An Iteration Method using Elliptical Arc Artificial Boundary for Exterior Problems

Yajun Chen, and Qikui Du

***Abstract*—In this paper, an iteration method using elliptical arc artificial boundary is designed to solve exterior Poisson problem with a concave angle. It is shown that the iteration method is equivalent to a Schwarz alternating method. The convergence of this method is given. The convergence rate is analyzed in details for a typical domain. Finally, some numerical examples are given, which show the effectiveness of the iteration method.**

***Index Terms*—Schwarz alternating method, elliptical arc artificial boundary, exterior problem.**

I. Introduction

THE problems in unbounded media are encountered in a variety of applications. To solve such problems in infinite region numerically, there is a variety of numerical methods. One commonly method is the method of artificial boundary conditions [1]-[7]. The method may be summarized as follows: (i) Introduce an artificial boundary Γ_μ, which divides the original unbounded domain into two non-overlapping subdomains: a bounded computational domain Ω_i and infinite residual domain Ω_e. (ii) By analyzing the problem in Ω_e, obtain a relation on Γ_μ (exact or approximate) involving the unknown function u and its derivatives. (iii) Using the relation as a boundary condition on Γ_μ, to obtain a well-posed problem in Ω_i. (iv) Solve the problem in Ω_i be the standard finite element methods or some other numerical methods. The relation obtained in Step (ii) and used as a boundary condition in Step (iii) is called an artificial boundary condition.

In the past three decades, artificial boundaries of various shapes have been derived for problems in unbounded domains [8]-[18]. Recently, the authors used a new elliptical arc artificial boundary to solve Poisson problems and anisotropic problems in concave angle domains [19]-[20]. In this paper, we design an iteration method to solve exterior Poisson problem with a concave angle. Using the results of [19]-[20] with an artificial boundary we change the original problem to an equivalent problem in a bounded domain. Then an iteration method is designed to solve the new problem. The convergence of the iteration is obtained by showing that the iteration is actually equivalent to the standard Schwarz alternating method.

Let Ω be an exterior concave angle domain with angle α, and $0 < \alpha \leq 2\pi$. The boundary of domain Ω is decomposed into three disjoint parts: Γ, Γ_0 and Γ_α(see Fig. 1), i.e. $\partial\Omega = \overline{\Gamma \cup \Gamma_0 \cup \Gamma_\alpha}$, $\Gamma_0 \cap \Gamma_\alpha = \emptyset$, $\Gamma \cap \Gamma_0 = \emptyset$, $\Gamma \cap \Gamma_\alpha = \emptyset$. The boundary Γ is a simple smooth curve part, Γ_0 and Γ_α are two half lines.

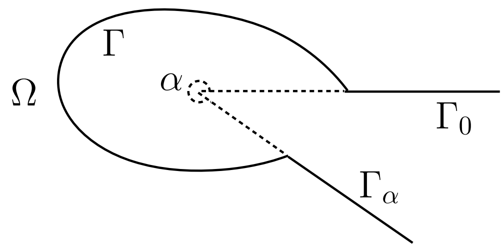

Fig. 1. The Illustration of Domain Ω

We consider the Poisson problem in two cases:

$$\begin{cases} -\Delta u = f, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \Gamma_0 \cup \Gamma_\alpha, \\ \dfrac{\partial u}{\partial n} = g, \quad \text{on } \Gamma, \\ u \text{ is vanish at infinity,} \end{cases} \tag{1}$$

and

$$\begin{cases} -\Delta u = f, \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0, \quad \text{on } \Gamma_0 \cup \Gamma_\alpha, \\ u = h, \quad \text{on } \Gamma, \\ u \text{ is bounded at infinity,} \end{cases} \tag{2}$$

where u is the unknown function, $f \in L^2(\Omega)$ and $g, h \in L^2(\Gamma)$ are given functions, supp(f) is compact.

The rest of the paper is organized as follows. In Section 2, we introduce two elliptical arc artificial boundaries which divide the original domain Ω into two subdomains, then we construct an iteration method which is equivalent to a Schwarz alternating method. In Section 3, we give the convergence of the method . In Section 4, we analyze the convergence rate for a typical domain. In Section 5, we give the discretization of the method. Finally, in Section 6 we present some numerical results, check its accuracy and the effectiveness of this method.

II. Iteration based on the elliptical arc artificial boundary condition

We introduce two elliptical arc artificial boundaries Γ_1 and Γ_2 with the same foci, $\Gamma_i = \{(\mu, \varphi) | \mu = \mu_i, 0 < \varphi < \alpha\}, i = 1, 2$, which enclose Γ such that dist$(\Gamma, \Gamma_2) > 0$ and $\mu_1 > \mu_2 > 0$. Here (μ, φ) denotes the elliptic coordinates, $x = f_0 \cosh\mu \cos\varphi$, $y = f_0 \sinh\mu \sin\varphi$. Then Ω is divided into two overlapping subdomains Ω_1 and Ω_2(see Fig. 2). Let Ω_1 be the bounded domain among Γ, Γ_0, Γ_α and Γ_1, and Ω_2 be the unbounded domain outside Γ_2, Γ_0 and Γ_α. Then the

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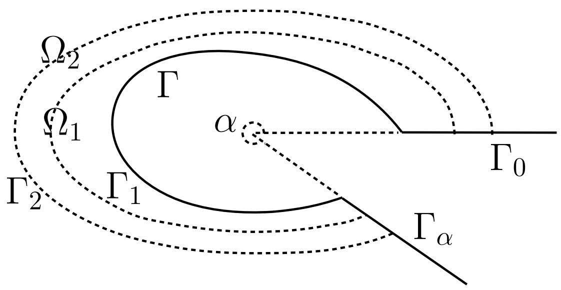

Fig. 2. The Illustration of Domain Ω_1 and Ω_2

original problem (1) is decomposed into two subproblems in domains Ω_1 and Ω_2 with $\Omega_1 \cap \Omega_2 \neq \emptyset$, $\partial\Omega_1 = \Gamma \cup \Gamma_1 \cup \Gamma_{01} \cup \Gamma_{\alpha 1}$, $\partial\Omega_2 = \Gamma_2 \cup \Gamma_{02} \cup \Gamma_{\alpha 2}$. where $\Gamma_{0i} = \overline{\Omega}_i \cap \Gamma_0$, $\Gamma_{\alpha i} = \overline{\Omega}_i \cap \Gamma_\alpha$, $i = 1, 2$. Assume that $f = 0$ in the domain Ω_2.

In the first case, we consider the following problem:

$$\begin{cases} -\Delta u = 0, \quad \text{in } \Omega_2, \\ u = 0, \quad \text{on } \Gamma_{02} \cup \Gamma_{\alpha 2}, \\ u = u_{\mu_2}, \quad \text{on } \Gamma_2, \\ u \text{ is vanish at infinity.} \end{cases} \tag{3}$$

It is well-known that the solution of this problem has the form

$$u(\mu, \varphi) = \sum_{n=1}^{+\infty} b_n e^{(\mu_2 - \mu)\frac{n\pi}{\alpha}} \sin \frac{n\pi\varphi}{\alpha}. \tag{4}$$

From (4) we obtain the coefficients b_n, $n = 1, 2, \ldots$

$$b_n = \frac{2}{\alpha} \int_0^\alpha u_{\mu_2}(\mu_2, \phi) \sin \frac{n\pi\phi}{\alpha} d\phi.$$

Thus (4) can be written as

$$\begin{aligned} & u(\mu, \varphi) \\ & = \frac{2}{\alpha} \sum_{n=1}^{+\infty} e^{(\mu_2 - \mu)\frac{n\pi}{\alpha}} \sin \frac{n\pi\varphi}{\alpha} \int_0^\alpha u_{\mu_2}(\mu_2, \phi) \sin \frac{n\pi\phi}{\alpha} d\phi \\ & \equiv H(u_{\mu_2}, \mu, \varphi). \end{aligned} \tag{5}$$

Using expression (5) we can construct an iteration method for problems (1). From (5) the solution u of (1) restricted on Γ_1 can be expressed as

$$u(\mu_1, \varphi) = H(u_{\mu_2}, \mu_1, \varphi).$$

Then in the domain Ω_1 problems (1) is equivalent to the following problem:

$$\begin{cases} -\Delta u = f, \quad \text{in } \Omega_1, \\ u = 0, \quad \text{on } \Gamma_{01} \cup \Gamma_{\alpha 1}, \\ \dfrac{\partial u}{\partial n} = g, \quad \text{on } \Gamma, \\ u = H(u_{\mu_2}, \mu_1, \varphi). \end{cases}$$

Since u_{μ_2} is unknown, we construct the following iteration method to solve this problem:

$$\begin{cases} -\Delta u_1^{(2k+1)} = f, \quad \text{in } \Omega_1, \\ u_1^{(2k+1)} = 0, \quad \text{on } \Gamma_{01} \cup \Gamma_{\alpha 1}, \\ \dfrac{\partial u_1}{\partial n}^{(2k+1)} = g, \quad \text{on } \Gamma, \\ u_1^{(2k+1)} = H(u_{\mu_2}^{(2k)}, \mu_1, \varphi), \quad \text{on } \Gamma_1, \end{cases} \tag{6}$$

where $u_{\mu_2}^{(2k)} = u^{(2k)}(\mu_2, \varphi)$.

It is not difficult to see that the above iteration method is actually equivalent to the following Schwarz alternating method:

$$\begin{cases} -\Delta u_1^{(2k+1)} = f, \quad \text{in } \Omega_1, \\ u_1^{(2k+1)} = 0, \quad \text{on } \Gamma_{01} \cup \Gamma_{\alpha 1}, \\ \dfrac{\partial u_1}{\partial n}^{(2k+1)} = g, \quad \text{on } \Gamma, \\ u_1^{(2k+1)} = u_2^{(2k)}, \quad \text{on } \Gamma_1, \quad k = 0, 1, \ldots \end{cases} \tag{7}$$

and

$$\begin{cases} -\Delta u_2^{(2k+2)} = f, \quad \text{in } \Omega_2, \\ u_2^{(2k+2)} = 0, \quad \text{on } \Gamma_{02} \cup \Gamma_{\alpha 2}, \\ u_2^{(2k+2)} = u_1^{(2k+1)}, \quad \text{on } \Gamma_2, \\ u_2^{(2k+2)} \text{ is vanish at infinity}, \quad k = 0, 1, \ldots \end{cases} \tag{8}$$

For the second case, we can also construct the following Schwarz alternating method:

$$\begin{cases} -\Delta u_1^{(2k+1)} = f, \quad \text{in } \Omega_1, \\ \dfrac{\partial u_1}{\partial n}^{(2k+1)} = 0, \quad \text{on } \Gamma_{01} \cup \Gamma_{\alpha 1}, \\ u_1^{(2k+1)} = h, \quad \text{on } \Gamma, \\ u_1^{(2k+1)} = u_2^{(2k)}, \quad \text{on } \Gamma_1, \quad k = 0, 1, \ldots \end{cases} \tag{9}$$

and

$$\begin{cases} -\Delta u_2^{(2k+2)} = f, \quad \text{in } \Omega_2, \\ \dfrac{\partial u_2}{\partial n}^{(2k+2)} = 0, \quad \text{on } \Gamma_{02} \cup \Gamma_{\alpha 2}, \\ u_2^{(2k+2)} = u_1^{(2k+1)}, \quad \text{on } \Gamma_2, \\ u_2^{(2k+2)} \text{ is bounded at infinity}, \quad k = 0, 1, \ldots \end{cases} \tag{10}$$

Taking some initial value of function u_0 on boundary Γ_1, e.g. $u|_{\Gamma_1} = 0$. Combining it with the given boundary condition on $\Gamma_{01} \cup \Gamma_{\alpha 1} \cup \Gamma$, we can solve the interior boundary value problem in domain Ω_1, get the value of solution $u_1|_{\Gamma_2}$ on Γ_2, and then solve the exterior boundary value problem in domain Ω_2, get the value of solution $u_2|_{\Gamma_1}$ on Γ_1, and then solve the problem in Ω_1 again, ..., and so on.

In the following sections, we just consider the convergence and convergence rate of problem (1), we can obtain corresponding result of problem (2) in the same way.

III. Convergence of the Method

The solution of problems (1) is in space

$$V = \{v \in W_0^1(\Omega) | v = 0, \text{ on } \Gamma_0 \cup \Gamma_\alpha\},$$

where

$$W_0^1(\Omega) = \{v| \frac{v}{\sqrt{x^2+y^2+1}\ln(x^2+y^2+2)}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \in L^2(\Omega)\}.$$

Functions $u_1^{(2k+1)} \in H(\Omega_1)$ and $u_2^{(2k)} \in W_0^1(\Omega_2)$ can be extended to functions in V. Let

$$V_1 = \{v \in H^1(\Omega_1)|v = 0, \text{ on } \Gamma_{01} \cup \Gamma_{\alpha 1} \cup \Gamma_1\},$$

$$V_2 = \{v \in W_0^1(\Omega_2)|v = 0, \text{ on } \Gamma_{02} \cup \Gamma_{\alpha 2} \cup \Gamma_2\}.$$

Then

$$u_1^{(2k+1)} - u_2^{(2k)} \in V_1, \quad u_2^{(2k+2)} - u_1^{(2k+1)} \in V_2.$$

We can look upon V_1 and V_2 as the subspaces of V. Define the bilinear form as follows

$$a(u,v) = \int_\Omega \nabla u \nabla v dx.$$

From this, the inner product $a(u,v)$ and the norm $\|v\|_1$ in V can be defined. Then (7) and (8) are equivalent to variational problems

$$\begin{cases} \text{Find } u_1^{(2k+1)} \in V_1 + u_2^{(2k)}, \quad \text{such that} \\ a(u_1^{(2k+1)} - u, v_1) = 0, \quad \forall v_1 \in V_1, \end{cases} \tag{11}$$

and

$$\begin{cases} \text{Find } u_2^{(2k+2)} \in V_2 + u_1^{(2k+1)}, \quad \text{such that} \\ a(u_2^{(2k+2)} - u, v_2) = 0, \quad \forall v_2 \in V_2. \end{cases} \tag{12}$$

Let $P_{V_i} : V \to V_i$, $i = 1, 2$ denote the orthogonal projectors under the inner product $a(\cdot,\cdot)$. We have

$$\begin{cases} u_1^{(2k+1)} - u_2^{(2k)} = P_{V_1}(u - u_2^{(2k)}), \\ u_2^{(2k+2)} - u_1^{(2k+1)} = P_{V_2}(u - u_1^{(2k+1)}), \quad k = 0, 1, \ldots, \end{cases} \tag{13}$$

or equivalently

$$\begin{cases} u - u_1^{(2k+1)} = P_{V_1^\perp}(u - u_2^{(2k)}), \\ u - u_2^{(2k+2)} = P_{V_2^\perp}(u - u_1^{(2k+1)}), \quad k = 0, 1, \ldots \end{cases} \tag{14}$$

where $V_i^\perp, i = 1, 2$ are the orthogonal complementary spaces of V_i in V. Let

$$\begin{cases} e_1^{(2k+1)} = u - u_1^{(2k+1)}, \quad k = 0, 1, \ldots, \\ e_2^{(2k)} = u - u_2^{(2k)}, \quad k = 1, 2, \ldots \end{cases}$$

be errors. Then (14) is

$$\begin{cases} e_1^{(2k+1)} = P_{V_1^\perp} e_2^{(2k)}, \quad k = 1, 2, \ldots, \\ e_2^{(2k+2)} = P_{V_2^\perp} e_1^{(2k+1)}, \quad k = 0, 1, \ldots \end{cases}$$

Therefore

$$\begin{cases} e_1^{(2k+1)} = P_{V_1^\perp} P_{V_2^\perp} e_1^{(2k-1)}, \quad k = 1, 2, \ldots, \\ e_2^{(2k+2)} = P_{V_2^\perp} P_{V_1^\perp} e_2^{(2k)}, \quad k = 0, 1, \ldots \end{cases}$$

This implies that, if $\{e_1^{(2k+1)}\}$ and $\{e_2^{(2k)}\}$ are convergent, then their limits are in $V_1^\perp \cap V_2^\perp$. Similar to the proofs given in [21]-[22] we can show the following results

Theorem 1. $\lim\limits_{k\to\infty} \|e_1^{(2k+1)}\|_1 = 0$, $\lim\limits_{k\to\infty} \|e_2^{(2k)}\|_1 = 0$.

Theorem 2. There exists a constant δ, $0 \leqslant \delta < 1$, such that

$$\|e_1^{(2k+1)}\|_1 \leqslant \delta^k \|e_1^{(1)}\|_1, \quad \|e_2^{(2k)}\|_1 \leqslant \delta^k \|e_2^{(0)}\|_1.$$

Theorems 1 and 2 show that the Schwarz alternating method converges geometrically, and the contraction factor is δ. We find it is quite difficult to analyze the rate of convergence δ for general unbounded domain Ω. However, it is possible to find δ when Γ is an elliptical arc, it will be given in next section.

IV. Analysis of convergence rate

For simplicity, we let Γ, Γ_1 and Γ_2 be elliptical arcs with the same foci, $\Gamma = \{(\mu,\varphi)|\mu = \mu_0, 0 < \varphi < \alpha\}$, $\Gamma_i = \{(\mu,\varphi)|\mu = \mu_i, 0 < \varphi < \alpha\}, i = 1, 2$, and $\mu_1 > \mu_2 > \mu_0$. Let

$$e_2^{(0)}(\mu_1,\varphi) = \sum_{n=1}^{+\infty} b_n \sin\frac{n\pi\varphi}{\alpha} \tag{15}$$

is given on the artificial boundary Γ_1 and

$$\frac{\partial e_1^{(1)}}{\partial\mu} = 0, \quad \text{on } \Gamma. \tag{16}$$

And let

$$e_1^{(1)}(\mu,\varphi) = \sum_{n=1}^{+\infty}(A_n e^{\frac{n\pi\mu}{\alpha}} + B_n e^{-\frac{n\pi\mu}{\alpha}})\sin\frac{n\pi\varphi}{\alpha}, \quad \text{in } \Omega_1.$$

From (15) and (16) we have

$$A_n = \frac{b_n e^{-\frac{n\pi\mu_0}{\alpha}}}{e^{\frac{n\pi}{\alpha}(\mu_1-\mu_0)} + e^{\frac{n\pi}{\alpha}(\mu_0-\mu_1)}},$$

$$B_n = \frac{b_n e^{\frac{n\pi\mu_0}{\alpha}}}{e^{\frac{n\pi}{\alpha}(\mu_1-\mu_0)} + e^{\frac{n\pi}{\alpha}(\mu_0-\mu_1)}}.$$

Hence

$$e_1^{(1)}(\mu,\varphi) = \sum_{n=1}^{+\infty}\frac{e^{\frac{n\pi}{\alpha}(\mu-\mu_0)} + e^{\frac{n\pi}{\alpha}(\mu_0-\mu)}}{e^{\frac{n\pi}{\alpha}(\mu_1-\mu_0)} + e^{\frac{n\pi}{\alpha}(\mu_0-\mu_1)}} b_n \sin\frac{n\pi\varphi}{\alpha}.$$

Therefore

$$e_1^{(1)}(\mu_2,\varphi) = \sum_{n=1}^{+\infty}\frac{e^{\frac{n\pi}{\alpha}(\mu_2-\mu_0)} + e^{\frac{n\pi}{\alpha}(\mu_0-\mu_2)}}{e^{\frac{n\pi}{\alpha}(\mu_1-\mu_0)} + e^{\frac{n\pi}{\alpha}(\mu_0-\mu_1)}} b_n \sin\frac{n\pi\varphi}{\alpha}.$$

Using (4), we can obtain the value of function on Γ_1

$$e_2^{(2)}(\mu_1,\varphi) = \sum_{n=1}^{+\infty} e^{\frac{n\pi}{\alpha}(\mu_2-\mu_1)} \frac{e^{\frac{n\pi}{\alpha}(\mu_2-\mu_0)} + e^{\frac{n\pi}{\alpha}(\mu_0-\mu_2)}}{e^{\frac{n\pi}{\alpha}(\mu_1-\mu_0)} + e^{\frac{n\pi}{\alpha}(\mu_0-\mu_1)}} b_n \sin\frac{n\pi\varphi}{\alpha}.$$

So

$$\begin{aligned} &\|e_2^{(2)}\|^2_{\frac{1}{2},\Gamma_1} \\ &= \sum_{n=1}^{+\infty}(1+n^2)^{\frac{1}{2}} |e^{\frac{n\pi}{\alpha}(\mu_2-\mu_1)} \frac{e^{\frac{n\pi}{\alpha}(\mu_2-\mu_0)} + e^{\frac{n\pi}{\alpha}(\mu_0-\mu_2)}}{e^{\frac{n\pi}{\alpha}(\mu_1-\mu_0)} + e^{\frac{n\pi}{\alpha}(\mu_0-\mu_1)}} b_n|^2 \\ &\leqslant \sum_{n=1}^{+\infty}(1+n^2)^{\frac{1}{2}} |e^{\frac{n\pi}{\alpha}(\mu_2-\mu_1)} b_n|^2 \\ &\leqslant e^{\frac{2\pi}{\alpha}(\mu_2-\mu_1)} \sum_{n=1}^{+\infty}(1+n^2)^{\frac{1}{2}}|b_n|^2 \\ &= e^{\frac{2\pi}{\alpha}(\mu_2-\mu_1)} \|e_2^{(0)}\|^2_{\frac{1}{2},\Gamma_1}. \end{aligned}$$

Similarly, we can obtain

$$\|e_1^{(3)}\|^2_{\frac{1}{2},\Gamma_2} \leqslant e^{\frac{2\pi}{\alpha}(\mu_2-\mu_1)}\|e_1^{(1)}\|^2_{\frac{1}{2},\Gamma_2}.$$

Using mathematics induction, we have

$$\|e_2^{(2k)}\|^2_{\frac{1}{2},\Gamma_1} \leqslant e^{\frac{2k\pi}{\alpha}(\mu_2-\mu_1)}\|e_2^{(0)}\|^2_{\frac{1}{2},\Gamma_1}, \quad k=1,2,\ldots,$$

$$\|e_1^{(2k+1)}\|^2_{\frac{1}{2},\Gamma_2} \leqslant e^{\frac{2k\pi}{\alpha}(\mu_2-\mu_1)}\|e_1^{(1)}\|^2_{\frac{1}{2},\Gamma_2}, \quad k=1,2,\ldots$$

Therefore, we have

Theorem 3. Let Γ, Γ_1 and Γ_2 be elliptical arcs with the same foci, $\Gamma = \{(\mu,\varphi)|\mu=\mu_0, 0<\varphi<\alpha\}$, $\Gamma_i = \{(\mu,\varphi)|\mu=\mu_i, 0<\varphi<\alpha\}, i=1,2$, and $\mu_1>\mu_2>\mu_0$. If we apply the Schwarz alternating method given in Section 2 to problem (1), then

$$\|e_2^{(2k)}\|_{\frac{1}{2},\Gamma_1} \leqslant \delta^k\|e_2^{(0)}\|_{\frac{1}{2},\Gamma_1}, \quad k=1,2,\ldots,$$

$$\|e_1^{(2k+1)}\|_{\frac{1}{2},\Gamma_2} \leqslant \delta^k\|e_1^{(1)}\|_{\frac{1}{2},\Gamma_2}, \quad k=1,2,\ldots,$$

where $\delta = e^{\frac{\pi}{\alpha}(\mu_2-\mu_1)}$.

Finally, using the trace theorem we have

$$\|e_2^{(2k)}\|_{1,\Omega_2} \leqslant C\delta^k, \quad k=1,2,\ldots,$$

$$\|e_1^{(2k+1)}\|_{1,\Omega_1} \leqslant C\delta^k, \quad k=1,2,\ldots$$

The smaller the $\mu_2-\mu_1$ is, the faster the convergence is.

V. Discretization

The bounded domain Ω_1 is divided into triangular finite element subdivisions. The subdivisions of Γ_0, Γ_α, Γ_1 and Γ_2 are Γ_{0h}, $\Gamma_{\alpha h}$, Γ_{1h} and Γ_{2h}, respectively. The subdivision of elliptical arc Γ_2 is $\Gamma_{2\varphi}$. Let $S^h(\Omega_{1h})$, $S^h(\Gamma_{1h})$ and $S^h(\Gamma_{2\varphi})$ are finite element function spaces, respectively, in Ω_{1h}, on Γ_{1h} and Γ_{2h}. Then we can obtain discrete Schwarz alternating algorithm as follows:

Step 0. Put any initial data $u_\varphi^0 \in C(\Gamma_1)$, $k:=0$.

Step 1. Find $u_{1h}^{(2k+1)} \in S^h(\Omega_{1h})$, such that

$$\begin{cases} a_h(u_{1h}^{(2k+1)}, v_h) = f_h(v_h), \quad v_h \in S^h(\Omega_{1h}), \\ u_{1h}^{(2k+1)} = 0, \quad \text{on } \Gamma_{01}\cup\Gamma_{\alpha 1}, \\ u_1^{(2k+1)} = \Pi_h u_{2\varphi}^{(2k)}, \quad \text{on } \Gamma_{1h}. \end{cases} \tag{17}$$

Step 2. Find $u_{2\varphi}^{(2k+2)} \in S^h(\Gamma_{2\varphi})$, such that

$$\begin{cases} -\Delta u_{2\varphi}^{(2k+2)} = 0, \quad \text{in } \Omega_2, \\ u_{2\varphi}^{(2k+2)} = 0, \quad \text{on } \Gamma_{02}\cup\Gamma_{\alpha 2}, \\ u_{2\varphi}^{(2k+2)} = \Pi_\varphi u_{1h}^{(2k+1)}, \quad \text{on } \Gamma_{2\varphi}, \\ u_{2\varphi}^{(2k+2)} \text{ is vanish at infinity.} \end{cases} \tag{18}$$

Step 3.

$$\varepsilon^k = \max\{\max_{\text{node}\in\Gamma_{1h}} |u_{1h}^{(2k+1)} - u_{1h}^{(2k-1)}|, \max_{\text{node}\in\Gamma_{2\varphi}} |u_{2\varphi}^{(2k+2)} - u_{2\varphi}^{(2k)}|\}.$$

Step 4. If ε^k is small, stop; else goto Step 1.

where $a_h(u,v) = \int_{\Omega_{1h}} \nabla u\nabla v dx$, $f_h(v) = \int_{\Omega_{1h}} fv dx + \int_{\Gamma_h} gv ds$, $\Pi_h : C(\Gamma_1) \to S^h(\Gamma_{1h})$ is the interpolation operator, $\Pi_\varphi : C(\Gamma_2) \to S^h(\Gamma_{2\varphi})$ is the interpolation operator, too. Here Step 1 is solved by finite element method in Ω_{1h}. Step 2 can be solved by artificial boundary condition.

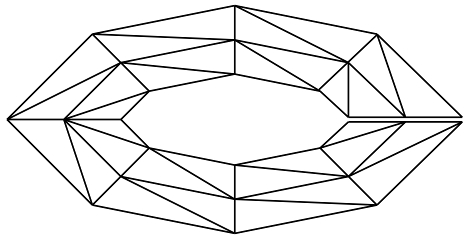

Fig. 3. Mesh h of Subdomain Ω_1 for Example 1

TABLE I
The Relation Between Convergence Rate and Mesh for Example 1 ($\mu_1=3$, $\mu_2=2$)

Mesh	k	1	2	3	4	5
$h/2$	$e(k)$	0.1182	0.0878	0.0746	0.0688	0.0664
	$e_h(k)$	0.1133	0.0491	0.0213	0.0093	0.0040
	$q_h(k)$		2.3049	2.3045	2.3044	2.3044
$h/4$	$e(k)$	0.1018	0.0483	0.0294	0.0227	0.0198
	$e_h(k)$	0.1214	0.0535	0.0236	0.0104	0.0046
	$q_h(k)$		2.2685	2.2685	2.2685	2.2685
$h/8$	$e(k)$	0.0996	0.0450	0.0208	0.0101	0.0069
	$e_h(k)$	0.1235	0.0547	0.0242	0.0107	0.0047
	$q_h(k)$		2.2593	2.2593	2.2593	2.2592

VI. Numerical Examples

In this section, we give two numerical examples to show the effectiveness of the Schwarz alternating method. In these examples, the exact solutions are known. The purpose of showing these examples is to check the convergence in terms of iteration k and mesh size h. The finite element method with liner elements is used in the computation. u_{1h} is the finite element solution in $\overline{\Omega}_1$, e and e_h denote the maximal error of all node functions in $\overline{\Omega}_1$, respectively, i.e.,

$$e(k) = \sup_{P_i\in\overline{\Omega}_1} |u(P_i) - u_{1h}^{2k+1}(P_i)|,$$

$$e_h(k) = \sup_{P_i\in\overline{\Omega}_1} |u_{1h}^{2k-1}(P_i) - u_{1h}^{2k+1}(P_i)|.$$

$q_h(k)$ is the approximation of the convergence rate, i.e.,

$$q_h(k) = \frac{e_h(k-1)}{e_h(k)}.$$

Example 1. We consider problem (1), where $\Omega = \{(\mu,\varphi)|\mu>1, 0<\varphi<2\pi\}$, $\Gamma = \{(1,\varphi)|0<\varphi<2\pi\}$, $\Gamma_0 = \{(\mu,0)|\mu>1\}$, $\Gamma_\alpha = \{(\mu,2\pi)|\mu>1\}$ and $f_0=2$. Let $u(\mu,\varphi) = \frac{\sin\frac{\varphi}{2}}{\cosh\frac{\mu}{2}+\sinh\frac{\mu}{2}}$ be the exact solution of original problem and $g = \frac{\partial u}{\partial n}|_\Gamma$. Let $\Gamma_{\mu_i} = \{(\mu_i,\varphi)|0<\varphi<2\pi\}$, $i=1,2$ be the artificial boundaries. Fig. 3 shows the mesh h of subdomain Ω_1, Table 1 shows the relation between convergence rate and mesh ($\mu_1=3$, $\mu_2=2$), Table 2 shows the relation between convergence rate and overlapping degree (mesh $h/4$, $\mu_1=3$), Fig. 4 shows $L^\infty(\Omega_1)$ errors with iteration k.

Example 2. We consider problem (1), where $\Omega = \{(\mu,\varphi)|\mu>1, 0<\varphi<\frac{3\pi}{2}\}$, $\Gamma = \{(1,\varphi)|0<\varphi<\frac{3\pi}{2}\}$, $\Gamma_0 = \{(\mu,0)|\mu>1\}$, $\Gamma_\alpha = \{(\mu,\frac{3\pi}{2})|\mu>1\}$ and $f_0=2$. Let $u(\mu,\varphi) = \frac{\sin\frac{2\varphi}{3}}{\cosh\frac{2\mu}{3}+\sinh\frac{2\mu}{3}}$ be the exact solution of original problem and $g = \frac{\partial u}{\partial n}|_\Gamma$. Let $\Gamma_{\mu_i} = \{(\mu_i,\varphi)|0<\varphi<2\pi\}$, $i=1,2$ be the artificial boundaries. Fig. 5 shows the

TABLE II
THE RELATION BETWEEN CONVERGENCE RATE AND OVERLAPPING DEGREE FOR EXAMPLE 1 (MESH $h/4$, $\mu_1 = 3$)

μ_2	k	1	2	3	4	5
1.5	$e(k)$	0.0740	0.0310	0.0217	0.0187	0.0178
	$e_h(k)$	0.1492	0.0467	0.0146	0.0046	0.0014
	$q_h(k)$		3.1953	3.1942	3.1949	3.1949
2	$e(k)$	0.1018	0.0483	0.0294	0.0227	0.0198
	$e_h(k)$	0.1214	0.0535	0.0236	0.0104	0.0046
	$q_h(k)$		2.2685	2.2685	2.2685	2.2685
2.5	$e(k)$	0.1476	0.0984	0.0663	0.0454	0.0339
	$e_h(k)$	0.0755	0.0492	0.0321	0.0209	0.0136
	$q_h(k)$		1.5344	1.5343	1.5342	1.5342

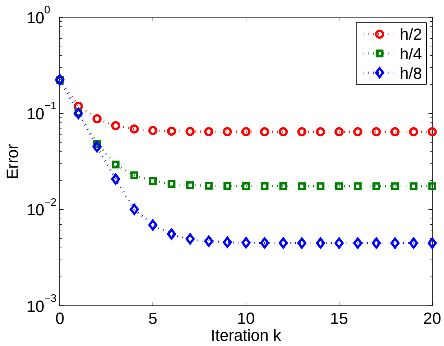

Fig. 4. $L^\infty(\Omega_1)$ Errors with Iteration k for Example 1

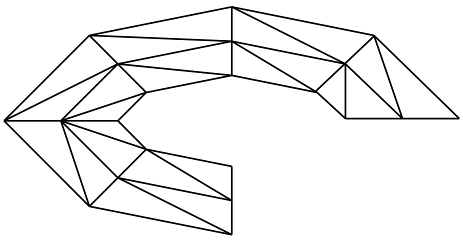

Fig. 5. Mesh h of Subdomain Ω_1 for Example 2

TABLE III
THE RELATION BETWEEN CONVERGENCE RATE AND MESH FOR EXAMPLE 2 ($\mu_1 = 3$, $\mu_2 = 2$)

Mesh	k	1	2	3	4	5
$h/2$	$e(k)$	0.0794	0.0682	0.0648	0.0638	0.0635
	$e_h(k)$	0.0824	0.0250	0.0076	0.0023	0.0007
	$q_h(k)$		3.2922	3.2922	3.2922	3.2922
$h/4$	$e(k)$	0.0450	0.0233	0.0192	0.0180	0.0176
	$e_h(k)$	0.0903	0.0279	0.0086	0.0027	0.0008
	$q_h(k)$		3.2321	3.2321	3.2321	3.2321
$h/8$	$e(k)$	0.0429	0.0141	0.0064	0.0051	0.0047
	$e_h(k)$	0.0925	0.0287	0.0089	0.0028	0.0009
	$q_h(k)$		3.2162	3.2162	3.2162	3.2162

mesh h of subdomain Ω_1, Table 3 shows the relation between convergence rate and mesh ($\mu_1 = 3$, $\mu_2 = 2$), Table 4 shows the relation between convergence rate and overlapping degree (mesh $h/4$, $\mu_1 = 3$), Fig. 6 shows $L^\infty(\Omega_1)$ errors with iteration k.

The numerical results show that this method is feasible and convergent quickly. Its convergence rate is related to the degree of overlapping of subdomains. The higher the

TABLE IV
THE RELATION BETWEEN CONVERGENCE RATE AND OVERLAPPING DEGREE FOR EXAMPLE 2 (MESH $h/4$, $\mu_1 = 3$)

μ_2	k	1	2	3	4	5
1.5	$e(k)$	0.0293	0.0196	0.0178	0.0175	0.0174
	$e_h(k)$	0.1061	0.0201	0.0038	0.0007	0.0001
	$q_h(k)$		5.2825	5.2825	5.2825	5.2825
2	$e(k)$	0.0450	0.0233	0.0192	0.0180	0.0176
	$e_h(k)$	0.0903	0.0279	0.0086	0.0027	0.0008
	$q_h(k)$		3.2321	3.2321	3.2321	3.2321
2.5	$e(k)$	0.0757	0.0432	0.0274	0.0228	0.0204
	$e_h(k)$	0.0597	0.0324	0.0176	0.0096	0.0052
	$q_h(k)$		1.8398	1.8398	1.8398	1.8398

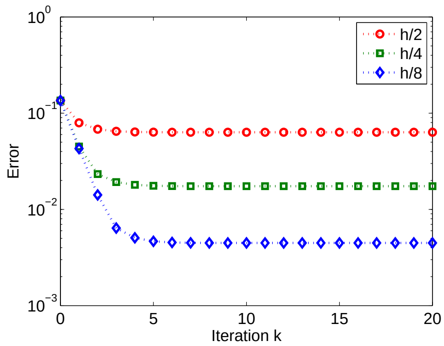

Fig. 6. $L^\infty(\Omega_1)$ Errors with Iteration k for Example 2

overlapping degree of the two subdomains is, the faster the convergence is. Moreover, the convergence rate is nearly not affected by finite element mesh.

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