$$x > \sum_{v=1}^{k} \|A(\alpha_{j_t}, \alpha_{i_v})\| \left(h \frac{P_{i_v}(A)}{\|[A(\alpha_{i_v}, \alpha_{i_v})]^{-1}\|^{-1}} + \varepsilon\right).$$

We construct a positive diagonal matrix $D = \mathrm{diag}(d_1, d_2, \cdots, d_{k+1})$, where

$$d_v = \begin{cases} 1, & v = 1, \\ h \frac{P_{i_{v-1}}(A)}{\|[A(\alpha_{i_{v-1}}, \alpha_{i_{v-1}})]^{-1}\|^{-1}} + \varepsilon, & 2 \leq v \leq k+1. \end{cases}$$

Denote $C_t = B_{j_t} D = (c_{sv})_{(k+1)\times(k+1)}$. If $s = 1$, then

$$|c_{ss}| - \sum_{v=1, v\neq s}^{k+1} |c_{sv}| = |c_{11}| - \sum_{v=2}^{k+1} |c_{1v}|$$
$$= x - \sum_{v=1}^{k} \|A(\alpha_{j_t}, \alpha_{i_v})\| \left(h \frac{P_{i_v}(A)}{\|[A(\alpha_{i_v}, \alpha_{i_v})]^{-1}\|^{-1}} + \varepsilon\right) > 0;$$

If $s = 2, 3, \cdots, k+1$, then it has

$$\begin{aligned} &|c_{ss}| - \sum_{v=1, v\neq s}^{k+1} |c_{sv}| \\ &= \|[A(\alpha_{i_{s-1}}, \alpha_{i_{s-1}})]^{-1}\|^{-1} (h \frac{P_{i_{s-1}}(A)}{\|[A(\alpha_{i_{s-1}}, \alpha_{i_{s-1}})]^{-1}\|^{-1}} + \varepsilon) \\ &\quad - \sum_{u=1}^{l} \|A(\alpha_{i_{s-1}}, \alpha_{j_u})\| \\ &\quad - \sum_{w=1, w\neq s-1}^{k} \|A(\alpha_{i_{s-1}}, \alpha_{i_w})\| (h \frac{P_{i_w}(A)}{\|[A(\alpha_{i_w}, \alpha_{i_w})]^{-1}\|^{-1}} + \varepsilon) \\ &= hP_{i_{s-1}}(A) + \varepsilon \|[A(\alpha_{i_{s-1}}, \alpha_{i_{s-1}})]^{-1}\|^{-1} - \sum_{u=1}^{l} \|A(\alpha_{i_{s-1}}, \alpha_{j_u})\| \\ &\quad - \sum_{w=1, w\neq s-1}^{k} \|A(\alpha_{i_{s-1}}, \alpha_{i_w})\| (h \frac{P_{i_w}(A)}{\|[A(\alpha_{i_w}, \alpha_{i_w})]^{-1}\|^{-1}} + \varepsilon). \end{aligned} \tag{7}$$

Since $A \in I - BSD_s$, it holds that $0 \leq r < 1$. Moreover, for $1 \leq u \leq k$, we have

$$r \geq \frac{\sum\limits_{v=1}^{l} \|A(\alpha_{i_u}, \alpha_{j_v})\|}{\|[A(\alpha_{i_u}, \alpha_{i_u})]^{-1}\|^{-1} - \sum\limits_{t=1, t\neq u}^{k} \|A(\alpha_{i_u}, \alpha_{i_t})\|},$$

i.e.,

$$r\|[A(\alpha_{i_u},\alpha_{i_u})]^{-1}\|^{-1} \geq \sum_{v=1}^{l}\|A(\alpha_{i_u},\alpha_{j_v})\| + r\sum_{t=1,t\neq u}^{k}\|A(\alpha_{i_u},\alpha_{i_t})\| = P_{i_u}(A).$$

From the above inequality, for $1 \leq u \leq k$, we obtain

$$0 \leq \frac{P_{i_u}(A)}{\|[A(\alpha_{i_u},\alpha_{i_u})]^{-1}\|^{-1}} \leq r < 1.$$

By the definition of $P_{i_w}(A)$, for $1 \leq w \leq k$, we have

$$\frac{\sum\limits_{v=1}^{l}\|A(\alpha_{i_w},\alpha_{j_v})\|}{P_{i_w}(A) - \sum\limits_{t=1,t\neq w}^{k}\|A(\alpha_{i_w},\alpha_{i_t})\|\frac{P_{i_t}(A)}{\|[A(\alpha_{i_t},\alpha_{i_t})]^{-1}\|^{-1}}} = \frac{P_{i_w}(A) - r\sum\limits_{t=1,t\neq w}^{k}\|A(\alpha_{i_w},\alpha_{i_t})\|}{P_{i_w}(A) - \sum\limits_{t=1,t\neq w}^{k}\|A(\alpha_{i_w},\alpha_{i_t})\|\frac{P_{i_t}(A)}{\|[A(\alpha_{i_t},\alpha_{i_t})]^{-1}\|^{-1}}} \leq 1,$$

which leads to $0 \leq h \leq 1$. Furthermore, for $1 \leq u \leq k$,

$$h \geq \frac{\sum\limits_{v=1}^{l}\|A(\alpha_{i_u},\alpha_{j_v})\|}{P_{i_u}(A) - \sum\limits_{t=1,t\neq u}^{k}\|A(\alpha_{i_u},\alpha_{i_t})\|\frac{P_{i_t}(A)}{\|[A(\alpha_{i_t},\alpha_{i_t})]^{-1}\|^{-1}}},$$

which can be rewritten as

$$hP_{i_u}(A) \geq \sum_{v=1}^{l}\|A(\alpha_{i_u},\alpha_{j_v})\| + h\sum_{t=1,t\neq u}^{k}\|A(\alpha_{i_u},\alpha_{i_t})\|\frac{P_{i_t}(A)}{\|[A(\alpha_{i_t},\alpha_{i_t})]^{-1}\|^{-1}}.$$

Thus, it follows from Equality (7) that for $s = 2, 3, \cdots, k+1$,

$$\begin{aligned}
&|c_{ss}| - \sum_{v=1,v\neq s}^{k+1}|c_{sv}| \\
&= hP_{i_{s-1}}(A) + \varepsilon\|[A(\alpha_{i_{s-1}},\alpha_{i_{s-1}})]^{-1}\|^{-1} - \sum_{u=1}^{l}\|A(\alpha_{i_{s-1}},\alpha_{j_u})\| \\
&\quad - \sum_{w=1,w\neq s-1}^{k}\|A(\alpha_{i_{s-1}},\alpha_{i_w})\|(h\frac{P_{i_w}(A)}{\|[A(\alpha_{i_w},\alpha_{i_w})]^{-1}\|^{-1}} + \varepsilon) \\
&\geq \sum_{u=1}^{l}\|A(\alpha_{i_{s-1}},\alpha_{j_u})\| + h\sum_{w=1,w\neq s-1}^{k}\|A(\alpha_{i_{s-1}},\alpha_{i_w})\| \\
&\quad \times \frac{P_{i_w}(A)}{\|[A(\alpha_{i_w},\alpha_{i_w})]^{-1}\|^{-1}} + \varepsilon\|[A(\alpha_{i_{s-1}},\alpha_{i_{s-1}})]^{-1}\|^{-1} \\
&\quad - \sum_{u=1}^{l}\|A(\alpha_{i_{s-1}},\alpha_{j_u})\| \\
&\quad - \sum_{w=1,w\neq s-1}^{k}\|A(\alpha_{i_{s-1}},\alpha_{i_w})\|(h\frac{P_{i_w}(A)}{\|[A(\alpha_{i_w},\alpha_{i_w})]^{-1}\|^{-1}} + \varepsilon) \\
&= \varepsilon\Big(\|[A(\alpha_{i_{s-1}},\alpha_{i_{s-1}})]^{-1}\|^{-1} - \sum_{w=1,w\neq s-1}^{k}\|A(\alpha_{i_{s-1}},\alpha_{i_w})\|\Big) > 0,
\end{aligned}$$

which means that C_t is a SD_{k+1}. By Lemma 2.1, $\mu(B_{j_t})$ is a M-matrix. Note that $\mu(B_{j_t}) = B_{j_t}$, then $\det B_{j_t} > 0$.

When the equality holds in (5), for any $\varepsilon > 0$, denote $B_\varepsilon = B + \mathrm{diag}(\varepsilon, 0, \cdots, 0)$. In a similar way to the above proof, we have $B_\varepsilon \in SD_{k+1}$ and hence $\det B_{j_t} > 0$. Let $\varepsilon \to 0^+$, we get $\det B_{j_t} \geq 0$ immediately.

For the case of $A \in II - BSD_s$, the proof is similar. ■

Lemma 2.6 [30] Let $A \in I-(II-)BSD_s$, $\alpha = \bigcup\limits_{u=1}^{k}\alpha_{i_u} \subset N$, $\alpha' = N - \alpha = \bigcup\limits_{v=1}^{l}\alpha_{j_v}$, and $k + l = s$. For any $t = 1, 2, \cdots, l$,,

$$\Psi_t = 1 - \left\|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}[A(\alpha_{j_t},\alpha_{i_1}), \cdots, A(\alpha_{j_t},\alpha_{i_k})][A(\alpha)]^{-1}\begin{pmatrix} A(\alpha_{i_1},\alpha_{j_t}) \\ \vdots \\ A(\alpha_{i_k},\alpha_{j_t})\end{pmatrix}\right\| > 0.$$

Lemma 2.7 [2] Let $A \in \mathbb{C}^{n\times n}$. If $\|A\| < 1$, then $I_n - A$ is nonsingular and

$$\|(I_n - A)^{-1}\| \leq \frac{1}{1 - \|A\|}.$$

where I_n is an identity matrix.

III. The diagonally dominant degree for Schur complement

In this section, we present several new estimates on the block diagonally dominant degree of the Schur complement of $I-(II-)BSD_s$, which improve the corresponding ones in [13], [20], [21], [35].

Theorem 3.1 Let $A \in I - BSD_s$, $\alpha = \bigcup\limits_{u=1}^{k}\alpha_{i_u} \subset N$, $\alpha' = N - \alpha = \bigcup\limits_{v=1}^{l}\alpha_{j_v}$, and $k + l = s$. Denote $A/\alpha = (\tilde{A}(\alpha_t,\alpha_r))$. Then

$$\begin{aligned}
&\|[\tilde{A}(\alpha_t,\alpha_t)]^{-1}\|^{-1} - R_t(A/\alpha) \\
&\geq \|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}\|^{-1} - R_{j_t}(A) + w_{j_t} \\
&\geq \|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}\|^{-1} - R_{j_t}(A) > 0 \qquad (8)
\end{aligned}$$

and

$$\begin{aligned}
&\|[\tilde{A}(\alpha_t,\alpha_t)]^{-1}\|^{-1} + R_t(A/\alpha) \\
&\leq \|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}\|^{-1} + R_{j_t}(A) - w_{j_t} \\
&\leq \|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}\|^{-1} + R_{j_t}(A), \qquad (9)
\end{aligned}$$

where

$$R_{j_t} = \sum_{m=1,m\neq j_t}^{s}\|A(\alpha_{j_t},\alpha_m)\|,$$

$$w_{j_t} = \sum_{v=1}^{k}\|A(\alpha_{j_t},\alpha_{i_v})\|\frac{\|[A(\alpha_{i_v},\alpha_{i_v})]^{-1}\|^{-1} - hP_{i_v}(A)}{\|[A(\alpha_{i_v},\alpha_{i_v})]^{-1}\|^{-1}},$$

and h and $P_{i_v}(A)$ $(v = 1, 2, \cdots, k)$ are defined as in Lemma 2.5.

Proof. Let

$$\Psi_{tr} = (A(\alpha_{j_t},\alpha_{i_1}), \cdots, A(\alpha_{j_t},\alpha_{i_k}))[A(\alpha)]^{-1}\begin{pmatrix} A(\alpha_{i_1},\alpha_{j_r}) \\ \vdots \\ A(\alpha_{i_k},\alpha_{j_r})\end{pmatrix},$$

$$G_t = (\|A(\alpha_{j_t},\alpha_{i_1})\|, \cdots, \|A(\alpha_{j_t},\alpha_{i_k})\|)^T,$$

$$H' = \left(\sum_{r=1}^{l}\|A(\alpha_{i_1},\alpha_{j_r})\|, \cdots, \sum_{r=1}^{l}\|A(\alpha_{i_k},\alpha_{j_r})\|\right)^T,$$

$t, r = 1, 2, \cdots, l.$

By the definition of Schur complement, denote by $J_t = |\alpha_{j_t}|$ and I_m the identity matrix. According to Lemma 2.5, we obtain $\|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\Psi_{tt}\| < 1$. It follows that

$$\begin{aligned}
&\|[\tilde{A}(\alpha_t, \alpha_t)]^{-1}\|^{-1} - \sum_{r=1, r\neq t}^{l} \|\tilde{A}(\alpha_t, \alpha_r)\| \\
=\ & \|\{A(\alpha_{j_t}, \alpha_{j_t}) - \Psi_{tt}\}^{-1}\|^{-1} - \sum_{r=1, r\neq t}^{l} \|A(\alpha_{j_t}, \alpha_{j_r}) - \Psi_{tr}\| \\
\geq\ & \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} \|\{I_{j_t} - [A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\Psi_{tt}\}^{-1}\|^{-1} \\
&- \sum_{r=1, r\neq t}^{l} \|A(\alpha_{j_t}, \alpha_{j_r}) - \Psi_{tr}\| \\
\geq\ & \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} \Big[1 - \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\Psi_{tt}\|\Big] \\
&- \sum_{r=1, r\neq t}^{l} \|A(\alpha_{j_t}, \alpha_{j_r})\| - \sum_{r=1, r\neq t}^{l} \|\Psi_{tr}\| \\
=\ & \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} - \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\Psi_{tt}\| \\
&- \sum_{r=1, r\neq t}^{l} \|A(\alpha_{j_t}, \alpha_{j_r})\| - \sum_{r=1, r\neq t}^{l} \|\Psi_{tr}\| \\
\geq\ & \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} - \|\Psi_{tt}\| \\
&- \sum_{r=1, r\neq t}^{l} \|A(\alpha_{j_t}, \alpha_{j_r})\| - \sum_{r=1, r\neq t}^{l} \|\Psi_{tr}\| \\
=\ & \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} - \sum_{r=1, r\neq t}^{l} \|A(\alpha_{j_t}, \alpha_{j_r})\| - \sum_{r=1}^{l} \|\Psi_{tr}\| \\
\geq\ & \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} - \sum_{r=1, r\neq t}^{l} \|A(\alpha_{j_t}, \alpha_{j_r})\| - G_t^T N[(A(\alpha))^{-1}]H' \\
\geq\ & \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} - \sum_{r=1, r\neq t}^{l} \|A(\alpha_{j_t}, \alpha_{j_r})\| \\
&- G_t^T [\mu_I(A)(\alpha)]^{-1} H' (\text{by Lemma 2.2}) \\
=\ & \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} - R_{j_t}(A) + w_{j_t} + \sum_{r=1}^{k} \|A(\alpha_{j_t}, \alpha_{i_r})\| \\
&- w_{j_t} - G_t^T [\mu_I(A)(\alpha)]^{-1} H' \\
=\ & \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} - R_{j_t}(A) + w_{j_t} - \varepsilon \\
&+ \frac{1}{\det[\mu_I(A)(\alpha)]} \det \begin{pmatrix} \sum_{r=1}^{k} \|A(\alpha_{j_t}, \alpha_{i_r})\| - w_{j_t} + \varepsilon & -G_t^T \\ -H' & \mu_I(A)(\alpha) \end{pmatrix} \\
=\ & \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} - R_{j_t}(A) + w_{j_t} - \varepsilon + \frac{\det B_1}{\det[\mu_I(A)(\alpha)]}. \qquad (10)
\end{aligned}$$

Inasmuch as $A \in I - BSD_s$, we have

$$\begin{aligned}
&\sum_{r=1}^{k} \|A(\alpha_{j_t}, \alpha_{i_r})\| - w_{j_t} + \varepsilon \\
&= h \sum_{r=1}^{k} \|A(\alpha_{j_t}, \alpha_{i_r})\| \frac{P_{i_r}(A)}{\|[A(\alpha_{i_r}, \alpha_{i_r})]^{-1}\|^{-1}} + \varepsilon \\
&> h \sum_{r=1}^{k} \|A(\alpha_{j_t}, \alpha_{i_r})\| \frac{P_{i_r}(A)}{\|[A(\alpha_{i_r}, \alpha_{i_r})]^{-1}\|^{-1}}.
\end{aligned}$$

It follows that $\det B_1 > 0$ by virtue of Lemma 2.5. By Lemma 2.1, we infer that $\mu_I(A)(\alpha)$ is *M*-matrix, and therefore $\det[\mu_I(A)(\alpha)] > 0$, which implies that

$$\begin{aligned}
&\|[\tilde{A}(\alpha_t, \alpha_t)]^{-1}\|^{-1} - R_t(A/\alpha) \\
&> \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} - R_{j_t}(A) + w_{j_t} - \varepsilon \\
&\geq \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} - R_{j_t}(A) - \varepsilon.
\end{aligned}$$

Let $\varepsilon \to 0$, thus we easily get

$$\begin{aligned}
&\|[\tilde{A}(\alpha_t, \alpha_t)]^{-1}\|^{-1} - R_t(A/\alpha) \\
&\geq \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} - R_{j_t}(A) + w_{j_t} \\
&\geq \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} - R_{j_t}(A) > 0,
\end{aligned}$$

which implies Inequality (8).

By making use of (2) and applying the same manner in the above proof, it has

$$\begin{aligned}
&\|[\tilde{A}(\alpha_t, \alpha_t)]^{-1}\|^{-1} + \sum_{r=1, r\neq t}^{l} \|\tilde{A}(\alpha_t, \alpha_r)\| \\
=\ & \|\{A(\alpha_{j_t}, \alpha_{j_t}) - \Psi_{tt}\}^{-1}\|^{-1} + \sum_{r=1, r\neq t}^{l} \|A(\alpha_{j_t}, \alpha_{j_r}) - \Psi_{tr}\| \\
=\ & \inf_{x\in\mathbb{C}^m, x\neq 0} \frac{\|\{A(\alpha_{j_t}, \alpha_{j_t}) - \Psi_{tt}\}x\|}{\|x\|} \\
&+ \sum_{r=1, r\neq t}^{l} \|A(\alpha_{j_t}, \alpha_{j_r}) - \Psi_{tr}\| \text{ (by (4))} \\
\leq\ & \inf_{x\in\mathbb{C}^m, x\neq 0} \frac{\|A(\alpha_{j_t}, \alpha_{j_t})x\| + \|\Psi_{tt}x\|}{\|x\|} \\
&+ \sum_{r=1, r\neq t}^{l} \|A(\alpha_{j_t}, \alpha_{j_r})\| + \sum_{r=1, r\neq t}^{l} \|\Psi_{tr}\| \\
\leq\ & \inf_{x\in\mathbb{C}^m, x\neq 0} \frac{\|A(\alpha_{j_t}, \alpha_{j_t})x\| + \|\Psi_{tt}\|\|x\|}{\|x\|} \\
&+ \sum_{r=1, r\neq t}^{l} \|A(\alpha_{j_t}, \alpha_{j_r})\| + \sum_{r=1, r\neq t}^{l} \|\Psi_{tr}\| \\
=\ & \inf_{x\in\mathbb{C}^m, x\neq 0} \frac{\|A(\alpha_{j_t}, \alpha_{j_t})x\|}{\|x\|} \\
&+ \sum_{r=1, r\neq t}^{l} \|A(\alpha_{j_t}, \alpha_{j_r})\| + \sum_{r=1}^{l} \|\Psi_{tr}\| \\
=\ & \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} + \sum_{r=1, r\neq t}^{l} \|A(\alpha_{j_t}, \alpha_{j_r})\| \\
&+ \sum_{r=1}^{l} \|\Psi_{tr}\| \text{ (by (4))} \\
\leq\ & \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} + R_{j_t}(A) - w_{j_t} + \varepsilon \\
&- \frac{1}{\det[\mu_I(A)(\alpha)]} \det B_1 \text{ (by (10))} \\
<\ & \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} + R_{j_t}(A) - w_{j_t} + \varepsilon \\
\leq\ & \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} + R_{j_t}(A) + \varepsilon.
\end{aligned}$$

Let $\varepsilon \to 0$, thus we can get

$$\begin{aligned}
&\|[\tilde{A}(\alpha_t, \alpha_t)]^{-1}\|^{-1} + R_t(A/\alpha) \\
&\leq \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} + R_{j_t}(A) - w_{j_t} \\
&\leq \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} + R_{j_t}(A).
\end{aligned}$$

Therefore, we obtain Inequality (9). This proof is completed. ■

Remark 3.1 Note that

$$\begin{aligned}
&h\frac{P_{i_u}(A)}{\|[A(\alpha_{i_u}, \alpha_{i_u})]^{-1}\|^{-1}} \leq \frac{P_{i_u}(A)}{\|[A(\alpha_{i_u}, \alpha_{i_u})]^{-1}\|^{-1}} \\
&\leq r \leq \max_{1\leq u\leq k} \frac{R_{i_u}(A)}{\|[A(\alpha_{i_u}, \alpha_{i_u})]^{-1}\|^{-1}},\ 1 \leq u \leq k.
\end{aligned}$$

This means that

$$\begin{aligned}
w_{j_t} &= \sum_{v=1}^{k}\|A(\alpha_{j_t},\alpha_{i_v})\|\frac{\|[A(\alpha_{i_v},\alpha_{i_v})]^{-1}\|^{-1}-hP_{i_v}(A)}{\|[A(\alpha_{i_v},\alpha_{i_v})]^{-1}\|^{-1}}\\
&\geq \sum_{v=1}^{k}\|A(\alpha_{j_t},\alpha_{i_v})\|\frac{\|[A(\alpha_{i_v},\alpha_{i_v})]^{-1}\|^{-1}-P_{i_v}(A)}{\|[A(\alpha_{i_v},\alpha_{i_v})]^{-1}\|^{-1}}\\
&\geq (1-r)\sum_{v=1}^{k}\|A(\alpha_{j_t},\alpha_{i_v})\|\\
&= \min_{1\leq u\leq k}\frac{\|[A(\alpha_{i_u},\alpha_{i_u})]^{-1}\|^{-1}-R_{i_u}(A)}{\|[A(\alpha_{i_u},\alpha_{i_u})]^{-1}\|^{-1}-\sum\limits_{t=1,t\neq u}^{k}\|A(\alpha_{i_u},\alpha_{i_t})\|}\\
&\quad\times\sum_{v=1}^{k}\|A(\alpha_{j_t},\alpha_{i_v})\|\\
&\geq \min_{1\leq u\leq k}\frac{\|[A(\alpha_{i_u},\alpha_{i_u})]^{-1}\|^{-1}-R_{i_u}(A)}{\|[A(\alpha_{i_u},\alpha_{i_u})]^{-1}\|^{-1}}\sum_{v=1}^{k}\|A(\alpha_{j_t},\alpha_{i_v})\|.
\end{aligned} \tag{11}$$

From Inequality (11), it's obvious that Theorem 1 improves the results of Theorem 3.1 in [13], Theorem 2.10 in [21] and Theorem 2.1.1 in [35].

Based on Theorem 3.1, the following corollary can be obtained immediately.

Corollary 3.1 Let $A\in I-BSD_s$, and take $\alpha=\bigcup\limits_{u=1}^{s-1}\alpha_u\subset N$. Then

$$\begin{aligned}
&\|(A/\alpha)^{-1}\|^{-1}\geq\|[A(\alpha_s,\alpha_s)]^{-1}\|^{-1}\\
&-h\sum_{v=1}^{s-1}\|A(\alpha_s,\alpha_v)\|\frac{P_v(A)}{\|[A(\alpha_v,\alpha_v)]^{-1}\|^{-1}},\\
&\|A/\alpha\|\leq\|A(\alpha_s,\alpha_s)\|\\
&+h\sum_{v=1}^{s-1}\|A(\alpha_s,\alpha_v)\|\frac{P_v(A)}{\|[A(\alpha_v,\alpha_v)]^{-1}\|^{-1}}.
\end{aligned}$$

Proof. Notice that $\alpha'=\alpha_s$. Thus, $A/\alpha=(\tilde{A}(\alpha_s,\alpha_s))$, and $R_s(A/\alpha)=0$, so by the definition of w_{j_t}, we have

$$\begin{aligned}
&w_{j_t}=w_s\\
&=\sum_{v=1}^{s-1}\|A(\alpha_s,\alpha_v)\|\frac{\|[A(\alpha_v,\alpha_v)]^{-1}\|^{-1}-hP_v(A)}{\|[A(\alpha_v,\alpha_v)]^{-1}\|^{-1}}\\
&=\sum_{v=1}^{s-1}\|A(\alpha_s,\alpha_v)\|-h\sum_{v=1}^{s-1}\|A(\alpha_s,\alpha_v)\|\frac{P_v(A)}{\|[A(\alpha_v,\alpha_v)]^{-1}\|^{-1}}.
\end{aligned} \tag{12}$$

Substituting Equation (12) into Inequality (8) and in a manner similar to that done for Theorem 3.1, the results are derived. ■

Theorem 3.2 Let $A\in II-BSD_s$, $\alpha=\bigcup\limits_{u=1}^{k}\alpha_{i_u}\subset N$, $\alpha'=N-\alpha=\bigcup\limits_{v=1}^{l}\alpha_{j_v}$, and $k+l=s$. Denote $A/\alpha=(\tilde{A}(\alpha_t,\alpha_r))$. Then

$$1-\hat{R}_t(A/\alpha)\geq 1-\hat{R}_{j_t}(A)+\hat{w}_{j_t}\geq 1-\hat{R}_{j_t}(A)>0 \tag{13}$$

and

$$1+\hat{R}_t(A/\alpha)\leq 1+\hat{R}_{j_t}(A)-\hat{w}_{j_t}\leq 1+\hat{R}_{j_t}(A), \tag{14}$$

where

$$\begin{aligned}
\hat{R}_{j_t}(A)&=\sum_{m=1,m\neq j_t}^{s}\|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}A(\alpha_{j_t},\alpha_m)\|,\\
\hat{w}_{j_t}&=\sum_{v=1}^{k}\|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}A(\alpha_{j_t},\alpha_{i_v})\|(1-f\tilde{P}_{i_v}(A)),
\end{aligned}$$

and f and $\tilde{P}_{i_v}(A)$ $(v=1,2,\cdots,k)$ are defined as in Lemma 2.4.

Proof. For $t,r=1,2,\cdots,l$, denote $J_t=|\alpha_{j_t}|$, let

$$\begin{aligned}
&D=\operatorname{diag}(A(\alpha_{i_1},\alpha_{i_1}),\cdots,A(\alpha_{i_k},\alpha_{i_k})),\\
&\Psi_{tr}=(A(\alpha_{j_t},\alpha_{i_1}),\cdots,A(\alpha_{j_t},\alpha_{i_k}))[A(\alpha)]^{-1}\begin{pmatrix}A(\alpha_{i_1},\alpha_{j_r})\\ \vdots\\ A(\alpha_{i_k},\alpha_{j_r})\end{pmatrix},\\
&\Upsilon_t=\{[A(\alpha_{j_t},\alpha_{j_t})]^{-1}A(\alpha_{j_t},\alpha_{i_1}),\cdots,[A(\alpha_{j_t},\alpha_{j_t})]^{-1}A(\alpha_{j_t},\alpha_{i_k})\},\\
&\Gamma_r=\{[A(\alpha_{i_1},\alpha_{i_1})]^{-1}A(\alpha_{i_1},\alpha_{j_r}),\cdots,[A(\alpha_{i_k},\alpha_{i_k})]^{-1}A(\alpha_{i_k},\alpha_{j_r})\}^T,\\
&L_t=\{\|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}A(\alpha_{j_t},\alpha_{i_1})\|,\\
&\cdots,\|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}A(\alpha_{j_t},\alpha_{i_k})\|\}^T,\\
&H'=\Big(\sum_{r=1}^{l}\|[A(\alpha_{i_1},\alpha_{i_1})]^{-1}A(\alpha_{i_1},\alpha_{j_r})\|,\\
&\cdots,\sum_{r=1}^{l}\|[A(\alpha_{i_k},\alpha_{i_k})]^{-1}A(\alpha_{i_k},\alpha_{j_r})\|\Big)^T.
\end{aligned}$$

It follows from the definition of Ψ_t in Lemma 2.6 that

$$\Psi_t=1-\|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}\Psi_{tt}\|=1-\|\Upsilon_t[A(\alpha)]^{-1}D\Gamma_t\|,$$

which is equivalent to

$$\frac{1}{\Psi_t}[1-\|\Upsilon_t[A(\alpha)]^{-1}D\Gamma_t\|]=1. \tag{15}$$

According to lemma 2.7, we obtain

$$\begin{aligned}
&\left\|\{I_{j_t}-[A(\alpha_{j_t},\alpha_{j_t})]^{-1}\Psi_{tt}\}^{-1}\right\|\\
&\leq\frac{1}{1-\|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}\Psi_{tt}\|}=\frac{1}{\Psi_t}.
\end{aligned} \tag{16}$$

By making use of the definition of the Schur complement, we deduce that

$$\begin{aligned}
&1-\hat{R}_t(A/\alpha)=1-\sum_{r=1,r\neq t}^{l}\|[\tilde{A}(\alpha_t,\alpha_t)]^{-1}\tilde{A}(\alpha_t,\alpha_r)\|\\
&=1-\sum_{r=1,r\neq t}^{l}\|[A(\alpha_{j_t},\alpha_{j_t})-\Psi_{tt}]^{-1}[A(\alpha_{j_t},\alpha_{j_r})-\Psi_{tr}]\|\\
&=1-\sum_{r=1,r\neq t}^{l}\|\{I_{j_t}-[A(\alpha_{j_t},\alpha_{j_t})]^{-1}\Psi_{tt}\}^{-1}\\
&\quad\times\{[A(\alpha_{j_t},\alpha_{j_t})]^{-1}A(\alpha_{j_t},\alpha_{j_r})-[A(\alpha_{j_t},\alpha_{j_t})]^{-1}\Psi_{tr}\}\|\\
&\geq 1-\sum_{r=1,r\neq t}^{l}\|\{I_{j_t}-[A(\alpha_{j_t},\alpha_{j_t})]^{-1}\Psi_{tt}\}^{-1}\|\\
&\quad\left\|\{[A(\alpha_{j_t},\alpha_{j_t})]^{-1}A(\alpha_{j_t},\alpha_{j_r})-\Upsilon_t[A(\alpha)]^{-1}D\Gamma_t\}\right\|\\
&\geq 1-\frac{1}{\Psi_t}\sum_{r=1,r\neq t}^{l}\{\|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}A(\alpha_{j_t},\alpha_{j_r})\|\\
&\quad+\|\Upsilon_t\|\|[A(\alpha)]^{-1}D\|\|\Gamma_t\|\}\ \text{(by (16))}\\
&=\frac{1}{\Psi_t}\Big\{1-\|\Upsilon_t[A(\alpha)]^{-1}D\Gamma_t\|
\end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{r=1,r\neq t}^{l} \|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}A(\alpha_{j_t},\alpha_{j_r})\| \\
&\quad \left. - \sum_{r=1,r\neq t}^{l} \|\Upsilon_t\|\|[A(\alpha)]^{-1}D\|\|\Gamma_t\| \right\} \text{(by (15))} \\
&\geq \frac{1}{\Psi_t}\left\{1 - \sum_{r=1,r\neq t}^{l} \|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}A(\alpha_{j_t},\alpha_{j_r})\| \right. \\
&\quad \left. - \sum_{r=1}^{l} \|\Upsilon_t\| N[(A(\alpha))^{-1}D]\|\Gamma_t\| \right\} \\
&\geq 1 - \sum_{r=1,r\neq t}^{l} \|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}A(\alpha_{j_t},\alpha_{j_r})\| \\
&\quad - L_t^T\{\mu_{II}[A(\alpha)]\}^{-1}H' \text{(by Lemma 2.4)} \\
&= 1 - \hat{R}_{j_t}(A) + \hat{w}_{j_t} + \sum_{r=1}^{k} \|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}A(\alpha_{j_t},\alpha_{i_r})\| \\
&\quad - \hat{w}_{j_t} - L_t^T\{\mu_{II}[A(\alpha)]\}^{-1}H' \\
&= 1 - \hat{R}_{j_t}(A) + \hat{w}_{j_t} - \varepsilon + \frac{1}{\det[\mu_{II}(A)(\alpha)]} \\
&\times \det\begin{pmatrix} \sum_{r=1}^{k} \|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}A(\alpha_{j_t},\alpha_{i_r})\| - \hat{w}_{j_t} + \varepsilon & -L_t^T \\ -H' & \mu_{II}[A(\alpha)] \end{pmatrix} \\
&= 1 - \hat{R}_{j_t}(A) + \hat{w}_{j_t} - \varepsilon + \frac{\det B_2}{\det[\mu_{II}(A)(\alpha)]}. \qquad (17)
\end{aligned}
$$

Since $A \in II - BSD_s$, it holds that

$$
\begin{aligned}
&\sum_{r=1}^{k} \|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}A(\alpha_{j_t},\alpha_{i_r})\| - \hat{w}_{j_t} + \varepsilon \\
&= f\sum_{r=1}^{k} \|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}A(\alpha_{j_t},\alpha_{i_r})\|\tilde{P}_{i_r}(A) + \varepsilon \\
&> f\sum_{r=1}^{k} \|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}A(\alpha_{j_t},\alpha_{i_r})\|\tilde{P}_{i_r}(A).
\end{aligned}
$$

By Lemma 2.3, it is easy to see that $\det B_2 > 0$. By Lemma 2.1, we deduce that $\mu_{II}(A)(\alpha)$ is nonsingular *M*-matrix, thus $\det[\mu_{II}(A)(\alpha)] > 0$, which yields that

$$1 - \hat{R}_t(A/\alpha) > 1 - \hat{R}_{j_t}(A) + \hat{w}_{j_t} - \varepsilon \geq 1 - \hat{R}_{j_t}(A) - \varepsilon.$$

Let $\varepsilon \to 0$, thus we can get

$$1 - \hat{R}_t(A/\alpha) \geq 1 - \hat{R}_{j_t}(A) + \hat{w}_{j_t} \geq 1 - \hat{R}_{j_t}(A) > 0,$$

which proves the desired Inequality (13). We can prove Inequality (14) with a quite similar strategy utilized in this theorem. ■

Remark 3.2 Note that

$$f\tilde{P}_{i_u}(A) \leq \tilde{P}_{i_u}(A) \leq \eta \leq \max_{1\leq u\leq k} \hat{R}_{i_u}(A),\ 1 \leq u \leq k,$$

which leads to

$$
\begin{aligned}
\hat{w}_{j_t} &= \sum_{v=1}^{k} \|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}A(\alpha_{j_t},\alpha_{i_v})\|(1 - f\tilde{P}_{i_v}(A)) \\
&\geq \sum_{v=1}^{k} \|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}A(\alpha_{j_t},\alpha_{i_v})\|(1 - \tilde{P}_{i_v}(A)) \\
&= \min_{1\leq u\leq k} \frac{1 - \hat{R}_{i_u}(A)}{1 - \sum_{t=1,t\neq u}^{k} \|[A(\alpha_{i_u},\alpha_{i_u})]^{-1}A(\alpha_{i_u},\alpha_{i_t})\|} \\
&\quad \times \sum_{v=1}^{k} \|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}A(\alpha_{j_t},\alpha_{i_v})\| \\
&\geq \min_{1\leq u\leq k}(1 - \hat{R}_{i_u}(A)) \sum_{v=1}^{k} \|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}A(\alpha_{j_t},\alpha_{i_v})\|.
\end{aligned} \tag{18}
$$

Evidently, from Inequality (18) we see that Theorem 2 improves the results of Theorem 3.2 in [13], Theorem 2.1.2 in [35] and Theorem 2.13 in [21].

IV. Distribution for eigenvalues of the Schur complement of I-(II-)BSD$_s$

In this section, as an application of our results in Section II and Section III, we establish some new locations for the eigenvalues of the Schur complements of $I-(II-)BSD_s$ by the elements of the original matrix. Without loss of generality, we assume that $\alpha = \bigcup_{u=1}^{k} \alpha_{i_u} \subset N$, $\alpha' = N - \alpha = \bigcup_{v=1}^{l} \alpha_{j_v} \subset N$, and $k + l = s$. Let $A/\alpha = (\tilde{A}(\alpha_r,\alpha_r))$, $|\alpha_t| = t$ and I_t be the identity matrix. Denote by $\lambda(A/\alpha)$ and $\lambda(A)$ the set of eigenvalues of A/α and A, respectively.

Lemma 4.1 [13] Let $A \in I - BSD_s$ and $\lambda(A)$ denote the set of eigenvalues of A. Then

$$\lambda(A) \subset G = \bigcup_{i=1}^{s}[G_i \cup \lambda(A(\alpha_i,\alpha_i))],$$

where

$$G_i = \left\{\lambda:\ \lambda \not\subseteq \lambda(A(\alpha_i,\alpha_i)) \text{ and } \|[A(\alpha_i,\alpha_i) - \lambda I(\alpha_i)]^{-1}\|^{-1} \leq \sum_{k\neq i}^{s} \|A(\alpha_i,\alpha_k)\|\right\}.$$

Theorem 4.1 Let $A \in I - BSD_s$ and w_{j_t} be defined as in Theorem 3.1. Then

$$\lambda(A) \subset G = \bigcup_{t=1}^{s}[G_t \cup \lambda(A(\alpha_{j_t},\alpha_{j_t}))],$$

where

$$G_i = \left\{\lambda:\ \lambda \not\subseteq \lambda(A(\alpha_{j_t},\alpha_{j_t})) \text{ and } \|[\lambda I_{j_t} - A(\alpha_{j_t},\alpha_{j_t})]^{-1}\|^{-1} \leq R_{j_t}(A) - w_{j_t}\right\}.$$

Proof. Let Ψ_{tr} be such as in Theorem 3.1. If $\lambda \not\subseteq \lambda[\tilde{A}(\alpha_t,\alpha_t)]$ and $\lambda \not\subseteq \lambda[A(\alpha_{j_t},\alpha_{j_t})]$, then combining Inequality (4) with Lemma 4.1 results in

$$\begin{aligned}
&\|[\lambda I_t-\tilde{A}(\alpha_t,\alpha_t)]^{-1}\|^{-1}\\
&=\left\{\sup_{x\in\mathbb{C}^m,x\neq 0}\frac{\|[\lambda I_t-\tilde{A}(\alpha_t,\alpha_t)]^{-1}x\|}{\|x\|}\right\}^{-1}\\
&=\inf_{x\in\mathbb{C}^m,x\neq 0}\frac{\|[\lambda I_t-\tilde{A}(\alpha_t,\alpha_t)]x\|}{\|x\|}\ \text{(by (4))}\\
&=\inf_{x\in\mathbb{C}^m,x\neq 0}\frac{\|\{\lambda I_{j_t}-[A(\alpha_{j_t},\alpha_{j_t})-\Psi_{tt}]\}x\|}{\|x\|}\\
&\geq\inf_{x\in\mathbb{C}^m,x\neq 0}\frac{\|[\lambda I_{j_t}-[A(\alpha_{j_t},\alpha_{j_t})]x\|}{\|x\|}-\sup_{x\in\mathbb{C}^m,x\neq 0}\frac{\|\Psi_{tt}x\|}{\|x\|}\\
&=\|[\lambda I_{j_t}-[A(\alpha_{j_t},\alpha_{j_t})]^{-1}\|^{-1}-\|\Psi_{tt}\|.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\|[\lambda I_{j_t}-[A(\alpha_{j_t},\alpha_{j_t})]^{-1}\|^{-1}\\
&\leq\|[\lambda I_t-[A(\alpha_t,\alpha_t)]^{-1}\|^{-1}+\|\Psi_{tt}\|\\
&\leq R_t(A/\alpha)+\|\Psi_{tt}\|\\
&=\sum_{r=1,r\neq t}^{l}\|A(\alpha_{j_t},\alpha_{j_r})-\Psi_{tr}\|+\|\Psi_{tt}\|\\
&\leq\sum_{r=1,r\neq t}^{l}\|A(\alpha_{j_t},\alpha_{j_r})\|+\sum_{r=1}^{l}\|\Psi_{tr}\|\\
&\leq R_{j_t}(A)-w_{j_t}-\sum_{r=1}^{k}\|A(\alpha_{j_t},\alpha_{i_r})\|\\
&\quad+w_{j_t}+G_t^T[\mu_I(A)(\alpha)]^{-1}H'\\
&=R_{j_t}(A)-w_{j_t}+\varepsilon-\frac{1}{\det[\mu_I(A)(\alpha)]}\\
&\quad\times\det\begin{pmatrix}\sum_{r=1}^{k}\|A(\alpha_{j_t},\alpha_{i_r})\|-w_{j_t}+\varepsilon & -G_t^T\\ -H' & \mu_I(A)(\alpha)\end{pmatrix}\\
&=R_{j_t}(A)-w_{j_t}+\varepsilon-\frac{\det B_1}{\det[\mu_I(A)(\alpha)]},
\end{aligned}$$

where H', B_1 and $\mu_I(A)(\alpha)$ are defined as in the proof of Theorem 3.1. Thus $\det B_1>0$ and $\det[\mu_I(A)(\alpha)]>0$. So

$$\|[\lambda I_{j_t}-[A(\alpha_{j_t},\alpha_{j_t})]^{-1}\|^{-1}<R_{j_t}(A)-w_{j_t}+\varepsilon,$$

Letting $\varepsilon\to 0$ yields

$$\|[\lambda I_{j_t}-[A(\alpha_{j_t},\alpha_{j_t})]^{-1}\|^{-1}\leq R_{j_t}(A)-w_{j_t}.$$

If $\lambda\subseteq\lambda[\tilde{A}(\alpha_t,\alpha_t)]$ and $\lambda\nsubseteq\lambda[A(\alpha_{j_t},\alpha_{j_t})]$, we assume that $\tilde{x}\neq 0$ is the eigenvector of A corresponding to λ. Then

$$\begin{aligned}
0&=\frac{\|[\lambda I_t-\tilde{A}(\alpha_t,\alpha_t)]x\|}{\|\tilde{x}\|}\\
&\geq\inf_{x\in\mathbb{C}^m,x\neq 0}\frac{\|[\lambda I_t-\tilde{A}(\alpha_t,\alpha_t)]x\|}{\|x\|}\\
&=\inf_{x\in\mathbb{C}^m,x\neq 0}\frac{\|[\lambda I_t-A(\alpha_{j_t},\alpha_{j_t})+\Psi_{tt}]x\|}{\|x\|}\\
&\geq\inf_{x\in\mathbb{C}^m,x\neq 0}\frac{\|[\lambda I_t-A(\alpha_{j_t},\alpha_{j_t})]x\|-\|\Psi_{tt}x\|}{\|x\|}\\
&\geq\inf_{x\in\mathbb{C}^m,x\neq 0}\left\{\frac{\|[\lambda I_{j_t}-A(\alpha_{j_t},\alpha_{j_t})]x\|}{\|x\|}-\frac{\|\Psi_{tt}x\|}{\|x\|}\right\}\\
&=\inf_{x\in\mathbb{C}^m,x\neq 0}\frac{\|[\lambda I_{j_t}-A(\alpha_{j_t},\alpha_{j_t})]x\|}{\|x\|}-\|\Psi_{tt}x\|\\
&=\|[\lambda I_{j_t}-A(\alpha_{j_t},\alpha_{j_t})]^{-1}\|^{-1}-\|\Psi_{tt}\|\ \text{(by (4))}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|[\lambda I_{j_t}-[A(\alpha_{j_t},\alpha_{j_t})]^{-1}\|^{-1}\\
&\leq\|\Psi_{tt}\|\leq R_t(A/\alpha)+\|\Psi_{tt}\|\leq R_{j_t}(A)-w_{j_t},
\end{aligned}$$

which proves this theorem. ■

Remark 4.1 By Remark 3.1, it is obvious that Theorem 4.1 improves the results of Theorem 4.1 in [13], Theorem 3.1.1 in [35] and Theorem 3.5 in [21].

By Theorem 3.2, similar to the proof of Theorem 4.1, the following theorem can be derived.

Theorem 4.2 Let $A\in II-BSD_s$ and w_{j_t} be defined as in Theorem 3.2. Then

$$\lambda(A)\subset G=\bigcup_{t=1}^{s}[G_t\cup\lambda(A(\alpha_{j_t},\alpha_{j_t}))],$$

where

$$G_i=\left\{\lambda:\ \lambda\nsubseteq\lambda(A(\alpha_{j_t},\alpha_{j_t}))\ \text{and}\ \|[\lambda I_{j_t}-A(\alpha_{j_t},\alpha_{j_t})]^{-1}\|^{-1}\leq\Upsilon_t\right\}$$

and $\Upsilon_t=\|A(\alpha_{j_t},\alpha_{j_t})\|[\hat{R}_{j_t}(A)-w_{j_t}]$.

Remark 4.2 Similar to the discussions in Remark 3.2, it can be seen that the results of Theorem 4.2 improve those in Theorem 4.2 of [13], Theorem 3.12 of [35] and Theorem 3.6 of [21].

V. Some new bounds for determinants of I-(II-)BSD$_s$

In this section, we make use of the results in Sections II-IV to exhibit some new upper and lower bounds for the determinants of $I-(II-)BSD_s$.

Lemma 5.1 [36] Let $A=(a_{ij})_{n\times n}$, $\emptyset\neq\alpha\subseteq N$, assume that $A(\alpha)$ is nonsingular. Then

$$\det A=\det A(\alpha)\det A/\alpha.$$

Lemma 5.2 [29] Let $A\in II-BSD_s$, then $D^{-1}A\in I-BSD_s$, where D is defined as in Lemma 2.4.

Let $\{j_1,j_2,\cdots,j_s\}$ be a rearrangement of the elements in $\{1,2,\cdots,s\}$. Denote $\beta_1=\alpha_{j_s}$, $\beta_2=\alpha_{j_s}\cup\alpha_{j_{s-1}}$, $\cdots$, $\alpha_s=\alpha_{j_s}\cup\alpha_{j_{s-1}}\cup\cdots\cup\alpha_{j_1}=N$. Then $\beta_{s-t+1}-\beta_{s-t}=\alpha_{j_t}$, $t=1,2,\cdots,s$, $\beta_0=\emptyset$, and

$$R_{j_t}[A(\beta_{s-t+1})]=\sum_{\alpha_u\subset\beta_{s-t}}\|A(\alpha_{j_t},\alpha_u)\|,$$

$$\hat{R}_{j_t}[A(\beta_{s-t+1})]=\sum_{\alpha_u\subset\beta_{s-t}}\|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}A(\alpha_{j_t},\alpha_u)\|.$$

Let φ represent any rearrangement $\{j_1,j_2,\cdots,j_s\}$ of the elements in $\{1,2,\cdots,s\}$ with $\beta_1,\beta_2,\cdots,\beta_s$ defined as above. Next, we establish some bounds for determinants of I-(II-)BSD$_s$ in the following theorems.

Theorem 5.1 Let $A\in I-BSD_s$ and be partitioned as in (1). Then

$$|\det A|\geq\max_{\varphi}\prod_{t=1}^{s}\left\{\|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}\|^{-1}-\Theta_{j_t}\right\}^{|\alpha_{j_t}|}\quad(19)$$

and

$$|\det A|\leq\min_{\varphi}\prod_{t=1}^{s}\left\{\|A(\alpha_{j_t},\alpha_{j_t})\|+\Theta_{j_t}\right\}^{|\alpha_{j_t}|},\quad(20)$$

where

$$\Theta_{j_t} = h[A(\beta_{s-t+1})] \sum_{\alpha_v \in \beta_{s-t}} \|A(\alpha_{j_t}, \alpha_v)\| \frac{P_v[A(\beta_{s-t+1})]}{\|[A(\alpha_v, \alpha_v)]^{-1}\|^{-1}},$$
$$r[A(\beta_{s-t+1})] = \max_{t+1 \le u \le s} \frac{\|A(\alpha_{j_u}, \alpha_{j_t})\|}{K_i},$$
$$K_i = \|[A(\alpha_{j_u}, \alpha_{j_u})]^{-1}\|^{-1} - \sum_{v=t+1, v\neq u}^{s} \|A(\alpha_{j_u}, \alpha_{j_v})\|,$$
$$P_{j_v}[A(\beta_{s-t+1})] = r[A(\beta_{s-t+1})] \sum_{u=t+1, u\neq v}^{s} \|A(\alpha_{j_v}, \alpha_{j_u})\|$$
$$+\|A(\alpha_{j_v}, \alpha_{j_t})\|, \quad \alpha_v \in \beta_{s-t},$$
$$h[A(\beta_{s-t+1})] = \max_{t+1 \le u \le s} \frac{\|A(\alpha_{j_u}, \alpha_{j_t})\|}{L_i},$$
$$L_i = P_{j_u}[A(\beta_{s-t+1})]$$
$$- \sum_{v=k+1, v\neq u}^{n} \|A(\alpha_{j_u}, \alpha_{j_v})\| \frac{P_{j_v}[A(\beta_{s-t+1})]}{\|[A(\alpha_{j_v}, \alpha_{j_v})]^{-1}\|^{-1}}.$$

Proof. Inasmuch as β_{s-t} is contained in β_{s-t+1} and $\beta_{s-t+1} - \beta_{s-t} = \alpha_{j_t}$, by Corollary 3.1, we have

$$\|[A(\beta_{s-t+1})/\beta_{s-t}]^{-1}\|^{-1} \ge \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} - \Theta_{j_t} > 0,$$
$$\|A(\beta_{s-t+1})/\beta_{s-t}\| \le \|A(\alpha_{j_t}, \alpha_{j_t})\| + \Theta_{j_t}.$$

By Lemma 5.1, it follows that

$$\begin{aligned}
&|\det A| \\
&= \left|\frac{\det A}{\det[A(\beta_{s-1})]}\right| \left|\frac{\det[A(\beta_{s-1})]}{\det[A(\beta_{s-2})]}\right| \cdots \left|\frac{\det[A(\beta_2)]}{\det[A(\beta_1)]}\right| |\det[A(\beta_1)]| \\
&= |\det(A/\beta_{s-1})|\, |\det[A(\beta_{s-1})/\beta_{s-2}]| \\
&\quad \cdots |\det[A(\beta_2)/\beta_1]|\, |\det[A(\beta_1)]| \\
&= |\prod_{i=1}^{|\alpha_{j_1}|} \lambda_i(A/\beta_{s-1}) \prod_{i=1}^{|\alpha_{j_2}|} \lambda_i(A(\beta_{s-1})/\beta_{s-2}) \\
&\quad \cdots \prod_{i=1}^{|\alpha_{j_{s-1}}|} \lambda_i(A(\beta_2)/\beta_1) \prod_{i=1}^{|\alpha_{j_s}|} \lambda_i(A(\beta_1))| \\
&= |\prod_{t=1}^{s-1} \prod_{i=1}^{|\alpha_{j_t}|} \lambda_i(A(\beta_{s-t+1})/\beta_{s-t}) \prod_{i=1}^{|\alpha_{j_s}|} \lambda_i(A(\beta_1))| \\
&\ge \prod_{t=1}^{s-1} \{\|[A(\beta_{s-t+1})/\beta_{s-t}]^{-1}\|^{-1}\}^{|\alpha_{j_t}|} \{\|[A(\beta_1)]^{-1}\|^{-1}\}^{|\alpha_{j_s}|}. \\
&\ge \max_{\varphi} \prod_{t=1}^{s} \left\{\|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} - \Theta_{j_t}\right\}^{|\alpha_{j_t}|},
\end{aligned}$$

which proves the desired bound (19). The bound (20) can be similarly proven. ■

Remark 5.1 Similar to the discussions in Remark 3.1, for $\alpha_v \in \beta_{s-t}$, we have

$$h[A(\beta_{s-t+1})] \frac{P_v[A(\beta_{s-t+1})]}{\|[A(\alpha_v, \alpha_v)]^{-1}\|^{-1}}$$
$$\le r[A(\beta_{s-t+1})] \le \max_{\alpha_v \in \beta_{s-t}} \frac{R_v[A(\beta_{s-t+1})]}{\|[A(\alpha_v, \alpha_v)]^{-1}\|^{-1}},$$

which results in

$$\begin{aligned}
&\max_{\varphi} \prod_{t=1}^{s} \left\{\|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} - \Theta_{j_t}\right\}^{|\alpha_{j_t}|} \\
&\ge \max_{\varphi} \prod_{t=1}^{s} \left\{\|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} - r[A(\beta_{s-t+1})] R_{j_t}[A(\beta_{s-t+1})]\right\}^{|\alpha_{j_t}|} \\
&\ge \max_{\varphi} \prod_{t=1}^{s} \left\{\|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1} - \max_{\alpha_v \in \beta_{s-t}} \frac{R_v[A(\beta_{s-t+1})]}{\|[A(\alpha_v, \alpha_v)]^{-1}\|^{-1}}\right\}^{|\alpha_{j_t}|}
\end{aligned}$$

and

$$\begin{aligned}
&\min_{\varphi} \prod_{t=1}^{s} \{\|A(\alpha_{j_t}, \alpha_{j_t})\| + \Theta_{j_t}\}^{|\alpha_{j_t}|} \\
&\le \min_{\varphi} \prod_{t=1}^{s} \{\|A(\alpha_{j_t}, \alpha_{j_t})\| + r[A(\beta_{s-t+1})] R_{j_t}[A(\beta_{s-t+1})]\}^{|\alpha_{j_t}|} \\
&\le \min_{\varphi} \prod_{t=1}^{s} \left\{\|A(\alpha_{j_t}, \alpha_{j_t})\| + \max_{\alpha_v \in \beta_{s-t}} \frac{R_v[A(\beta_{s-t+1})]}{\|[A(\alpha_v, \alpha_v)]^{-1}\|^{-1}}\right\}^{|\alpha_{j_t}|}.
\end{aligned}$$

The above discussions verify that Theorem 5.1 improves Theorem 3.2.3 in [35] and Theorem 3.6.1 in [34].

Theorem 5.2 Let $A \in II - BSD_s$ and be partitioned as in (1). Then

$$|\det A| \ge \max_{\varphi} \prod_{t=1}^{s} \left\{\|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1}\|^{-1}(1 - \Delta_{j_t})\right\}^{|\alpha_{j_t}|} \quad (21)$$

and

$$|\det A| \le \min_{\varphi} \prod_{t=1}^{s} \{\|A(\alpha_{j_t}, \alpha_{j_t})\|(1 + \Delta_{j_t})\}^{|\alpha_{j_t}|}, \quad (22)$$

where

$$D = \mathrm{diag}(A(\alpha_1, \alpha_1), \cdots, A(\alpha_s, \alpha_s)),$$
$$\Delta_{j_t} = h[(D^{-1}A)(\beta_{s-t+1})]$$
$$\times \sum_{\alpha_v \in \beta_{s-t}} \|[A(\alpha_{j_t}, \alpha_{j_t})]^{-1} A(\alpha_{j_t}, \alpha_v)\| P_v[(D^{-1}A)(\beta_{s-t+1})].$$

Proof. Combining Lemma 5.2 and Theorem 5.1 yields that $D^{-1}A \in I - BSD_s$ and

$$|\det(D^{-1}A)| \le \min_{\varphi} \prod_{t=1}^{s} \{1 + \Delta_{j_t}\}^{|\alpha_{j_t}|},$$

i.e.,

$$|\det A| \le |\det D| \min_{\varphi} \prod_{t=1}^{s} \{1 + \Delta_{j_t}\}^{|\alpha_{j_t}|}$$
$$\le \min_{\varphi} \prod_{t=1}^{s} ||A(\alpha_{j_t}, \alpha_{j_t})||^{|\alpha_{j_t}|} \{1 + \Delta_{j_t}\}^{|\alpha_{j_t}|}.$$

So Inequality (22) is obtained, similarly, we can prove the Inequality (21). ■

VI. Numerical examples

In this section, we present some numerical examples to illustrate the theory results in this paper and show the advantages of our derived results.

Example 6.1 Let

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{pmatrix},$$
$$A_{11} = \mathrm{diag}(16, \cdots, 16)_{20\times 20}, \; A_{22} = \mathrm{diag}(15, \cdots, 15)_{20\times 20},$$
$$A_{33} = \mathrm{diag}(18, \cdots, 18)_{30\times 30}, \; A_{44} = \mathrm{diag}(8, \cdots, 8)_{15\times 15},$$
$$A_{55} = \mathrm{diag}(9, \cdots, 9)_{15\times 15}, \; A_{12} = \mathrm{diag}(-1, \cdots, -1)_{20\times 20},$$
$$A_{14} = A_{15} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -2 & -2 & \cdots & -2 \end{pmatrix}_{20\times 15},$$
$$A_{31} = A_{32} = \begin{pmatrix} -2 & \cdots & -2 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{30\times 20},$$

$$A_{45}=\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ -2 & \cdots & -2 \end{pmatrix}_{15\times 15},\quad A_{54}=\begin{pmatrix} -2 & -1 & & \\ -1 & -2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & -2 \end{pmatrix}_{15\times 15},$$

$$A_{51}=\begin{pmatrix} 0 & \cdots & -5 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{15\times 20},\quad A_{43}=\begin{pmatrix} -1/3 & \cdots & -1/3 \\ \vdots & \ddots & \vdots \\ -1/3 & \cdots & -1/3 \end{pmatrix}_{15\times 30},$$

$$A_{53}=\begin{pmatrix} 0 & \cdots & -2 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{15\times 30},\quad A_{42}=\begin{pmatrix} -3 & \cdots & -3 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{15\times 20},$$

$$A_{41}=\begin{pmatrix} -1/3 & \cdots & -1/3 \\ \vdots & \ddots & \vdots \\ -1/3 & \cdots & -1/3 \end{pmatrix}_{15\times 20},\quad A_{52}=\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ -2 & \cdots & -2 \end{pmatrix}_{20\times 15},$$

$$A_{34}=\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ -1 & \cdots & 0 \end{pmatrix}_{30\times 15},\quad A_{23}=\begin{pmatrix} -0.1 & \cdots & -0.1 \\ \vdots & \ddots & \vdots \\ -0.1 & \cdots & -0.1 \end{pmatrix}_{20\times 30},$$

$$A_{24}=\begin{pmatrix} -0.2 & \cdots & -0.2 \\ \vdots & \ddots & \vdots \\ -0.2 & \cdots & -0.2 \end{pmatrix}_{20\times 15},\quad A_{21}=\begin{pmatrix} -3 & \cdots & -3 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{20\times 20},$$

$$A_{13}=2.5A_{23},\ A_{25}=0.75A_{24},\ A_{35}=4A_{34}.$$

In the following, we choose $\alpha = \{1, 3\}$ in the Schur complement A/α. Without loss of generality, we assume $\|.\| = \|.\|_\infty$, $\alpha = \bigcup_{r=1}^{2} \alpha_{i_r}$, $\alpha' = \bigcup_{t=1}^{3} \alpha_{j_t}$, $i_1 = 1, i_2 = 3$, $j_1 = 2, j_2 = 4, j_3 = 5$ and $A/\alpha = (\tilde{A}(\alpha_t, \alpha_t))$. By computation, $A \in I - BSD_s$. According to Theorem 4.1, any eigenvalues λ of A/α satisfies

$$\lambda \in \{\lambda: \ |\lambda - 15| \le 7.9186\} \cup \{\lambda : |\lambda - 8| \le 6.8310\} \cup \{\lambda : |\lambda - 9| \le 6.2910\} = \Gamma_1.$$

By Theorem 3.13 in [20] and Theorem 3.5 in [21], any eigenvalues λ of A/α satisfies

$$\lambda \in \{\lambda: \ |\lambda - 15| \le 9.2727\} \cup \{\lambda : |\lambda - 8| \le 9.5455\} \cup \{\lambda : |\lambda - 9| \le 8.1818\} = \Gamma_2.$$

By Theorem 3.1.1 in [35], any eigenvalues λ of A/α satisfies

$$\lambda \in \{\lambda: \ |\lambda - 15| \le 9.2424\} \cup \{\lambda : |\lambda - 8| \le 9.4697\} \cup \{\lambda : |\lambda - 9| \le 8.1515\} = \Gamma_3.$$

By Theorem 4.1 in [13], any eigenvalues λ of A/α satisfies

$$\lambda \in \{\lambda: \ |\lambda - 15| \le 10.1250\} \cup \{\lambda : |\lambda - 8| \le 11.2500\} \cup \{\lambda : |\lambda - 9| \le 9.3750\} = \Gamma_4.$$

To further confirm the facts in the above results, Figures 1-3 depict the eigenvalue distributions of the Schur complement.

From these numerical results and figures, we have the following observations:

- As observed in the comparison results, the Theorem 3.13 in [20], Theorem 3.5 in [21], Theorem 3.1.1 in [35], Theorem 4.1 in [13] and Theorem 4.1 can succeed in computing and determining the the eigenvalue distributions of the Schur complement by using the elements of the original matrix, whereas the eigenvalue distributions derived by Theorem 4.1 are sharper than the ones computed by Theorem 3.13 in [20], Theorem 3.5 in [21], Theorem 3.1.1 in [35] and Theorem 4.1 in [13], that is, $\Gamma_1 \subset \Gamma_2$, $\Gamma_1 \subset \Gamma_3$ and $\Gamma_1 \subset \Gamma_4$.
- From Figures 1-3, we clearly find that Γ_1 is the tightest among all eigenvalue distributions, which demonstrates the validity of the conclusion given in Remark 4.1.

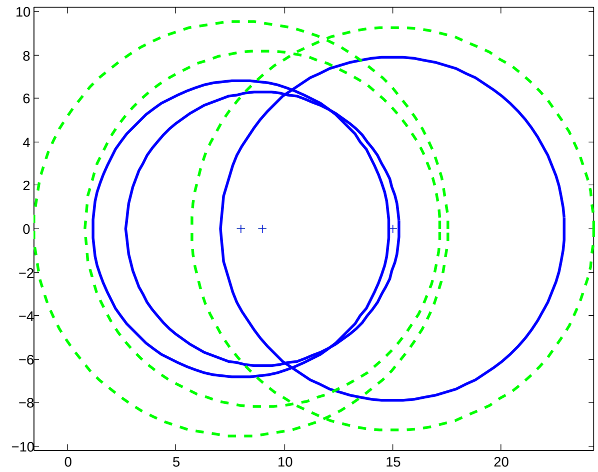

Fig. 1. The blue solid line and the green dashed line denote the corresponding discs Γ_1 and Γ_2, respectively

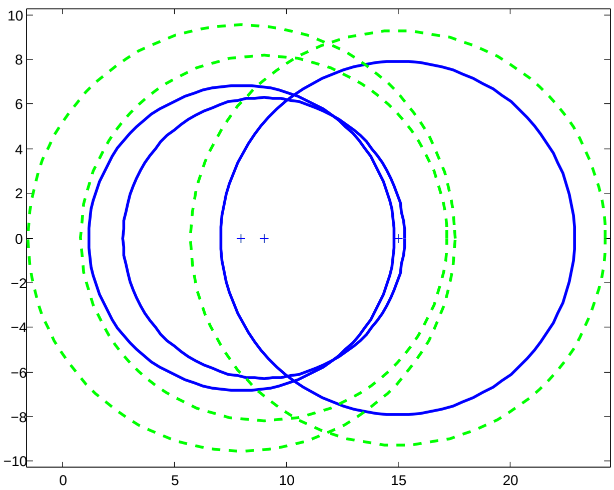

Fig. 2. The blue solid line and the green dashed line denote the corresponding discs Γ_1 and Γ_3, respectively

Example 6.2 Let

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

where

$$A_{11} = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix},\ A_{12} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ A_{13} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$
$$A_{21} = \begin{pmatrix} 0 & 2 \end{pmatrix},\ A_{22} = 10,\ A_{23} = \begin{pmatrix} 3 & 0 \end{pmatrix},$$
$$A_{31} = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix},\ A_{32} \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ A_{33} = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}.$$

It is easy to see that $A \in I - BSD_3$. We compare the bounds in Theorem 5.1 with those in Theorem 3.61 of [34]

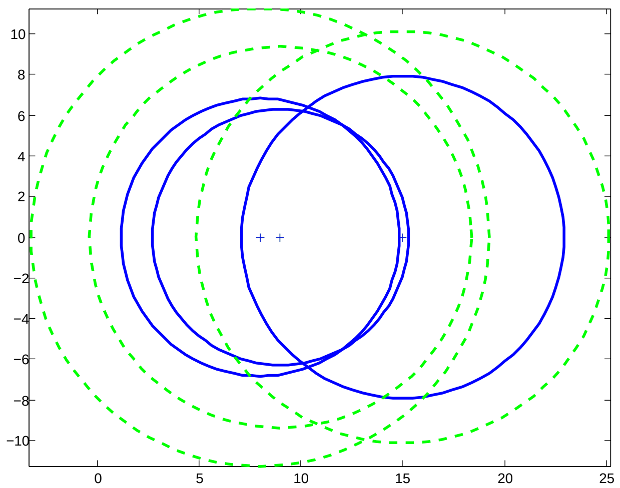

Fig. 3. The blue solid line and the green dashed line denote the corresponding discs Γ_1 and Γ_4, respectively

and Theorem 3.2.3 of [35]. By utilizing Theorem 3.61 in [34], we have

$$35403 \leq |\det A| \leq 70596.$$

By making use of Theorem 3.2.3 in [35], we have

$$43643 \leq |\det A| \leq 60938.$$

Now, by applying Theorem 5.1, we derive

$$43841 \leq |\det A| \leq 60703,$$

which is an improvement on the bounds in Theorem 3.61 of [34] and Theorem 3.2.3 of [35]. This example shows that the upper and lower bounds in Theorem 5 are better than those in Theorem 3.61 of [34] and Theorem 3.2.3 of [35]. In fact, $\det A = 47448$.

VII. Conclusions

To estimates diagonally dominant degree on the Schur complement of matrices, we first exhibit some new estimations of diagonally dominant degree on the Schur complement of I(II)-block diagonally dominant matrices in this paper, which are proved to be sharper than the ones in [13], [35], [21]. As applications, some new distributions for the eigenvalues of the Schur complement of matrices as well as the new upper and lower bounds for determinants of the I(II)-block diagonally dominant matrices are derived, these results are better compared with those of [13], [35], [21], [34]. Numerical examples are also given to illustrate these facts.

It would be nice if we can find more precise estimates of I(II)-block diagonally dominant degree for Schur complement of matrices, distributions for the eigenvalues of the Schur complement of matrices and upper and lower bounds for determinants of the I(II)-block diagonally dominant matrices compared those proposed in this paper. We will continue to research this topic in our further work.

References

[1] D. Carlson and T. Markham, “Schur complements on diagonally dominant matrices,” *Czechoslov. Math. J.*, vol. 29, no. 1, pp. 246–251, 1979.
[2] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*. Cambridge University Press, 1991.
[3] K. D. Ikramov, “Invariance of the Brauer diagonal dominance in Gaussian elimination,” *Vestn. Mosk. Univ., Ser. 15 Vyčisl. Mat. kibern*, vol. 2, pp. 91–94, 1989.
[4] R. Smith, “Some interlacing properties of the Schur complement of a Hermitian matrix,” *Linear Algebra Appl.*, vol. 177, pp. 137–144, 1992.
[5] F.-Z. Zhang, *The Schur Complement and Its Applications*. Springer, New York, 2005.
[6] J.-Z. Liu and Y.-Q. Huang, “The Schur complements of generalized doubly diagonally dominant matrices,” *Linear Algebra Appl.*, vol. 378, pp. 231–244, 2004.
[7] J.-Z. Liu, Z.-J. Huang, and J. Zhang, “The dominant degree and disc theorem for the Schur complement,” *Appl. Math. Comput.*, vol. 215, pp. 4055–4066, 2010.
[8] J.-Z. Liu and Z.-J. Huang, “The Schur complement of γ-diagonally and product γ-diagonally dominant matrix and their disc seperation,” *Linear Algebra Appl.*, vol. 432, pp. 1090–1104, 2010.
[9] M.-Z. Zhu and Y.-E. Qi, “On the Eigenvalues Distribution of Preconditioned Block Two-by-two Matrix,” *IAENG International Journal of Applied Mathematics*, vol. 46, no. 4, pp. 500–504, 2016.
[10] Z.-G. Huang, L.-G. Wang, Z. Xu, and J.-J. Cui, “Convergence Analysis of Some New Preconditioned AOR Iterative Methods for L-matrices,” *IAENG International Journal of Applied Mathematics*, vol. 46, no. 2, pp. 202–209, 2016.
[11] A.-J. Li, “Improving AOR Iterative Methods For Irreducible L-matrices,” *Engineering Letters*, vol. 19, no. 1, pp. 46–49, 2011.
[12] J.-G. Hu, “The estimate of $\|M^{-1}N\|_\infty$ and the optimally scaled matrix,” *JCM Sinica*, vol. 3, pp. 122–129, 1982.
[13] J.-Z. Liu, Z.-H. Huang, L. Zhu, and Z.-J. Huang, “Theorems on schur complement of block diagonally dominant matrices and their application in reducing the order for the solution of large scale linear systems,” *Linear Algebra Appl.*, vol. 435, pp. 3085–3100, 2011.
[14] J.-Z. Liu, J.-C. Li, and Z.-H. Huang, “Some properties on Schur complement and diagonal Schur compelment of some diagonally dominant matrices,” *Linear Algebra Appl.*, vol. 428, no. 1, pp. 1009–1030, 2008.
[15] B. Li and M. Tsatsomeros, “Doubly diagonally dominant matrices,” *Linear Algebra Appl.*, vol. 261, pp. 221–235, 1997.
[16] J.-Z. Liu and Y.-Q. Huang, “Some properties on Schur complement of H-matrices and diagonally dominant matrices,” *Linear Algebra Appl.*, vol. 389, pp. 365–380, 2004.
[17] J.-Z. Liu and F.-Z. Zhang, “Disc separation of the Schur complements of diagonally dominant matrices and determinantal bounds,” *SIAM J. Matrix Anal. Appl.*, vol. 27, pp. 665–674, 2005.
[18] J. Zhang, J. Z. Liu, and G. Tu, “The improved disc theorems for the Schur complements of diagonally dominant matrices,” *J. Inequal. Appl.*, vol. 2013, no. 2, 2013.
[19] Y.-T. Li and F. Wang, “The estimates of diagonally dominant degree and eigenvalues distributions for Schur complements of matrices,” *J. Inequal. Appl.*, vol. 2013, no. 431, 2013.
[20] F. Wang, *Criteria for H-matrices (tensors) and research of Schur complement of H-matrices (tensors)*. Yunnan University, Kunming, 2014.
[21] F. Wang and D.-S. Sun, “The estimates of diagonally dominant degree and eigenvalues inclusion regions for the Schur complement of block diagonally dominant matrices,” *Open Math.*, vol. 13, pp. 781–794, 2015.
[22] R. Vein and P. Dale, *Determinants and Their Applications in Mathematical Physics*. Applied Mathematical Sciences, Springer, 1999.
[23] M. I. Gil′, *Operator Functions and Localization of Spectra*. Lecture Notes in Mathematics, Springer, Berlin, 2003.
[24] C. Krattenthaler, “Advanced determinant calculus: a complement,” *Linear Algebra Appl.*, vol. 411, pp. 68–166, 2005.
[25] T.-Z. Huang and X.-P. Liu, “Estimations for Certain Determinants,” *Comput. Math. Appl.*, vol. 2015, no. 50, pp. 1677–1684, 2005.
[26] H.-B. Li, T.-Z. Huang, and H. Li, “Some New Results on Determinantal Inequalities and Applications,” *J. Inequal. Appl.*, vol. Article ID 847357, doi:10.1155/2010/847357, 2010.
[27] D. G. Feingold and R. S. Varga, “Block diagonally dominant matrices and generalizations of the gerschgorin circle theorem,” *Pacific J. Math.*, vol. 12, no. 1, pp. 1241–1250, 1962.
[28] Z.-Y. You and Z.-Q. Jiang, “The diagonal dominance of block matrices,” *J. Xian Jiaotong Univ.*, vol. 18, no. 3, pp. 123–125, 1984.

[29] S.-H. Xiang and Z.-Y. You, "Weak block diagonally dominant matrices, weak block h-matrix and their applications," *Linear Algebra Appl.*, vol. 282, pp. 263–274, 1998.
[30] C.-Y. Zhang, Y.-T. Li, and F. Chen, "On schur complement of block diagonally dominant matrices," *Linear Algebra Appl.*, vol. 414, pp. 533–546, 2006.
[31] B. Polman, "Incomplete blockwise factorizations of (block) H-matrices," *Linear Algebra Appl.*, vol. 90, pp. 119–132, 1987.
[32] F. Robert, "Block H-matrices et convergence des methods iterations classiques par blocks," *Linear Algebra Appl.*, vol. 6, pp. 223–265, 1969.
[33] L. Y. Kolotilina, "Nonsingularity/singularity criteria for nonstrictly block diagonally dominant matrices," *Linear Algebra Appl.*, vol. 359, pp. 133–159, 2003.
[34] L. Zhu, *Properties, Criteria and Application for Block Diagonally Dominant Matrices*. Xiangtan University, Xiangtan, 2007.
[35] N.-N. Xu, *The Block Dominant Degree and Disc Theorem on Schur Complements of Block Diagonally Dominant matrices*. Xiangtan University, Xiangtan, 2009.
[36] Z. Xu, Q. Lu, K. Y. Zhang, and X. H. An, *Theory and Applications of H-matrices*. Science Press, 2013.