A Spatial Sixth Order Finite Difference Scheme for Time Fractional Sub-diffusion Equation with Variable Coefficient

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Abstract—**In this paper, we present a finite difference scheme for a class of time fractional diffusion equation with variable coefficient, where the fractional derivative is defined by the Caputo derivative. The present algorithm is unconditionally stable, and possess spatial sixth order and temporal $2-\alpha$ order accuracy, which is an improvement of the spatial fourth order accuracy in the existing results. Theoretical analysis including local truncating error, unique solvability, stability and convergence for this algorithm is fulfilled. Then based on this finite difference scheme, we also investigate the construction of unconditionally stable finite difference scheme for a class of time fractional parabolic equation with spatial fourth derivative. In order to testify the efficiency of the algorithms as well as the convergence orders, some numerical examples are presented.**

Index Terms—**fractional sub-diffusion equation, difference scheme, variable coefficient, spatial sixth order accuracy, unconditional stability.**

I. Introduction

FRACTIONAL differential equations containing the fractional derivative are widely used in various domains including physics, biology, engineering, signal processing, systems identification, control theory, finance, fractional dynamics and so on (see [1,2] for example). In particular, the fractional derivative has proved to be very useful in describing the memory and hereditary properties of materials and processes. For the basic theory, readers can refer to the works [3,4]. One of its most important applications is to model the process of subdiffusion and superdiffusion of particles in physics, where the fractional diffusion equation is usually used for modeling this movement [5-7]. Recently, various aspects for fractional diffusion equations have been researched by many authors. In [8,9], the authors proposed certain methods for finding analytical solutions of fractional differential equations. In [10-12], qualitative and quantitative properties of solutions of fractional differential equations are investigated. In [13-15], the applications and physical meaning of fractional diffusion equations have been discussed, while numerical solutions for fractional diffusion equations are obtained by use of various methods including the finite element method [16,17], the meshless method [18,19], the finite difference method [20-30], the Bernstein polynomials method and so on [31]. Among the works for solving fractional diffusion equations, we notice that most of the current research for fractional diffusion equations is in constant coefficient case, and less attention has been paid to the research for fractional diffusion equations with variable coefficient. In [32], Chen et al. presented a fast semi-implicit difference method with convergence order $O(\tau+h)$ for a nonlinear two-sided space-fractional diffusion equation with variable diffusivity coefficients, and also developed a fast accurate iterative method by decomposing the dense coefficient matrix into a combination of Toeplitz-like matrices. In [33,34], the authors presented compact finite difference schemes with convergence order $O(\tau^{2-\alpha}+h^4)$ (here $0<\alpha<1$) for fractional sub-diffusion equation with the spatially variable coefficient subject to both Dirichlet boundary conditions and Neumann boundary conditions, and proved the stability and convergence order for the difference schemes. In [35], Wang established a compact finite difference method with convergence order $O(\tau^{3-\alpha}+\tau^2+h^4)$ (here $1<\alpha<2$) for a class of time fractional convection-diffusion-wave equations with variable coefficients, while in [36], Wang et al. proposed a Petrov-Galerkin finite element method for variable-coefficient fractional diffusion equations, and proved the well-posedness and optimal-order convergence of this method. In [37], Anatoly et al. researched numerical methods for solving inverse problems for time fractional diffusion equation with variable coefficient.

In this paper, we consider the following time fractional sub-diffusion equation with variable coefficient and nonhomogeneous source term

$$ {}_0^C D_t^\alpha u(x,t)=\frac{\partial}{\partial x}(b(x)\frac{\partial u(x,t)}{\partial x})+f(x,t),\ 0<\alpha<1, \quad (1)$$

which is subject to the following initial and periodic boundary value conditions

$$\begin{cases} u(x,0)=\varphi(x),\ x\in\mathbf{R}, \\ u(x,t)=u(x+L,t),\ x\in\mathbf{R},\ t\in[0,T], \end{cases} \quad (2)$$

where the fractional derivative ${}_0^C D_t^\alpha u(x,t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{u_t'(x,s)}{(t-s)^\alpha}ds$ is defined in the sense of Caputo derivative, L is the period of $u(x,t)$ with respect to the variable x, and $b(x)$ is assumed to be smooth enough satisfying $b(x)\geq L_0>0$.

The current work is devoted to deriving a finite difference scheme with spatial sixth order and temporal $2-\alpha$ order accuracy for the above problems, and based on this finite

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difference scheme we will also investigate the construction of unconditionally stable finite difference scheme for a class of time fractional parabolic equation with spatial fourth derivative as follows

$$ {}_{0}^{C}D_t^{\alpha}u(x,t) + u_{xxxx} = f(x,t),\ 0<\alpha<1, \tag{3} $$

which is subject to the following initial and periodic boundary value conditions (2).

We organize the rest of this paper as follows. In Section 2, we give the derivation of the finite difference scheme for solving fractional sub-diffusion equation (1) under the conditions (2). Then in Section 3, theoretical analysis of local truncating error, unique solvability, stability and convergence for the finite difference scheme are fulfilled. In Section 4, we apply the method in Section 2 to construct unconditionally stable finite difference scheme with spatial fourth order accuracy for Eq. (3). In Section 5, we give some numerical examples to verify the theoretically analytical results. In Section 6, some concluding statements are presented.

II. Establishment of the high order finite difference scheme for Eq. (1)

Since the periodic boundary value condition is considered here, it is sufficient to assume $x\in[0,L]$. Let $M,\ N$ be two positive integers, and $h=\frac{L}{M},\ \tau=\frac{T}{N}$ denote the spatial and temporal step size respectively. Define $x_i=i*h(0\le i\le M),\ t_n=n\tau(0\le n\le N),\ \Omega_h=\{x_i|0\le i\le M\},\ \Omega_\tau=\{t_n|0\le n\le N\},\ (i,n)=(x_i,t^n)$, and then the domain $[0,L]\times[0,T]$ is covered by $\Omega_h\times\Omega_\tau$. Let $V_h=\{u_i^n|0\le i\le M,\ 0\le n\le N\}$ be the grid function on the mesh $\Omega_h\times\Omega_\tau$. $U_i^n=u(x_i,t^n)$ and u_i^n denote the exact solution and numerical solution at the point (i,n) respectively. $U^n=(U_1^n,\ U_2^n,...,\ U_M^n)^T,\ u^n=(u_1^n,\ u_2^n,...,\ u_M^n)^T$.

In order to approximate the time derivative, the following lemmas are listed for further use.

Lemma 1 [33, Lem. 2.1](The $L1$ formula). Suppose $0<\alpha<1$, and $u(t)\in C^2[0,t_n]$. Then it holds that

$$ \Big|\frac{1}{\Gamma(1-\alpha)}\int_0^{t_n}\frac{u'(s)}{(t_n-s)^{\alpha}}ds-\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} $$
$$ [a_0^{(\alpha)}u(t_n)-\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)}-a_{n-k}^{(\alpha)})u(t_k)-a_{n-1}^{(\alpha)}u(t_0)]\Big|\le $$
$$ \frac{1}{\Gamma(2-\alpha)}[\frac{1-\alpha}{12}+\frac{2^{2-\alpha}}{2-\alpha}-(1+2^{-\alpha})]\max_{t_0\le t\le t_n}|u^{''}(t)|\tau^{2-\alpha}, \tag{4} $$

where $t_0=0,\ a_k^{(\alpha)}=(k+1)^{1-\alpha}-k^{1-\alpha},\ k\ge 0$.

Lemma 2. [38, Lem. 1.4.8] Suppose $0<\alpha<1$, and $a_k^{(\alpha)},\ k\ge0$ are defined as in Lemma 2, then

$$ \begin{cases} 1=a_0^{(\alpha)}>a_1^{(\alpha)}>a_2^{(\alpha)}>...>a_k^{(\alpha)}>...>0,\\ \lim\limits_{k\to\infty}a_k^{(\alpha)}=0, \end{cases} \tag{5} $$

and

$$ (1-\alpha)k^{-\alpha}<a_{k-1}^{(\alpha)}<(1-\alpha)(k-1)^{-\alpha}. \tag{6} $$

Lemma 3. Suppose $u(x,t)\in C^{(8,3)}([x_{i-3},x_{i+3}]\times[0,T])$, and define two operators $H_1,\ H_2$ such that

$$ \begin{cases} H_1U_i^n=\frac{1}{h}(-\frac{1}{60}U_{i-3}^n+\frac{3}{20}U_{i-2}^n-\frac{3}{4}U_{i-1}^n\\ \qquad +\frac{3}{4}U_{i+1}^n-\frac{3}{20}U_{i+2}^n+\frac{1}{60}U_{i+3}^n),\\ H_2U_i^n=\frac{1}{h^2}(\frac{1}{90}U_{i-3}^n-\frac{3}{20}U_{i-2}^n+\frac{3}{2}U_{i-1}^n-\frac{49}{18}U_i^n\\ \qquad +\frac{3}{2}U_{i+1}^n-\frac{3}{20}U_{i+2}^n+\frac{1}{90}U_{i+3}^n), \end{cases} \tag{7} $$

where $U_i^n=u(x_i,t^n)$. Then u_x and u_{xx} can be approximated by $H_1U_i^n$ and $H_2U_i^n$ respectively with sixth order accuracy, that is,

$$ \begin{cases} |u_x(x_i,t^n)-H_1U_i^n|\le\frac{84}{5\times7!}\max\limits_{x_{i-3}\le x\le x_{i+3}}|u_x^{(7)}(x,t)|h^6,\\ |u_{xx}(x_i,t^n)-H_2U_i^n|\le\frac{72}{8!}\max\limits_{x_{i-3}\le x\le x_{i+3}}|u_x^{(8)}(x,t)|h^6, \end{cases} \tag{8} $$

where $C_1,\ C_2$ are two constants.

The proof of Lemma 3 can be completed by the expansion of the Taylor's formula.

In order to establish the difference scheme, we rewrite Eq. (1) as

$$ {}_{0}^{C}D_t^{\alpha}u(x,t)=b'(x)u_x+b(x)u_{xx}+f(x,t),\ 0<\alpha<1, \tag{9} $$

Denote $\Delta_\tau^\alpha U_i^n=\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[a_0^{(\alpha)}U_i^n-\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)}-a_{n-k}^{(\alpha)})U_i^k-a_{n-1}^{(\alpha)}U_i^0]$, where $a_k^{(\alpha)},\ k=0,1,...,n-1$ are defined as in Lemma 1. Then it follows from Lemmas 1 and 3 that

$$ \Delta_\tau^\alpha U_i^n=b_i'H_1U_i^n+b_iH_2U_i^n+f_i^n+R_i^n(\tau,h), \tag{10} $$

where $b_i'=b'(x_i),\ b_i=b(x_i),\ f_i^n=f(x_i,t^n)$, and

$$ |R_i^n(\tau,h)|\le\frac{1}{\Gamma(2-\alpha)}[\frac{1-\alpha}{12}+\frac{2^{2-\alpha}}{2-\alpha} $$
$$ -(1+2^{-\alpha})]\max_{t_0\le t\le t_n}|u_t^{''}(x,t)|\tau^{2-\alpha} $$
$$ +\frac{84}{5\times7!}\max_{0\le x\le L}|u_x^{(7)}(x,t)|h^6+\frac{72}{8!}\max_{0\le x\le L}|u_x^{(8)}(x,t)|h^6. $$

Then the difference scheme approximating Eq. (1) at the point (i,n) under the conditions (2) can be established as follows

$$ \begin{cases} \Delta_\tau^\alpha u_i^n=b_i'H_1u_i^n+b_iH_2u_i^n+f_i^n,\ 1\le n\le N,\\ u_i^0=\varphi(x_i),\\ u_i^n=u_{i\pm M}^n,\ 1\le i\le M,\ 1\le n\le N. \end{cases} \tag{11} $$

III. Theoretical analysis of the difference scheme

In this section, we fulfill analysis of local truncating error, unique solvability, stability and convergence for the finite difference scheme (11) by use of the Fourier analysis method.

A. Local truncating error and unique solvability

First the local truncating error for the difference scheme (11) can be directly obtained as $O(\tau^{2-\alpha}+h^6)$.

In order to facilitate fulfilling the analysis, we rewrite the first equation of the finite difference scheme (11) as the following form

$$[\mu-180h^2(b_i'H_1+b_iH_2)]u_i^n=\sum_{k=0}^{n-1}B_k^n u_i^k+180h^2f_i^n,\ 1\le i\le M, \tag{12}$$

where $\mu=\dfrac{180h^2\tau^{-\alpha}}{\Gamma(2-\alpha)}$, $B_0^n=\mu a_{n-1}^{(\alpha)}$, $B_k^n=\mu(a_{n-k-1}^{(\alpha)}-a_{n-k}^{(\alpha)})$, $k=1,2,...,n-1$, $(b_i'H_1+b_iH_2)u_i^n=-\dfrac{1}{180h^2}[g_{i,i-3}u_{i-3}^n+g_{i,i-2}u_{i-2}^n+g_{i,i-1}u_{i-1}^n+g_{i,i}u_i^n+g_{i,i+1}u_{i+1}^n+g_{i,i+2}u_{i+2}^n+g_{i,i+3}u_{i+3}^n]$, $1\le i\le M$, $u_i^n=u_{i\pm M}^n$, and

$$\left\{\begin{array}{ll} g_{i,i}=490b_i, & \\ g_{i,i+1}=-135hb_i'-270b_i, & \\ g_{i,i+2}=27hb_i'+27b_i, & \\ g_{i,i+3}=-3hb_i'-2b_i, & i=1,2,...,M. \\ g_{i,i-1}=135hb_i'-270b_i, & \\ g_{i,i-2}=-27hb_i'+27b_i, & \\ g_{i,i-3}=3hb_i'-2b_i, & \end{array}\right. \tag{13}$$

For the sake of using the Fourier analysis method, define

$$v^n(x)=\left\{\begin{array}{l} u_i^n,\ x\in[x_{i-1},x_{i+\frac{1}{2}}),\ i=1,2,...,M-1, \\ u_M^n,\ x\in[x_{M-1},x_M], \end{array}\right.$$

$$y^n(x)=\left\{\begin{array}{l} f_i^n,\ x\in[x_{i-1},x_i),\ i=1,2,...,M-1, \\ f_M^n,\ x\in[x_{M-1},x_M], \end{array}\right.$$

and a periodic extension is applied to $v^n(x)$, $y^n(x)$. Then from Eq. (12) one can obtain that

$$[\mu-180h^2(b_i'H_1+b_iH_2)]v^n(x)=\sum_{k=0}^{n-1}B_k^n v^k(x)+180h^2y^n(x). \tag{14}$$

where

$$(b_i'H_1+b_iH_2)v^n(x)=\frac{b_i'}{h}[-\frac{1}{60}v^n(x-3h)+\frac{3}{20}v^n(x-2h)$$
$$-\frac{3}{4}v^n(x-h)+\frac{3}{4}v^n(x+h)-\frac{3}{20}v^n(x+2h)+\frac{1}{60}v^n(x+3h)]$$
$$+\frac{b_i}{h^2}[\frac{1}{90}v^n(x-3h)-\frac{3}{20}v^n(x-2h)+\frac{3}{2}v^n(x-h)$$
$$-\frac{49}{18}v^n(x)+\frac{3}{2}v^n(x+h)-\frac{3}{20}v^n(x+2h)+\frac{1}{90}v^n(x+3h)].$$

The functions $v^n(x)$ and $y^n(x)$ can be denoted by the Fourier series forms as follows

$$v^n(x)=\sum_{l=-\infty}^{\infty}\widetilde{v}_l^n\exp(\frac{2\pi lxj}{L}),\ y^n(x)=\sum_{l=-\infty}^{\infty}\widetilde{y}_l^n\exp(\frac{2\pi lxj}{L}), \tag{15}$$

where

$$\widetilde{v}_l^n=\frac{1}{L}\int_0^L v^n(x)\exp(-\frac{2\pi lxj}{L})dx,$$
$$\widetilde{y}_l^n=\frac{1}{L}\int_0^L y^n(x)\exp(-\frac{2\pi lxj}{L})dx,$$

and j denotes the imaginary unit.

Define the discrete L_2 norm by $\|u^n\|_2=(\sum_{i=1}^{M}h|u_i^n|^2)^{\frac{1}{2}}$. Then by use of the Parseval's equality one can obtain that

$$\|u^n\|_2=(\int_0^L|v^n(x)|^2dx)^{\frac{1}{2}}=(\sum_{l=-\infty}^{\infty}|\widetilde{v}_l^n|^2)^{\frac{1}{2}},$$
$$\|f^n\|_2=(\int_0^L|y^n(x)|^2dx)^{\frac{1}{2}}=(\sum_{l=-\infty}^{\infty}|\widetilde{y}_l^n|^2)^{\frac{1}{2}}.$$

Substituting (15) into (14), and denoting $p=\dfrac{2\pi l}{L}$, we can get that

$$\sum_{l=-\infty}^{\infty}\{[\mu-180h^2(b_i'H_1+b_iH_2)]\exp(pxj)\}\widetilde{v}_l^n$$
$$=\sum_{l=-\infty}^{\infty}[\sum_{k=0}^{n-1}B_k^n\widetilde{v}_l^k\exp(pxj)+180h^2\widetilde{y}_l^n\exp(pxj)]. \tag{16}$$

Furthermore, after some basic computation, one can see that the following relations hold

$$\begin{aligned}&-180h^2(b_i'H_1+b_iH_2)\exp(pxj)=\\ &\exp(pxj)[g_{i,i-3}\exp(-3phj)+g_{i,i-2}\exp(-2phj)\\ &+g_{i,i-1}\exp(-phj)+g_{i,i}+g_{i,i+1}\exp(phj)\\ &+g_{i,i+2}\exp(2phj)+g_{i,i+3}\exp(3phj)]\\ &=\exp(pxj)(q_1+h^2q_2j),\end{aligned} \tag{17}$$

where

$$q_1=-4[4\cos^3(ph)-27\cos^2(ph)+132\cos(ph)-109]b_i,$$
$$q_2=\frac{[-270\sin(ph)+54\sin(2ph)-6\sin(3ph)]b_i'}{h}.$$

By a close observation on q_1, q_2 one can deduce that $q_1\ge 0$, q_2 is continuous without $h=0$, and $\lim_{h\to 0}q_2=-180pb_i'$.

So it follows from above that

$$\sum_{l=-\infty}^{\infty}\{[\mu+(q_1+h^2q_2j)]\widetilde{v}_l^n\}\exp(pxj)$$
$$=\sum_{l=-\infty}^{\infty}\{\sum_{k=0}^{n-1}B_k^n\widetilde{v}_l^k+180h^2\widetilde{y}_l^n\}\exp(pxj). \tag{18}$$

On the other hand, due to the orthogonality of $\exp(\frac{2\pi lxj}{L})$, $l=0,\pm1,\pm2,...,\pm\infty$, multiplying $\exp(-pxj)$ on both sides of (18), and integrating from 0 to L we get that

$$[\mu+(q_1+h^2q_2j)]\widetilde{v}_l^n=\sum_{k=0}^{n-1}B_k^n\widetilde{v}_l^k+180h^2\widetilde{y}_l^n,$$
$$l=0,\pm1,\pm2,...,\pm\infty. \tag{19}$$

Theorem 1. The finite difference scheme denoted by (11) is uniquely solvable.

Proof. In order to prove the unique solvability of (11), it is sufficient to prove that there is only zero solution for the corresponding homogeneous difference equation.

From the process above one can see that after fulfilling Fourier transformation, the following equation can be obtained due to the homogeneous difference equation

$$[\mu+(q_1+h^2q_2j)]\widetilde{v}_l^n=0, \tag{20}$$

which implies $\widetilde{v}_l^n=0$ due to $|\mu+(q_1+h^2q_2j)|>0$.

So by (15) one has $v^n(x)=0$, and then $\|u^n\|_2=0$, which implies $u_i^n=0,\ i=1,2,...,M$. The proof is complete.

B. Stability

Theorem 2. For Eq. (19), it holds that

$$|\widetilde{v}_l^n|\le|\widetilde{v}_l^0|+(1+\tau)^n\max_{1\le s\le n}|\widetilde{y}_l^s|\le|\widetilde{v}_l^0|+e^T\max_{1\le s\le n}|\widetilde{y}_l^s|. \tag{21}$$

Furthermore, the finite difference scheme denoted by (11) is unconditionally stable on the initial value and the right source term $f(x,t)$.

Proof. We will use the mathematical induction method to prove (21).

When $n=1$, from (19) it holds that

$$[\mu+(q_1+h^2q_2j)]\widetilde{v}_l^1=B_0^1\widetilde{v}_l^0+180h^2\widetilde{y}_l^1=\mu\widetilde{v}_l^0+180h^2\widetilde{y}_l^1,$$

which implies that

$$\begin{aligned}|\widetilde{v}_l^1|&=\left|\frac{\mu\widetilde{v}_l^0+180h^2\widetilde{y}_l^1}{\mu+(q_1+h^2q_2j)}\right|\\&\le\frac{\mu}{\sqrt{(\mu+q_1)^2+h^4q_2^2}}|\widetilde{v}_l^0|+\frac{180h^2}{\sqrt{(\mu+q_1)^2+h^4q_2^2}}|\widetilde{y}_l^1|\\&\le|\widetilde{v}_l^0|+\Gamma(2-\alpha)\tau^\alpha|\widetilde{y}_l^1|\le|\widetilde{v}_l^0|+\tau^\alpha|\widetilde{y}_l^1|.\end{aligned}$$

In the case $\tau\ge1$, one has $\tau^\alpha\le\tau\le1+\tau$, while in the case $0<\tau<1$, one has $\tau^\alpha\le1\le1+\tau$. So $\tau^\alpha\le1+\tau$ holds for $\forall\tau>0$. Then

$$|\widetilde{v}_l^1|\le|\widetilde{v}_l^0|+(1+\tau)|\widetilde{y}_l^1|\le|\widetilde{v}_l^0|+e^T|\widetilde{y}_l^1|.$$

So (21) holds for $n=1$.

Suppose (21) holds for the time levels $1,2,...,n-1$. Then for the time level n, from (19) one can deduce that

$$\begin{aligned}|\widetilde{v}_l^n|&=\left|\frac{\sum_{k=0}^{n-1}B_k^n\widetilde{v}_l^k+180h^2\widetilde{y}_l^n}{\mu+(q_1+h^2q_2j)}\right|\\&\le\frac{1}{|\mu+(q_1+h^2q_2j)|}\sum_{k=0}^{n-1}|B_k^n||\widetilde{v}_l^k|+\left|\frac{180h^2\widetilde{y}_l^n}{\mu+(q_1+h^2q_2j)}\right|\\&\le\frac{1}{|\mu+(q_1+h^2q_2j)|}\sum_{k=0}^{n-1}|B_k^n||\widetilde{v}_l^0|+\frac{1}{|\mu+(q_1+h^2q_2j)|}\\&\quad\sum_{k=0}^{n-1}[|B_k^n|(1+\tau)^k\max_{1\le s\le k}|\widetilde{y}_l^s|]+\left|\frac{180h^2\widetilde{y}_l^n}{\mu+(q_1+h^2q_2j)}\right|\\&=\frac{1}{\sqrt{(\mu+q_1)^2+h^4q_2^2}}\sum_{k=0}^{n-1}|B_k^n||\widetilde{v}_l^0|+\frac{1}{\sqrt{(\mu+q_1)^2+h^4q_2^2}}\\&\quad\sum_{k=0}^{n-1}[|B_k^n|(1+\tau)^k\max_{1\le s\le k}|\widetilde{y}_l^s|]+\left|\frac{180h^2\widetilde{y}_l^n}{\sqrt{(\mu+q_1)^2+h^4q_2^2}}\right|\\&\le\frac{1}{\sqrt{(\mu+q_1)^2+h^4q_2^2}}\sum_{k=0}^{n-1}|B_k^n||\widetilde{v}_l^0|\\&\quad+\frac{(1+\tau)^{n-1}\max_{1\le s\le n-1}|\widetilde{y}_l^s|}{\sqrt{(\mu+q_1)^2+h^4q_2^2}}\sum_{k=0}^{n-1}|B_k^n|+\left|\frac{180h^2\widetilde{y}_l^n}{\sqrt{(\mu+q_1)^2+h^4q_2^2}}\right|.\end{aligned}$$

From Lemma 2 one can obtain $\sum_{k=0}^{n-1}|B_k^n|=\mu a_0^\alpha=\mu$. So

$$\begin{aligned}|\widetilde{v}_l^n|&\le\frac{\mu}{\sqrt{(\mu+q_1)^2+h^4q_2^2}}|\widetilde{v}_l^0|+\frac{\mu}{\sqrt{(\mu+q_1)^2+h^4q_2^2}}\\&\quad(1+\tau)^{n-1}\max_{1\le s\le n-1}|\widetilde{y}_l^s|+\left|\frac{180h^2\widetilde{y}_l^n}{\sqrt{(\mu+q_1)^2+h^4q_2^2}}\right|\\&\le|\widetilde{v}_l^0|+(1+\tau)^{n-1}\max_{1\le s\le n-1}|\widetilde{y}_l^s|+\left|\frac{180h^2\widetilde{y}_l^n}{\mu}\right|\\&=|\widetilde{v}_l^0|+(1+\tau)^{n-1}\max_{1\le s\le n-1}|\widetilde{y}_l^s|+\Gamma(2-\alpha)\tau^\alpha|\widetilde{y}_l^n|\\&\le|\widetilde{v}_l^0|+(1+\tau)^{n-1}\max_{1\le s\le n-1}|\widetilde{y}_l^s|+(1+\tau)|\widetilde{y}_l^n|\\&\le|\widetilde{v}_l^0|+(1+\tau)^n\max_{1\le s\le n}|\widetilde{y}_l^s|\\&\le|\widetilde{v}_l^0|+e^{n\tau}\max_{1\le s\le n}|\widetilde{y}_l^s|\\&\le|\widetilde{v}_l^0|+e^T\max_{1\le s\le n}|\widetilde{y}_l^s|\end{aligned}$$

So (21) holds according to the mathematical induction method.

By the Parseval's equality, one can deduce that $\|u^n\|_2\le\|u^0\|_2+e^T\max_{1\le s\le n}\|f^s\|_2$. So the finite difference scheme denoted by (11) is unconditionally stable on the initial value and the right source term $f(x,t)$. The proof is complete.

C. Convergence

Theorem 3. The finite difference scheme denoted by (11) is convergent.

Proof. Let $\eta_i^n=U_i^n-u_i^n,\ i=1,2,...,M,\ n=0,1,...,N$ denote the absolute error between the exact solutions and the numerical solutions, and $\eta^n=(\eta_1^n,\ \eta_2^n,...,\ \eta_M^n)^T$. Then $\eta_i^0=0$, and from (10)-(12) one can deduce that

$$[\mu c_0^n-180h^2(b_i'H_1+b_iH_2)]\eta_i^n=\sum_{k=0}^{n-1}B_k^nu_i^k+180h^2R_i^n(\tau,h), \tag{22}$$

where $|R_i^n(\tau,h)|=O(\tau^{2-\alpha}+h^6)$.

Define

$$\theta^n(x)=\begin{cases}\eta_i^n,\ x\in[x_{i-1},x_i),\ i=1,2,...,M-1,\\ \eta_M^n,\ x\in[x_{M-1},x_M],\end{cases}$$

$$r^n(x)=\begin{cases}R_i^n(\tau,h),\ x\in[x_{i-1},x_i),\ i=1,2,...,M-1,\\ R_M^n(\tau,h),\ x\in[x_{M-1},x_M],\end{cases}$$

and a periodic extension is applied to $\theta^n(x),\ \ r^n(x)$. The functions $\theta^n(x)$ and $r^n(x)$ can be denoted by the

Fourier series form as follows

$$\theta^n(x) = \sum_{l=-\infty}^{\infty} \widetilde{\theta}_l^n \exp(\frac{2\pi lxj}{L}),$$
$$r^n(x) = \sum_{l=-\infty}^{\infty} \widetilde{r}_l^n \exp(\frac{2\pi lxj}{L}),$$

where

$$\widetilde{\theta}_l^n = \frac{1}{L}\int_0^L \theta^n(x)\exp(-\frac{2\pi lxj}{L})dx,$$
$$\widetilde{r}_l^n = \frac{1}{L}\int_0^L r^n(x)\exp(-\frac{2\pi lxj}{L})dx.$$

Following in a similar manner as the proof of Theorem 2 one can get that

$$|\widetilde{\theta}_l^n| \le |\widetilde{\theta}_l^0| + e^T \max_{1\le s\le n} |\widetilde{r}_l^s|.$$

By $\eta_i^0 = 0$ we have $\widetilde{\theta}_l^0 = 0$, and then

$$|\widetilde{\theta}_l^n| \le e^T \max_{1\le s\le n} |\widetilde{r}_l^s|.$$

Furthermore,

$$\|\eta^n\|_2 \le e^T \max_{1\le s\le n} \|R^s(\tau,h)\|_2 \le C(\tau^{2-\alpha} + h^6), \qquad (23)$$

where C is a positive constant.

The convergence of the finite difference scheme (11) follows from (23), and the proof is complete.

IV. The high order finite difference scheme for Eq. (3)

In this section, we apply the concept of constructing high order finite difference scheme in Section 2 to a class of time fractional parabolic equation with spatial fourth derivative denoted by Eq. (3), and try to construct unconditionally stable finite difference scheme for it under the conditions (2).

In order to approximate the spatial derivative, the following lemma will be used.

Lemma 4. Suppose $u(x,t) \in C^{(8,2)}([x_{i-3}, x_{i+3}] \times [0,T])$, and define one operator ϕ such that

$$\phi U_i^n = \frac{1}{h^4}(-\frac{1}{6}U_{i-3}^n + 2U_{i-2}^n - \frac{13}{2}U_{i-1}^n + \frac{28}{3}U_i^n - \frac{13}{2}U_{i+1}^n + 2U_{i+2}^n - \frac{1}{6}U_{i+3}^n), \qquad (24)$$

where $U_i^n = u(x_i, t^n)$. Then it holds that

$$|u_{xxxx}(x_i,t^n) - \phi U_i^n| \le \frac{7}{240} \max_{x_{i-3}\le x\le x_{i+3}} |u_x^{(8)}(x,t)| h^4. \qquad (25)$$

The proof of Lemma 4 can be completed by applying the expansion of the Taylor's formula to the right term of Eq. (24).

By use of Lemmas 1 and 4, one has the following observation at the point (i,n)

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[a_0^{(\alpha)}U_i^n - \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})U_i^k - a_{n-1}^{(\alpha)}U_i^0] + \phi U_i^n = f_i^n + R_i^n(\tau,h), \qquad (26)$$

where ϕ is define as in Lemma 1, $f_i^n = f(x_i, t^n)$, and

$$|R_i^n(\tau,h)| \le \frac{1}{\Gamma(2-\alpha)}[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1+2^{-\alpha})] \max_{t_0\le t\le t_n} |u_t^{''}(x,t)|\tau^{2-\alpha} + \frac{7}{240}\max_{0\le x\le L}|u_x^{(8)}(x,t)|h^4.$$

So the finite difference scheme approximating Eq. (3) at the point (i,n) under the conditions (2) can be established as follows

$$\begin{cases} \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[a_0^{(\alpha)}u_i^n - \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})u_i^k - a_{n-1}^{(\alpha)}u_i^0] \\ \quad +\phi u_i^n = f_i^n, \ 1\le n\le N, \\ u_i^0 = \varphi(x_i), \\ u_i^n = u_{i\pm M}^n, \ 1\le i\le M, \ 1\le n\le N. \end{cases} \qquad (27)$$

By use of the Fourier analysis method, similar to the process of Theorems 1-3, we have the following theorems.

Theorem 4. The finite difference scheme denoted by (27) is uniquely solvable.

Theorem 5. For the difference scheme (27), it holds that

$$\|u^n\|_2 \le \|u^0\|_2 + e^T \max_{1\le s\le n} \|f^s\|_2, \ n = 1,2,...,N, \qquad (28)$$

that is, the finite difference scheme denoted by (27) is unconditionally stable on the initial value and the right term $f(x,t)$.

Theorem 6. The finite difference scheme denoted by (27) is convergent with spatial fourth order and temporal $2-\alpha$ order accuracy.

V. Numerical experiments

In this section, we present some numerical examples for the present finite difference schemes. In the following, the maximum error at all grid points is denoted by

$$e(\tau,h) = \max_{1\le n\le N} |U^n - u^n|,$$

and the convergence orders in temporal direction and spatial direction are defined by

$$Rate_\tau = \frac{\ln(e(\tau_1,h)/e(\tau_2,h))}{\ln(\tau_1/\tau_2)},$$
$$Rate_h = \frac{\ln(e(\tau,h_1)/e(\tau,h_2))}{\ln(h_1/h_2)}$$

respectively.

Example 1. Consider the problems (1)-(2) with an exact analytical solution $u(x,t) = (t^3+1)\sin(2\pi x)$, where the period with respect to the variable x is $L = 1$, and satisfies

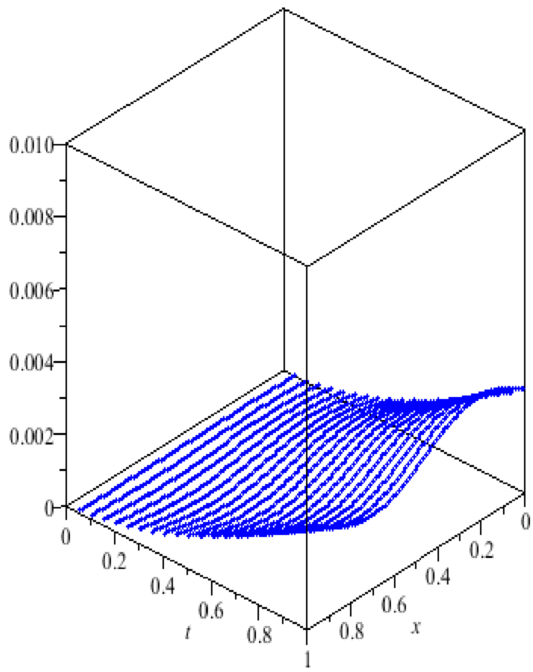

Fig 1. Errors with h=1/80;τ=0.05;α=0.7

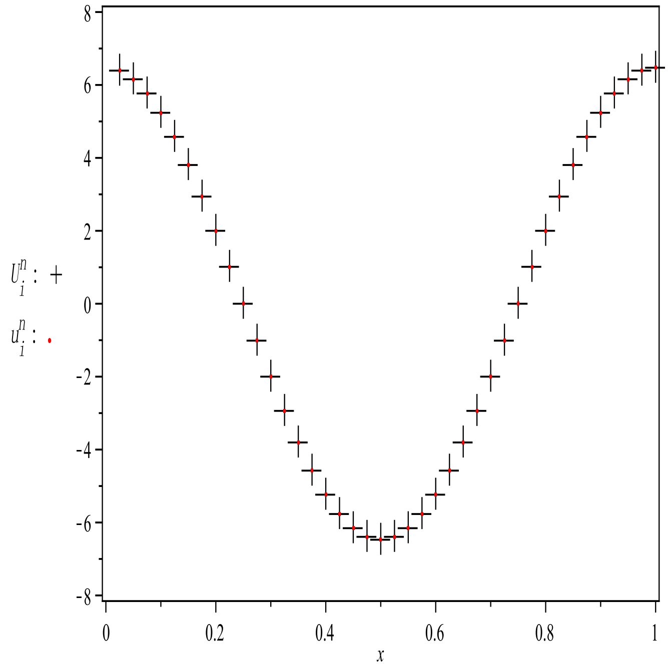

Fig 2. Comparison between exact solutions and numerical solutions with α=0.3,h=1/40,τ=0.1,t=0.3

$$\begin{cases} b(x) = x^2 + 1, \\ u(x,0) = \varphi(x) = \sin(2\pi x), \\ f(x,t) = [\dfrac{\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\alpha} + 4\pi^2(t^3+1)(x^2+1)] \\ \qquad \sin(2\pi x) - 4\pi x(t^3+1)\cos(2\pi x). \end{cases}$$

In Fig. 1, the errors between the exact solutions and the numerical solutions under certain conditions are demonstrated, from which one can see that the errors between the numerical solutions and the exact solutions can be restricted to a low level. Even with the increasing of the time steps, the errors can also be maintained to an accepted level, which coincides with the stability analysis in Section 3.

The maximum errors and convergence orders in spatial direction after five time steps are listed in Tables 1-2.

Table 1: The maximum errors and convergence orders in spatial direction at $\tau = 0.01$

	$\alpha = 0.3$		$\alpha = 0.5$	
h	$e(\tau,h)$	$Rate_h$	$e(\tau,h)$	$Rate_h$
$\frac{1}{10}$	1.245×10^{-4}	-	1.109×10^{-4}	-
$\frac{1}{12}$	4.040×10^{-5}	6.173	3.693×10^{-5}	6.031
$\frac{1}{14}$	1.578×10^{-5}	6.097	1.460×10^{-5}	6.018
$\frac{1}{16}$	7.022×10^{-6}	6.066	6.620×10^{-6}	5.924
$\frac{1}{18}$	3.372×10^{-6}	6.227	3.229×10^{-6}	6.095
$\frac{1}{20}$	1.786×10^{-6}	6.033	1.754×10^{-6}	5.795

Table 2: The maximum errors and convergence orders in spatial direction at $\tau = 0.001$

	$\alpha = 0.3$		$\alpha = 0.5$	
h	$e(\tau,h)$	$Rate_h$	$e(\tau,h)$	$Rate_h$
$\frac{1}{10}$	1.153×10^{-4}	-	8.137×10^{-5}	-
$\frac{1}{12}$	3.820×10^{-5}	6.062	2.764×10^{-5}	5.922
$\frac{1}{14}$	1.506×10^{-5}	6.036	1.082×10^{-5}	6.084
$\frac{1}{16}$	6.798×10^{-6}	5.958	4.964×10^{-6}	5.836
$\frac{1}{18}$	3.289×10^{-6}	6.164	2.442×10^{-6}	6.022
$\frac{1}{20}$	1.760×10^{-6}	5.935	1.296×10^{-6}	6.011

The results in Tables 1-2 show that the convergence orders in spatial direction are about sixth order, which coincide with the theoretical analysis in Section 3.

Example 2. Consider the problems (2)-(3) with an exact analytical solution $u(x,t) = t^{1.7}\cos(2\pi x)$, where the period $L = 1$ with respect to the variable x, and satisfies

$$\begin{cases} u(x,0) = 0, \\ f(x,t) = [\dfrac{1.7\Gamma(1.7)}{\Gamma(2.7-\alpha)} t^{\beta-\alpha} - 16t^{1.7}\pi^4]\cos(2\pi x). \end{cases}$$

The accuracy of the finite difference scheme is checked by comparing the exact solutions and the numerical solutions, which can be seen from the maximum error. Also the convergence orders in both spatial direction and temporal direction are obtained.

In Fig. 2, comparison between the exact solutions and the numerical solutions with different conditions is made, which shows that the numerical solutions can approximate the exact solutions satisfactorily.

The maximum errors and convergence orders in spatial and temporal directions are listed in the following tables respectively, where $t \in [0,1]$ in Table 3, while $t \in [0,3]$ in Table 4.

Table 3: The maximum errors and convergence orders in spatial direction at $\alpha = 0.3,\ \tau = 0.1$

h	$e(\tau,h)$	$Rate_h$
$\frac{1}{8}$	3.152×10^{-3}	-
$\frac{1}{10}$	1.325×10^{-3}	3.88523
$\frac{1}{15}$	2.673×10^{-4}	3.94707
$\frac{1}{20}$	8.385×10^{-5}	4.03054
$\frac{1}{25}$	3.308×10^{-5}	4.16818

Table 4: The maximum errors and convergence orders in temporal direction at $\alpha = 0.3,\ h = 0.1$

τ	$e(\tau, h)$	$Rate_\tau$
0.01	2.641×10^{-5}	-
0.1	1.325×10^{-3}	1.70039
0.5	2.044×10^{-2}	1.70021
0.6	2.787×10^{-2}	1.70016
1	6.642×10^{-2}	1.70014

From the results in Tables 3-4 one can see that the convergence orders are fourth order and $2-\alpha$ order roughly in spatial direction and temporal direction respectively, which coincide with the theoretical analysis in Section 4.

VI. Conclusions

In this paper, a new high order finite difference algorithm for solving a class of time fractional sub-diffusion equation with variable coefficient was developed. The present algorithm is of spatial sixth order and temporal $2-\alpha$ order accuracy. The unconditional stability and convergence for this algorithm were proved by use of the Fourier analysis method. This concept of constructing high order finite difference scheme was applied to a class of time fractional parabolic equation with spatial fourth derivative, and a high order unconditionally stable finite difference scheme for it was also proposed. In order to verify the validity of the present algorithms, numerical experiments were carried out, and the numerical results show their coincidence with the theoretical analysis. Finally, we note that this handling process can be applied to other fractional differential equations to develop corresponding finite difference algorithms with high order.

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