On Global Generalized Solution for a Generalized Zakharov Equations

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Abstract—**This paper considers the existence of the generalized solution to the initial vale problem for a generalized Zakharov equation in dimension two. By a priori integral estimates and the Galerkin method, one can arrive at the global generalized solution to the problem.**

Index Terms—**generalized solution, generalized Zakharov equations, initial value problem, Zakharov equations.**

I. Introduction

The Zakharov equations, derived by Zakharov in 1972 [1], describes the propagation of Langmuir waves in an unmagnetized plasma. The usual Zakharov system defined in space time $\mathbb{R}^{d+1}$ is given by

$$iE_t + \Delta E = nE,$$
$$n_{tt} - \Delta n = \Delta |E|^2 ,$$

where $E : \mathbb{R}^{d+1} \to \mathbb{C}^d$ is the slowly varying amplitude of the high-frequency electric field, and $n : \mathbb{R}^{d+1} \to \mathbb{R}$ denotes the fluctuation of the ion-density from its equilibrium.

This system attracted the wide interest of many scientists [2]-[10]. In [7], Dem zo. Jahrestag der DDR gewidmet studied the following generalized Zakharov system, and established the global existence for the Cauchy problem.

$$i\varepsilon_t + \varepsilon_{xx} + (\alpha - n)\varepsilon = 0,$$
$$v_t + \left(\frac{1}{2}v^2 - \beta v_x + n + |E|^2\right)_x = 0,$$
$$n_t + v_x = 0.$$

In this paper, we are interested in the following generalized Zakharov system.

$$i\varepsilon_t + \Delta\varepsilon - n\varepsilon = 0, \tag{1}$$

$$v_t + \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\mathrm{grad}\varphi(v) - \Delta v + \nabla\left(n + |E|^2\right) = 0, \tag{2}$$

$$n_t + \nabla \cdot v = 0, \tag{3}$$

with initial data

$$\varepsilon|_{t=0} = \varepsilon_0(x), \quad v|_{t=0} = v_0(x), \quad n|_{t=0} = n_0(x), \tag{4}$$

where $\varepsilon(x,t) = (\varepsilon_1(x,t),\ \varepsilon_2(x,t),\ \cdots,\ \varepsilon_N(x,t))$ is an N-dimensional complex valued unknown functional vector, $v(x,t) = (v_1(x,t),\ v_2(x,t))$ is a 2-dimensional real-valued unknown functional vector, $n(x,t)$ is a real-valued unknown function, $\varphi(s)$ is a real function, and $x \in \mathbb{R}^2$, $t \geq 0$.

We study the generalized Zakharov system in dimension two with the initial data. First, a priori estimates of the problem is made. Next, using the Galerkin method, the global generalized solution of the problem is shown. In fact, nonlinear partial differential equations have also been studied by others using different approaches, as seen in [14]-[39]. The main results of this paper are as follows.

Theorem 1. *Suppose that*

(1) $\varepsilon_0(x) \in H^1(\mathbb{R}^2), v_0(x) \in L^2(\mathbb{R}^2), n_0(x) \in L^2(\mathbb{R}^2)$,

(2) $\varphi(s) \in C^2, \varphi(0) = 0$.

(3) $\|\varepsilon_0(x)\|_{L^2}^2 < \sigma \|\psi(x)\|_{L^2}^2$,

(4) $\left|\mathrm{grad}\varphi(s)\right| \leq C\left(|s| + 1\right)$.

where $0 < \sigma < 1$*,* $\psi(x)$ *is a solution of the equation*

$$\Delta\psi - \psi + \psi^3 = 0.$$

Then there is the global generalized solution of the initial problem (1)-(4)*.*

$$\varepsilon(x,t) \in L^\infty(\mathbb{R}^+; H^1) \cap W^{1,\infty}(\mathbb{R}^+; H^{-1}),$$
$$v(x,t) \in L^\infty(\mathbb{R}^+; L^2) \cap W^{1,\infty}(\mathbb{R}^+; H^{-2}),$$
$$n(x,t) \in L^\infty(\mathbb{R}^+; L^2) \cap W^{1,\infty}(\mathbb{R}^+; H^{-1}),$$

For the sake of convenience of the following contexts, we set some notations. For $1 \leq q \leq \infty$, we denote $L^q(\mathbb{R}^d)$ the space of all q times integrable functions in $\mathbb{R}^d$ equipped with norm $\|\cdot\|_{L^q(\mathbb{R}^d)}$ or simply $\|\cdot\|_{L^q}$ and $H^{s,p}(\mathbb{R}^d)$ the Sobolev space with norm $\|\cdot\|_{H^{s,p}(\mathbb{R}^d)}$. If $p = 2$, we write $H^s(\mathbb{R}^d)$ instead of $H^{s,2}(\mathbb{R}^d)$. Let $(f,g) = \int_{\mathbb{R}^n} f(x)\cdot\overline{g(x)}dx$, where $\overline{g(x)}$ denotes the complex conjugate function of $g(x)$. We use C to represent various constants that can depend on the initial data.

This paper is organized as follows. In Section II, we make a priori estimates of the problem (1)-(4). In Section III, we establish the global generalized solution of the problem (1)-(4) by the Galerkin method.

II. A PRIORI ESTIMATES

In this section, we will derive a priori estimates for the solution of the system (1)-(4).

Lemma 1. *Suppose that* $\varepsilon_0(x) \in L^2(\mathbb{R}^2)$*. Then for the solution of problem* (1)-(4)*, we have*

$$\|\varepsilon(x,t)\|_{L^2(\mathbb{R}^2)}^2 = \|\varepsilon_0(x)\|_{L^2(\mathbb{R}^2)}^2 .$$

Proof: Taking the inner product of (1) and ε, it follows that

$$(i\varepsilon_t + \Delta\varepsilon - n\varepsilon,\ \varepsilon) = 0. \tag{5}$$

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and since

$$\mathrm{Im}\,(i\varepsilon_t,\,\varepsilon) = \frac{1}{2}\frac{d}{dt}\,\|\varepsilon\|_{L^2}^2\,,$$
$$\mathrm{Im}\,(\Delta\varepsilon - n\varepsilon,\,\varepsilon) = 0,$$

hence from (5), we get

$$\frac{d}{dt}\,\|\varepsilon(x,t)\|_{L^2}^2 = 0.$$

We thus get Lemma 1. ■

Lemma 2. *Supposing that*

(1) $\varepsilon_0(x) \in H^1(\mathbb{R}^2)$, $v_0(x) \in L^2(\mathbb{R}^2)$, $n_0(x) \in L^2(\mathbb{R}^2)$,
(2) $\varphi(s) \in C^2$, $\varphi(0) = 0$.

Then we have

$$\mathscr{F}(t) + \int_0^t \|\nabla v(x,\tau)\|_{L^2}^2 d\tau = \mathscr{F}(0).$$

where

$$\mathscr{F}(t) = \frac{1}{2}\,\|v\|_{L^2}^2 + \frac{1}{2}\,\|n\|_{L^2}^2 + \|\nabla\varepsilon\|_{L^2}^2 + \int_{\mathbb{R}^2} n\,|\varepsilon|^2\,dx.$$

Proof: Taking the inner products of (2) and v, it follows that

$$\left(v_t + \sum_{j=1}^{2}\frac{\partial}{\partial x_j}\mathrm{grad}\varphi(v) - \Delta v + \nabla\left(n + |E|^2\right),\, v\right) = 0. \quad (6)$$

And since

$$(v_t,\,v) = \frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2, \quad (-\Delta v,\,v) = \|\nabla v\|_{L^2}^2,$$

$$\left(\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\mathrm{grad}\varphi(v),\,v\right) = -\sum_{j=1}^{2}\left(\mathrm{grad}\varphi(v),\,\frac{\partial v}{\partial x_j}\right)$$
$$= -\sum_{j=1}^{2}\left((\mathrm{grad}\varphi(v))_{x_j},\,1\right) = 0,$$

$$(\nabla n,\,v) = -(n,\,\nabla\cdot v) = (n,\,n_t) = \frac{1}{2}\frac{d}{dt}\|n\|_{L^2}^2,$$

$$\left(\nabla|\varepsilon|^2,\,v\right) = -\left(|\varepsilon|^2,\,\nabla\cdot v\right)$$
$$= \left(|\varepsilon|^2,\,n_t\right) = \int_{\mathbb{R}^2} n_t|\varepsilon|^2dx,$$

thus from (6) it follows that

$$\frac{1}{2}\frac{d}{dt}\left(\|v\|_{L^2}^2 + \|n\|_{L^2}^2\right) + \|\nabla v\|_{L^2}^2 + \int_{\mathbb{R}^2} n_t|\varepsilon|^2dx = 0. \quad (7)$$

Taking the inner products of (1) and $-\varepsilon_t$, it follows that

$$(i\varepsilon_t + \Delta\varepsilon - n\varepsilon,\,-\varepsilon_t) = 0. \quad (8)$$

And since

$$\mathrm{Re}\,(i\varepsilon_t,\,-\varepsilon_t) = 0,$$
$$\mathrm{Re}\,(\Delta\varepsilon,\,-\varepsilon_t) = \mathrm{Re}\,(\nabla\varepsilon,\,\nabla\varepsilon_t) = \frac{1}{2}\frac{d}{dt}\,\|\nabla\varepsilon\|_{L^2}^2\,,$$

$$\mathrm{Re}\,(-n\varepsilon,\,-\varepsilon_t) = \frac{1}{2}\int_{\mathbb{R}^2} n\left(|\varepsilon|^2\right)_t dx$$
$$= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} n|\varepsilon|^2dx - \frac{1}{2}\int_{\mathbb{R}^2} n_t|\varepsilon|^2dx.$$

Thus from (8) it follows that

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla\varepsilon\|_{L^2}^2 + \int_{\mathbb{R}^2} n|\varepsilon|^2dx\right) - \frac{1}{2}\int_{\mathbb{R}^2} n_t|\varepsilon|^2dx = 0. \quad (9)$$

Hence from (7) and (9), we get

$$\frac{d}{dt}\left(\frac{1}{2}\,\|v\|_{L^2}^2 + \frac{1}{2}\,\|n\|_{L^2}^2 + \|\nabla\varepsilon\|_{L^2}^2 + \int_{\mathbb{R}^2} n|\varepsilon|^2dx\right) + \|\nabla v\|_{L^2}^2 = 0.$$

Letting

$$\mathscr{F}(t) = \frac{1}{2}\,\|v\|_{L^2}^2 + \frac{1}{2}\,\|n\|_{L^2}^2 + \|\nabla\varepsilon\|_{L^2}^2 + \int_{\mathbb{R}^2} n|\varepsilon|^2dx.$$

It follows that

$$\mathscr{F}(t) + \int_0^t \|\nabla v(x,\tau)\|_{L^2}^2 d\tau = \mathscr{F}(0).$$

■

Lemma 3 (Gagliardo-Nirenberg inequality [11]). *Assume that $u \in L^q(\mathbb{R}^n)$, $D^m u \in L^r(\mathbb{R}^n)$, $1 \le q, r \le \infty, 0 \le j \le m$, we have the estimations*

$$\|D^j u\|_{L^p(\mathbb{R}^n)} \le C\|D^m u\|_{L^r(\mathbb{R}^n)}^{\alpha}\|u\|_{L^q(\mathbb{R}^n)}^{1-\alpha},$$

where C is a positive constant, $0 \le \frac{j}{m} \le \alpha \le 1$,

$$\frac{1}{p} = \frac{j}{n} + \alpha\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\alpha)\frac{1}{q}.$$

Lemma 4 (Sobolev's best constant estimates [12]). *Suppose that $f(x) \in H^1(\mathbb{R}^N)$. Then we have*

$$\|f\|_{L^{2p+2}(\mathbb{R}^N)}^{2p+2} \le C_{p,N}^{2p+2}\|\nabla f\|_{L^2(\mathbb{R}^N)}^{pN}\|f\|_{L^2(\mathbb{R}^N)}^{2+p(2-N)},$$
$$0 < p < \frac{2}{N-2},\ N \ge 2,$$

where the constant

$$C_{p,N} = \left(\frac{p+1}{\|\psi\|_{L^2(\mathbb{R}^N)}^{2p}}\right)^{\frac{1}{2p+2}}$$

and $\psi(x)$ is a ground state solution for the equation

$$\frac{pN}{2}\Delta\psi - \left(1 + \frac{p}{2}(2-N)\right)\psi + \psi^{2p+1} = 0. \quad (10)$$

Obviously, the solution of equation (10) exists, and $\psi(x) \ne 0$.

Lemma 5. *Supposing that the conditions of Lemma 2 are satisfied and*

$$\|\varepsilon_0(x)\|_{L^2}^2 < \sigma\,\|\psi(x)\|_{L^2}^2\,,$$

where $0 < \sigma < 1$, ψ is a solution of the equation

$$\Delta\psi - \psi + \psi^3 = 0.$$

Then we have

$$\|\nabla\varepsilon\|_2^2 + \|n\|_2^2 + \|v\|_2^2 + \int_0^t \|\nabla v(x,\tau)\|_{L^2}^2 d\tau \le C.$$

Proof: By Hölder inequality and Young inequality, there holds

$$\left|\int_{\mathbb{R}^2} n\,|\varepsilon|^2\,dx\right| \le \|n\|_{L^2}\|\varepsilon\|_{L^4}^2$$
$$\le \frac{\sigma}{2}\|n\|_{L^2}^2 + \frac{1}{2\sigma}\|\varepsilon\|_{L^4}^4. \quad (11)$$

Using Gagliardo-Nirenberg inequality and Lemma 4, we write

$$\|\varepsilon\|_{L^4}^4 \le \frac{2}{\|\psi\|_{L^2}^2}\|\nabla\varepsilon\|_{L^2}^2\|\varepsilon\|_{L^2}^2. \quad (12)$$

Note that Lemma 2 and Equations (11), (12), one has

$$\frac{1}{2}\|v\|_{L^2}^2 + \frac{1-\sigma}{2}\|n\|_{L^2}^2 + \left(1 - \frac{\|\varepsilon_0\|_{L^2}^2}{\sigma\|\psi\|_{L^2}^2}\right)\|\nabla\varepsilon\|_{L^2}^2 + \int_0^t \|\nabla v(x,\tau)\|_{L^2}^2 d\tau \le |\mathscr{F}(0)|.$$

Note that $\|\varepsilon_0(x)\|_{L^2}^2 < \sigma\|\psi(x)\|_{L^2}^2$ and $0 < \sigma < 1$, we thus get Lemma 5. ■

Lemma 6. *Supposing that the conditions of Lemma 5 are satisfied and* $\left|\mathrm{grad}\varphi(s)\right| \le C(|s|+1)$. *Then we have*

$$\|\varepsilon_t\|_{H^{-1}} + \|v_t\|_{H^{-2}} + \|n_t\|_{H^{-1}} \le C.$$

Proof: Taking the inner product of Eq. (1) and Φ, Eq. (2) and Γ, Eq. (3) and η, it follows that

$$(i\varepsilon_t + \Delta\varepsilon - n\varepsilon,\ \Phi) = 0, \tag{13}$$

$$\left(v_t + \sum_{j=1}^{2}\frac{\partial}{\partial x_j}\mathrm{grad}\varphi(v) - \Delta v + \nabla\left(n + |E|^2\right),\ \Gamma\right) = 0, \tag{14}$$

$$(n_t + \nabla\cdot v,\ \eta) = 0, \tag{15}$$

where $\eta, \eta_j, \zeta_k \in H_0^2$ $(j = 1,\cdots,N,\ k = 1,2)$, $\Phi = (\eta_1,\cdots,\eta_N)$, $\Gamma = (\zeta_1,\zeta_2)$.

By Hölder inequality, it follows from Eq. (13) that

$$\begin{aligned}|(\varepsilon_t,\Phi)| &\le |(\Delta\varepsilon,\Phi)| + |(n\varepsilon,\Phi)| \\ &= |(\nabla\varepsilon,\nabla\Phi)| + |(n\varepsilon,\Phi)| \\ &\le \|\nabla\varepsilon\|_{L^2}\|\nabla\Phi\|_{L^2} + \|n\|_{L^2}\|\varepsilon\|_{L^4}\|\Phi\|_{L^4}.\end{aligned} \tag{16}$$

By Gagliardo-Nirenberg inequality, we know that

$$\|\varepsilon\|_{L^4} \le C\|\nabla\varepsilon\|_{L^2}^{\frac{1}{2}}\|\varepsilon\|_{L^2}^{\frac{1}{2}} \le C, \tag{17}$$

$$\|\Phi\|_{L^4} \le C\|\nabla\Phi\|_{L^2}^{\frac{1}{2}}\|\Phi\|_{L^2}^{\frac{1}{2}} \le C\left(\|\nabla\Phi\|_{L^2} + \|\Phi\|_{L^2}\right).$$

Hence from Eq. (16) we get

$$|(\varepsilon_t,\Phi)| \le C\|\Phi\|_{H_0^1}. \tag{18}$$

Using Hölder inequality, from Eq. (14), there is

$$\begin{aligned}|(v_t,\ \Gamma)| &\le \left|\left(\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\mathrm{grad}\varphi(v),\ \Gamma\right)\right| + |(\Delta v,\ \Gamma)| \\ &\quad + \left|\left(\nabla\left(n + |E|^2\right),\ \Gamma\right)\right| \\ &= \left|\sum_{j=1}^{2}\left(\mathrm{grad}\varphi(v),\ \frac{\partial\Gamma}{\partial x_j}\right)\right| + |(v,\ \Delta\Gamma)| \\ &\quad + \left|\left(n + |E|^2,\ \nabla\cdot\Gamma\right)\right| \\ &\le C\left(\|v\|_{L^2} + 1\right)\|\Gamma\|_{H_0^1} + \|v\|_{L^2}\|\Delta\Gamma\|_{L^2} \\ &\quad + \|n\|_{L^2}\|\nabla\cdot\Gamma\|_{L^2} + \|\varepsilon\|_{L^4}^2\|\nabla\cdot\Gamma\|_{L^2}.\end{aligned} \tag{19}$$

From Eq. (17) and (19) we get

$$|(v_t,\ \Gamma)| \le C\|\Gamma\|_{H_0^2}. \tag{20}$$

From Eq. (15) and Hölder inequality, we have

$$\begin{aligned}|(n_t,\eta)| &= |(\nabla\cdot v,\eta)| = |(v,\nabla\eta)| \\ &\le \|v\|_{L^2}\|\nabla\eta\|_{L^2} \le C\|\eta\|_{H_0^1}.\end{aligned} \tag{21}$$

Hence from (18), (20) and (21), we obtain

$$\|\varepsilon_t\|_{H^{-1}} + \|v_t\|_{H^{-2}} + \|n_t\|_{H^{-1}} \le C.$$

■

III. THE EXISTENCE OF GLOBAL GENERALIZED SOLUTION

In this section, we formulate the proof of Theorem 1. First we give the definition of generalized solution for problems (1)-(4).

Definition 1. *The functions*

$$\varepsilon_m(x,t) \in L^\infty(\mathbb{R}^+;H^1) \cap W^{1,\infty}(\mathbb{R}^+;H^{-1}), \quad m = 1,2,\cdots,N,$$
$$v_\lambda(x,t) \in L^\infty(\mathbb{R}^+;L^2) \cap W^{1,\infty}(\mathbb{R}^+;H^{-2}), \quad \lambda = 1,2,$$
$$n(x,t) \in L^\infty(\mathbb{R}^+;L^2) \cap W^{1,\infty}(\mathbb{R}^+;H^{-1})$$

are called generalized solution of problems (1)-(4), *if for any* $\xi \in H_0^2$ *they satisfy the integral equality*

$$(i\varepsilon_{mt},\xi) - (\nabla\varepsilon_m,\nabla\xi) - (n\varepsilon_m,\xi) = 0,$$
$$(v_{\lambda t},\xi) - \sum_{j=1}^{2}\left(\frac{\partial\varphi(v)}{\partial v_\lambda},\frac{\partial\xi}{\partial x_j}\right) - (v_\lambda,\Delta\xi) - \left(n + |\varepsilon|^2,\frac{\partial\xi}{\partial x_\lambda}\right) = 0,$$
$$(n_t,\xi) - (v,\nabla\xi) = 0,$$
$$m = 1,2,\cdots,N, \quad \lambda = 1,2.$$

with initial data

$$\varepsilon|_{t=0} = \varepsilon_0(x), \quad n|_{t=0} = n_0(x), \quad v|_{t=0} = v_0(x),$$

Next, we give two lemmas recalled in [13].

Lemma 7. *Let* B_0, B, B_1 *be three reflexive Banach spaces and assume that the embedding* $B_0 \to B$ *is compact. Let*

$$W = \left\{V \in L^{p_0}((0,T);B_0),\ \frac{\partial V}{\partial t} \in L^{p_1}((0,T);B_1)\right\},$$
$$T < \infty,\ 1 < p_0, p_1 < \infty.$$

W is a Banach space with norm

$$\|V\|_W = \|V\|_{L^{p_0}((0,T);B_0)} + \|V_t\|_{L^{p_1}((0,T);B_1)}.$$

Then the embedding $W \to L^{p_0}((0,T);B)$ *is compact.*

Lemma 8. *Let* Ω *be an open set of* $\mathbb{R}^n$ *and let* $g, g_\varepsilon \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, *such that*

$$g_\varepsilon \to g \quad \text{a.e. in } \Omega \qquad \text{and} \qquad \|g_\varepsilon\|_{L^p(\Omega)} \le C.$$

Then $g_\varepsilon \to g$ *weakly in* $L^p(\Omega)$.

Now, one can estimate Theorem 1.

Proof: By using the Galerkin method, choose the basic periodic functions $\{\omega_j(x)\}$ as follows:

$$-\Delta\omega_j(x) = \lambda_j\omega_j(x),\ \omega_j(x) \in H_0^2(\Omega),\ j = 1,2,\cdots,l.$$

The approximate solution of problem (1)-(4) can be written as

$$\varepsilon^l(x,t) = \sum_{j=1}^{l}\alpha_j^l(t)\omega_j(x), \quad v^l(x,t) = \sum_{j=1}^{l}\beta_j^l(t)\omega_j(x),$$
$$n^l(x,t) = \sum_{j=1}^{l}\gamma_j^l(t)\omega_j(x),$$

where

$$\varepsilon^l = \left(\varepsilon_1^l,\cdots,\varepsilon_N^l\right), \quad \alpha_j^l(t) = \left(\alpha_{j1}^l(t),\cdots,\alpha_{jN}^l(t)\right),$$
$$v^l = \left(v_1^l,v_2^l\right), \quad \beta_j^l(t) = \left(\beta_{j1}^l(t),\beta_{j2}^l(t)\right).$$

and Ω is a 2-dimensional cube with $2D$ in each direction, that is, $\overline{\Omega} = \{x = (x_1, x_2) \,|\, |x_i| \le 2D, i = 1, 2\}$. According to Galerkin's method, these undetermined coefficients $\alpha_j^l(t)$, $\beta_j^l(t)$ and $\gamma_j^l(t)$ must satisfy the following initial value problem of the system of ordinary differential equations.

$$\left(i\varepsilon_{mt}^l, \omega_\kappa\right) - \left(\nabla\varepsilon_m^l, \nabla\omega_\kappa\right) - \left(n^l\varepsilon_m^l, \omega_\kappa\right) = 0, \quad (22)$$

$$\left(v_{\lambda t}^l, \omega_\kappa\right) - \sum_{j=1}^{2}\left(\frac{\partial\varphi(v^l)}{\partial v_\lambda^l}, \frac{\partial\omega_\kappa}{\partial x_j}\right) - \left(v_\lambda^l, \Delta\omega_\kappa\right) - \left(n^l + \left|\varepsilon^l\right|^2, \frac{\partial\omega_\kappa}{\partial x_\lambda}\right) = 0, \quad (23)$$

$$\left(n_t^l, \omega_\kappa\right) - \left(v^l, \nabla\omega_\kappa\right) = 0, \quad (24)$$

$$m = 1, 2, \cdots, N, \quad \lambda = 1, 2, \quad \kappa = 1, 2, \cdots, l.$$

with initial data

$$\varepsilon^l|_{t=0} = \varepsilon_0(x), \quad n^l|_{t=0} = n_0(x), \quad v^l|_{t=0} = v_0(x), \quad (25)$$

Suppose

$$\varepsilon_0^l(x) \xrightarrow{H^1} \varepsilon_0(x), \quad v_0^l(x) \xrightarrow{L^2} v_0(x),$$
$$n_0^l(x) \xrightarrow{L^2} n_0(x), \quad l \to \infty.$$

Similar to the proof of Lemma 1-6, for the solution $\varepsilon^l(x,t)$, $v^l(x,t)$ and $n^l(x,t)$ of problem (22)-(25), we can establish the following estimations.

$$\left\|\varepsilon^l\right\|_{H^1} + \left\|v^l\right\|_{L^2} + \left\|n^l\right\|_{L^2} \le C \quad (26)$$

$$\left\|\varepsilon_t^l\right\|_{H^{-1}} + \left\|v_t^l\right\|_{H^{-2}} + \left\|n_t^l\right\|_{H^{-1}} \le C \quad (27)$$

where the constant C is independent of l and D. By compact argument, some subsequence of $\left(\varepsilon^l, v^l, n^l\right)$, also labeled as l, has a weak limit (ε, v, n). More precisely,

$$\varepsilon^l(x,t) \to \varepsilon(x,t) \text{ in } L^\infty(\mathbb{R}^+; H^1) \text{ weakly star}, \quad (28)$$

$$v^l(x,t) \to v(x,t) \text{ in } L^\infty(\mathbb{R}^+; L^2) \text{ weakly star}, \quad (29)$$

$$n^l(x,t) \to n(x,t) \text{ in } L^\infty(\mathbb{R}^+; L^2) \text{ weakly star}. \quad (30)$$

Eq. (27) implies that

$$\varepsilon_t^l \to \varepsilon_t \text{ in } L^\infty(\mathbb{R}^+; H^{-1}) \text{ weakly star}, \quad (31)$$
$$v_t^l \to v_t \text{ in } L^\infty(\mathbb{R}^+; H^{-2}) \text{ weakly star},$$
$$n_t^l \to n_t \text{ in } L^\infty(\mathbb{R}^+; H^{-1}) \text{ weakly star}.$$

Moreover, it should be noted that the following maps are continuous.

$$H^1(\mathbb{R}^2) \to L^4(\mathbb{R}^2), \quad u \mapsto u,$$
$$H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \quad (u,v) \mapsto uv.$$

It then follows from Eq. (28) and (30) that

$$\left|\varepsilon^l\right|^2 \to w \text{ in } L^\infty(\mathbb{R}^+; L^2) \text{ weakly star}, \quad (32)$$

$$n^l\varepsilon^l \to z \text{ in } L^\infty(\mathbb{R}^+; L^2) \text{ weakly star}. \quad (33)$$

First, we prove $w = |\varepsilon|^2$. Let Ω be any bounded subdomain of $\mathbb{R}^2$. We notice that

$$\text{the embedding } H^1(\Omega) \to L^4(\Omega) \text{ is compact}.$$

and for any Banach space X,

$$\text{the embedding } L^\infty(\mathbb{R}^+; X) \to L^2(0, T; X) \text{ is continuous}.$$

Hence, according to Eq. (28), (32) and Lemma 7, applied to $B_0 = H^1(\Omega), B = L^4(\Omega), B_1 = H^{-1}(\Omega)$, and says that some subsequence of $\varepsilon^l|_\Omega$ (also referred to as l) converges strongly to $\varepsilon|_\Omega$ in $L^2(0, T; L^4(\Omega))$. So we can assume that

$$\varepsilon^l \to \varepsilon \text{ strongly in } L^2(0, T; L^4_{loc}(\Omega)), \quad (34)$$

and thus

$$\varepsilon^l \to \varepsilon \text{ a.e. in } [0, T] \times \Omega.$$

Then, using Lemma 8 and Eq. (32) imply that $w = |\varepsilon|^2$

Second, we prove $z = n\varepsilon$. Let χ be a test function in $L^2(0, T; H^1)$, $\text{supp}\chi \subset \Omega \subset \mathbb{R}^2$.

$$\int_0^T \int_{\mathbb{R}^2} \left(n^l\varepsilon^l - n\varepsilon\right)\chi dxdt = \int_0^T \int_\Omega n^l\left(\varepsilon^l - \varepsilon\right)\chi dxdt + \int_0^T \int_\Omega \left(n^l - n\right)\varepsilon\chi dxdt.$$

Firstly,

$$\left|\int_0^T \int_\Omega n^l\left(\varepsilon^l - \varepsilon\right)\chi dxdt\right| \le \left\|n^l\right\|_{L^\infty(0,T;L^2(\Omega))} \left\|\varepsilon^l - \varepsilon\right\|_{L^2(0,T;L^4(\Omega))} \|\chi\|_{L^2(0,T;L^4(\Omega))},$$

Since Ω is bounded, we deduce from Eq. (30) and (34) that

$$\left|\int_0^T \int_\Omega n^l\left(\varepsilon^l - \varepsilon\right)\chi dxdt\right| \to 0 \quad (l \to +\infty).$$

Secondly, let us note that $\varepsilon\chi \in L^1(0, T; L^2)$. In fact,

$$\|\varepsilon\chi\|_{L^1(0,T;L^2)} \le \|\varepsilon\|_{L^2(0,T;L^4)} \|\chi\|_{L^2(0,T;L^4)} < \infty.$$

Therefore we deduce from Eq. (30) that

$$\int_0^T \int_\Omega \left(n^l - n\right)\varepsilon\chi dxdt \to 0 \quad (l \to +\infty).$$

Thus $n^l\varepsilon^l \to n\varepsilon$ in $L^2(o, T; H^{-1})$. So $z = n\varepsilon$.

Hence taking $l \to \infty$ from Eq. (22)-(25), by using the density of ω_j in $H_0^2(\Omega)$ we arrive at the local generalized solution for the periodic initial value problem (1)-(4); letting $D \to \infty$, the local solution for the initial value problem (1)-(4) can be obtained. By the continuation extension principle and a priori estimates, we can arrive at the global generalized solution for problem (1)-(4).

We thus complete the proof of Theorem 1. ■

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