# A Rapid Iterative Algorithm for Solving Split Variational Inclusion Problems and Fixed Point Problems

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***Abstract*—In this study, we introduce a rapid iterative algorithm to find a common element of the solution set of split variational inclusion problems and the set of fixed points of a nonexpansive mapping by using the hybrid steepest descent method. The strong convergence results of presented algorithms have been obtained under some mild conditions. The proposed results are the supplement, extension and generalization of the previously known results in this area. Finally, preliminary numerical results indicate the feasibility and efficiency of the proposed methods.**

***Index Terms*—Split variational inclusion problem, Fixed point problem, Nonexpansive mappings**

## I. Introduction

In 2011, A. Moudafi [1] first introduced the split monotone variational inclusion problem(SMVIP) as follow: Find a point $x^* \in H_1$ such that

$$0 \in f_1(x^*) + B_1(x^*) \tag{1}$$

and

$$y^* = Ax^* \in H_2 \text{ solves } 0 \in f_2(y^*) + B_2(y^*) \tag{2}$$

where $H_1$ and $H_2$ are two real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$, respectively. The mappings $B_1: H_1 \to 2^{H_1}$ and $B_2: H_2 \to 2^{H_2}$ are multi-valued maximal mappings.

A. Moudafi [1] revealed that SMVIP (1)-(2) included the split common fixed point problem, the split zero problem, the split variational inequality problem and split feasibility problem [1-8] as special cases, which have wide applications to intensity modulated radiation therapy treatment planning, see [6,7, 9,10].

If $f_1 \equiv 0$ and $f_2 \equiv 0$, then SMVIP (1)-(2) degenerate exactly into the split variational inclusion problem (SVIP): Look for $x^* \in H_1$ such that

$$0 \in B_1(x^*) \tag{3}$$

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and

$$y^* = Ax^* \in H_2 \text{ solves } 0 \in B_2(y^*) \tag{4}$$

We denote the solution set $\Gamma$ of SVIP (3)-(4) by $\Gamma = \{x^* \in H_1 : x^* \in \mathrm{SOLVIP}(B_1) \text{and } Ax^* \in \mathrm{SOLVIP}(B_2).\}$

Many works were devoted to SVIP (3)-(4) [4, 12, 14, 24, 25, 26, 27, 28]. In 2012, C. Byrne et al. [4] revealed the weak and strong convergence of the iterative method

$$x_{n+1} = J_\lambda^{B_1}\left(x_n + \gamma A^*\left(J_\lambda^{B_2} - I\right)Ax_n\right) \tag{5}$$

where $A^*$ is the adjoint of A, and $\gamma \in (0, \frac{2}{L}), \lambda > 0$.

In 2014, Kazmi and Rizvi [12] proposed the following iterative procedure

$$\begin{cases} u_n = J_\lambda^{B_1}\left(x_n + \gamma A^*\left(J_\lambda^{B_2} - I\right)Ax_n\right) \\ x_{n+1} = a_n f(x_n) + (1 - a_n)Su_n) \end{cases} \tag{6}$$

Then, the sequence $\{x_n\}$ converges strongly to the solution set $\Gamma$ andthe fixed point of nonexpansive mapping S.

In 2015, using the hybrid steepest descent method, K. Sitthithakerngkietnet al. [14] considered theconvergence of the following iterative procedure

$$\begin{cases} u_n = J_\lambda^{B_1}\left(x_n + \gamma A^*\left(J_\lambda^{B_2} - I\right)Ax_n\right) \\ x_{n+1} = a_n f(x_n) + (I - a_n D)S_n u_n) \end{cases} \tag{7}$$

where $S_n$ is a sequence of nonexpansive mappings and D is a strongly positive bounded linear operator.

Following the work of Moudafi [1], Kazmi and Rizvi [12], Sitthithakerngkiet et al. [14], we introduce a rapid iterative algorithm for finding a common element of the solution set of split variational inclusion problems and the set of fixed points of a nonexpansive mapping. Under suitable conditions, the strong convergence for the sequences generated by the algorithm to a solution of the problems is proved. As applications, we apply our iterative algorithms to split feasibility problem. Preliminary numerical results indicate that our algorithm is more effective for SVIP (3)-(4) than the proposed algorithms in [4], [12] and [14].

## II. Preliminaries

Before proceeding further, we give a few concepts.

A mapping$T: H_1 \to H_1$ is called contraction, if there exists a constant$\alpha \in (0,1)$such that

$$\|Tx - Ty\| \le \alpha\|x - y\|, \forall(x, y) \in H_1 \tag{8}$$

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holds.

If $\alpha = 1$, then T is called nonexpansive.

A mapping $T: H_1 \to H_1$ is said to be firmly nonexpansive, if

$$\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2, \forall (x, y) \in H_1 \quad (9)$$

A set-valued mapping $Q: H_1 \to 2^{H_1}$ is called monotone if for all $x, y \in H_1$, $f \in Qx$ and $g \in Qy$ imply $\langle x - y, f - g\rangle \geq 0$.

A monotone mapping $Q: H_1 \to 2^{H_1}$ is called maximal if the graph $G(Q)$ of Q is not properly contained in the graph of any other monotone mapping. It is well known that a monotone mapping Q is maximal if and only if for the following relation $(x, f) \in H \times H, \langle x - y, f - g\rangle \geq 0, (y, g) \in G(Q)$ implies $f \in Qx$.

To obtain our results, we need the following technical lemmas.

**Lemma 2.1** [18] If $x, y, z \in H$, then

(a) $\|x + y\|^2 \leq \|x\|^2 + \langle y, x + y\rangle$.

(b) For any $\lambda \in [0,1]\|\lambda x + (1 + \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$.

**Lemma 2.2** [20] Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, n \geq 0$, where $\gamma_n \in (0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$, such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$

(ii) $\lim_{n\to\infty} \sup \frac{\delta_n}{\gamma_n} \leq 0$ or $\sum_{n=0}^{\infty} \delta_n < \infty$

Then, $\lim_{n\to\infty} a_n = 0$.

**Lemma 2.3** [12] Assume A is a strongly positive linear bounded operator on a Hilbert space H with the coefficient $\bar{\gamma} > 0$ and $0 < \rho < \|A\|^{-1}$. Then $\|I - \rho A\| < 1 - \rho\bar{\gamma}$.

## III. MAIN RESULTS

In this section, we first show some important lemmas and propose our algorithm. Then the convergence analysis of the algorithm is proved.

The following results are some important tools in this paper.

**Lemma 3.1** [13] Let H be a real Hilbert space. Let $M: H \to 2^H$ be a multi-valued maximal monotone mapping and let the resolving mapping $J_\lambda^M: H \to H$ be defined by $J_\lambda^M(x) = (I + \lambda M)^{-1}(x)$ associated with $M > 0, \lambda > 0$. Then the following are satisfied:

(i) For each $\lambda > 0$, $J_\lambda^M$ is a single-valued and firmly nonexpansive mapping.

(ii) Suppose that $M^{-1}(0) \neq \emptyset$, then $\|x - J_\lambda^M(x)\|^2 + \|J_\lambda^M(x) - \bar{x}\|^2 \leq \|x - \bar{x}\|^2$ for each $x \in H$, $\bar{x} \in M^{-1}(0)$ and $\lambda > 0$.

(iii) Suppose that $M^{-1}(0) \neq \emptyset$, then $\langle x - J_\lambda^M(x), J_\lambda^M(x) - w\rangle \geq 0$ for each $x \in H$, $w \in M^{-1}(0)$ and $\lambda > 0$.

(iv) $\|J_\lambda^M(x) - J_\lambda^M(y)\|^2 \leq \langle x - y, J_\lambda^M(x) - J_\lambda^M(y)\rangle$ for each $x, y \in H$ and $\lambda > 0$.

**Lemma 3.2** If the resolving mapping $J_\lambda^M: H \to H$ is defined by Lemma 3.1, for each $x, y \in H$ and $\lambda > 0$, then we have

(i) $\|J_\lambda^M(x) - J_\lambda^M(y)\|^2 \leq \|x - y\|^2 - \|J_\lambda^M(x) - x + y - J_\lambda^M(y)\|^2$.

(ii) $\|J_\lambda^M(x) - x + y - J_\lambda^M(y)\| \leq \|x - y\|$.

(iii) $\|J_\lambda^M(x) - x\|^2 \leq \langle J_\lambda^M(x) - x, w - x\rangle$, for $w \in M^{-1}(0)$.

Proof. For (i), from lemma 3.1 (iv), for $x, y \in H$, we have

$$\begin{aligned}\|J_\lambda^M(x) - J_\lambda^M(y)\|^2 &\leq \langle x - y, J_\lambda^M(x) - J_\lambda^M(y)\rangle \\ &= \|x - y\|^2 \\ &\quad + \langle x - y, J_\lambda^M(x) - x + y - J_\lambda^M(y)\rangle \\ &= \|x - y\|^2 - \|J_\lambda^M(x) - x + y - J_\lambda^M(y)\|^2 \\ &\quad + \langle J_\lambda^M(x) - J_\lambda^M(y), J_\lambda^M(x) - x + y - J_\lambda^M(y)\rangle \\ &\leq \|x - y\|^2 - \|J_\lambda^M(x) - x + y - J_\lambda^M(y)\|^2.\end{aligned}$$

(ii) is straightforward from (i), which means $J_\lambda^M(x) - x$ is 1-Lipschitz.

For (iii), from lemma 3.1 (iii), for $x \in H$ one has $\langle J_\lambda^M(x) - x, w - x\rangle = \langle J_\lambda^M(x) - x, w - J_\lambda^M(x)\rangle + \langle J_\lambda^M(x) - x, J_\lambda^M(x) - x\rangle \geq \|J_\lambda^M(x) - x\|^2$.

**Lemma 3.3** [1] SVIP(3)-(4) is equivalent to the problem

$$x^* \in H^* \text{ such that } y^* = Ax^* \in H_2$$

$$x^* = J_\lambda^{B_1}x^*, y^* = J_\lambda^{B_2}y^*, \text{for } \lambda > 0$$

**Lemma 3.4** Let $H_1$ and $H_2$ be two real Hilbert spaces and $A: H_1 \to H_1$ be a bounded linear operator. Assume that $B_1: H_1 \to 2^{H_1}$ and $B_2: H_2 \to 2^{H_2}$ are maximal monotone mappings. Let S be a nonexpansive mappings on $H_1$ such that $\Gamma \cap F(S) \neq \emptyset$. Suppose $f: H_1 \to H_1$ is a contraction mapping with constant $\beta \in (0,1)$ and D is a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and $0 < \xi < \frac{\bar{\gamma}}{\beta}$. For any $t \in (0,1)$ and $x \in H_1$, let the mapping $W_t$ on $H_1$ be defined by $W_t x = J_\lambda^{B_1}(I + \tau A^*(J_\lambda^{B_2} - I)A J_\lambda^{B_1}(t\xi f(x) + (I - tD)Sx)$ where $\lambda > 0$, and $\tau \in (0, \frac{1}{L_1})$, $L_1$ is the spectral radius of the operator $AA^*$ and $A^*$ is the adjoint of A. Then the mapping $W_t$ is contraction and has a unique fixed point.

Proof. Note that $J_\lambda^{B_1}(I + \tau A^*(J_\lambda^{B_2} - I)A$ is nonexpansive and $\|I - tD\| \leq 1 - t\bar{\gamma}$. For any $x, y \in H_1$, we have

$$\begin{aligned}\|W_t x - W_t y\| &= \|J_\lambda^{B_1}(I \\ &\quad + \tau A^*(J_\lambda^{B_2} - I)A)J_\lambda^{B_1}(t\xi f(x) \\ &\quad + (I - tD)Sx) - J_\lambda^{B_1}(I + \tau A^*(J_\lambda^{B_2} \\ &\quad - I)A)J_\lambda^{B_1}(t\xi f(y) + (I - tD)Sy)\| \\ &\leq \|(t\xi f(x) + (I - tD)Sx) - (t\xi f(y) + (I \\ &\quad - tD)Sy)\| \\ &\leq t\xi\beta\|x - y\| + (1 - t\bar{\gamma})\|x - y\| \\ &= (1 - t(\bar{\gamma} - \xi\beta))\|x - y\|\end{aligned}$$

Thus, $W_t$ is a contraction when $t \in (0, \frac{1}{\bar{\gamma} - \xi\beta})$. Furthermore, it follows from the Banach contraction principle that $W_t$ has a unique fixed point. This completes the proof.

**Remark 3.1** Lemma 3.4 means that $x_t$ is the unique solution of the fixed point equation $x_t = J_\lambda^{B_1}(I + \tau A^*(J_\lambda^{B_2} - I)AJ_\lambda^{B_1}(t\xi f(x_t) + (I - tD)Sx_t)$.

Now, we are in a position to show the description of our algorithm.

**Algorithm 3.1** For an arbitrary initial point $x_0 \in H_1$, we define $\{x_n\}$ by

$$\begin{cases} y_n = J_\lambda^{B_1}(a_n\xi f(x_n) + (I - a_n D)Sx_n) \\ x_{n+1} = J_\lambda^{B_1}\big(y_n + \tau A^*(J_\lambda^{B_2} - I)Ay_n\big) \end{cases} \tag{10}$$

where $\lambda > 0$, $a_n \in (0,1)$, and $\tau \in (0, \frac{1}{L_1})$, $L_1$ is the spectral radius of the operator $AA^*$ and $A^*$ is the adjoint of A.

Next, we will show the convergence analysis of the algorithm (10) for approximating a common solution of SVIP (3)-(4).

**Theorem 3.1** Let $H_1$ and $H_2$ be two real Hilbert spaces and $A: H_1 \to H_1$ be a bounded linear operator. Assume that $B_1: H_1 \to 2^{H_1}$ and $B_2: H_2 \to 2^{H_2}$ are maximal monotone mappings. Let S be a nonexpansive mappings on $H_1$. Let $f: H_1 \to H_2$ be a contraction mapping with constant $\beta \in (0,1)$ and D be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and $0 < \xi < \frac{\bar{\gamma}}{\beta}$. Assume that $\Omega = \Gamma \cap F(S) \neq \emptyset$. Let the sequences $\{x_n\}$ be generated by (10). Assume that the sequence $\{a_n\}$ satisfies the following conditions:

(i) $\lim_{n\to\infty} a_n = 0$.
(ii) $\sum_{n=0}^{\infty} a_n = \infty$.
(iii) $\sum_{n=0}^{\infty} |a_{n+1} - a_n| < \infty$.

Then $\{x_n\}$ converges strongly to a point z, where $z = P_\Omega(I - D + \xi f)$ is a unique solution of the variational inequalities

$$\langle (D - \xi f)z, z - x\rangle \leq 0. \tag{11}$$

Proof. We divide the proof into five steps.

Step1.We prove that $\{x_n\}$ is bounded.

Set $p \in \Omega$. From $p = J_\lambda^{B_1}p$, $Ap = J_\lambda^{B_2}(Ap)$, $Sp \in p$, and Lemma 3.3, we have

$$\|x_{n+1} - p\|^2 = \left\|J_\lambda^{B_1}(y_n + \tau A^*(J_\lambda^{B_2} - I)Ay_n) - p\right\|^2 = \left\|J_\lambda^{B_1}(y_n + \tau A^*(J_\lambda^{B_2} - I)Ay_n) - J_\lambda^{B_1}p\right\|^2 \leq \left\|y_n + \tau A^*(J_\lambda^{B_2} - I)Ay_n - p\right\|^2 = \|y_n - p\|^2 + \tau^2\left\|A^*(J_\lambda^{B_2} - I)Ay_n\right\|^2 + 2\tau\langle y_n - p, A^*(J_\lambda^{B_2} - I)Ay_n\rangle \tag{12}$$

Since

$$2\tau\langle y_n - p, A^*(J_\lambda^{B_2} - I)Ay_n\rangle = 2\tau\langle A(y_n - p), A^*(J_\lambda^{B_2} - I)Ay_n\rangle = 2\tau\langle Ay_n - Ap + (J_\lambda^{B_2} - I)Ay_n - (J_\lambda^{B_2} - I)Ay_n, (J_\lambda^{B_2} - I)Ay_n\rangle = 2\tau\langle J_\lambda^{B_2}Ay_n - p, (J_\lambda^{B_2} - I)Ay_n\rangle - 2\tau\left\|(J_\lambda^{B_2} - I)Ay_n\right\|^2 = \tau\left(\left\|J_\lambda^{B_2}Ay_n - p\right\|^2 + \left\|(J_\lambda^{B_2} - I)Ay_n\right\|^2 - \|Ay_n - Ap\|^2\right) - 2\tau\left\|(J_\lambda^{B_2} - I)Ay_n\right\|^2 \leq \tau\left\|(J_\lambda^{B_2} - I)Ay_n\right\|^2 - 2\tau\left\|(J_\lambda^{B_2} - I)Ay_n\right\|^2 - 2\tau\left\|(J_\lambda^{B_2} - I)Ay_n\right\|^2 \tag{13}$$

and

$$\tau^2\left\|A^*(J_\lambda^{B_2} - I)Ay_n\right\|^2 = \tau^2\langle A^*(J_\lambda^{B_2} - I)Ay_n, A^*(J_\lambda^{B_2} - I)Ay_n\rangle = \tau^2\langle (J_\lambda^{B_2} - I)Ay_n, AA^*(J_\lambda^{B_2} - I)Ay_n\rangle \leq L_1\tau^2\langle (J_\lambda^{B_2} - I)Ay_n, (J_\lambda^{B_2} - I)Ay_n\rangle \leq L_1\tau^2\left\|(J_\lambda^{B_2} - I)Ay_n\right\|^2 \tag{14}$$

From (13), (14) and (12), we deduce

$$\|x_{n+1} - p\|^2 \leq \|y_n - p\|^2 + \tau(L_1\tau - 1)\left\|(J_\lambda^{B_2} - I)Ay_n\right\|^2 \leq \|y_n - p\|^2 \tag{15}$$

The definition of $y_n$ yields

$$\|y_n - p\| = \left\|J_\lambda^{B_1}(a_n\xi f(x_n) + (I - a_n D)Sx_n) - p\right\| \leq \|a_n\xi f(x_n)(I - a_n D)Sx_n - p\| \leq a_n\xi\|f(x_n) - f(p)\| + \|I - a_n D\|\|Sx_n - p\| + a_n\|f(p) - Dp\| \leq a_n\xi\beta\|x_n - p\| + (1 - a_n\bar{\gamma})\|x_n - p\| + a_n\|f(p) - Dp\| \leq (1 - a_n(\bar{\gamma} - \xi\beta)\|x_n - p\| + a_n\|f(p) - Dp\| \tag{16}$$

Thus, following from (15), (14) and Lemma 2.3, we have

$$\|x_{n+1} - p\| \leq \|y_n - p\| \leq (1 - a_n(\bar{\gamma} - \xi\beta)\|x_n - p\| + a_n\|f(p) - Dp\| \leq \max\{\|x_n - p\|, \tfrac{\|f(p) - Dp\|}{\bar{\gamma} - \xi\beta} \tag{17}$$

By induction, we have

$$\|x_{n+1} - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|f(p) - Dp\|}{\bar{\gamma} - \xi\beta}\right\}$$

which indicates that $\{x_n\}$ is bounded. It is easily to show that $\{f(x_n)\}$,$\{Sx_n\}$,$\{Dx_n\}$ and $\{y_n\}$ are all bounded.

Step 2. We reveal that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Since $J_\lambda^{B_1}(I + \lambda A^*(J_\lambda^{B_2} - I)A$ is nonexpansive, one has

$$\|x_{n+1} - x_n\| = \left\|J_\lambda^{B_1}(I + \tau A^*(J_\lambda^{B_2} - I)A)y_n - J_\lambda^{B_1}(I + \tau A^*(J_\lambda^{B_2} - I)A)y_{n-1}\right\| \leq \|y_n - y_{n-1}\| \tag{18}$$

In what follows, we will estimate $\|y_n - y_{n-1}\|$. It follows from (10) that

$$\|y_n - y_{n-1}\| = \left\|J_\lambda^{B_1}(a_n\xi f(x_n) + (I - a_n D)Sx_n) - J_\lambda^{B_1}(a_{n-1}\xi f(x_{n-1}) + (I - a_{n-1}D)Sx_{n-1})\right\| \leq \|a_n\xi f(x_n) + (I - a_n D)Sx_n - a_{n-1}\xi f(x_{n-1}) + (I - a_{n-1}D)Sx_{n-1} \leq a_n\xi\beta\|x_n - x_{n-1}\| + |a_n - a_{n-1}|\|\xi f(x_{n-1})\| + (1 - a_n\bar{\gamma})\|x_n - x_{n-1}\| + |a_n - a_{n-1}|\|DSx_{n-1}\| \leq \big(1 - a_n(\bar{\gamma} - \xi\beta)\big)\|x_n - x_{n-1}\| + |a_n - a_{n-1}|(\|\xi f(x_{n-1})\| + \|DSx_{n-1}\|) \tag{19}$$

Then, one has

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \|y_n - y_{n-1}\| \\ &\leq \big(1 - a_n(\bar{\gamma} - \xi\beta)\big)\|x_n - x_{n-1}\| \\ &\quad + |a_n - a_{n-1}|(\|\xi f(x_{n-1})\| + \|DSx_{n-1}\|)\end{aligned}$$

Furthermore, by Lemma 2.2, we obtain

$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ (20)

Step 3. We will obtain $\lim_{n\to\infty}\|Sx_n-Sx_n\|=0$.
Set $z_n=a_n\xi f(x_n)+(I-a_nD)Sx_n$ ). One has $y_n=J_\lambda^{B_1}z_n$.
It follows from Lemma 2.1 that

$$\|y_n-p\|^2=\left\|J_\lambda^{B_1}(a_n\xi f(x_n)+(I-a_nD)Sx_n)-p\right\|^2\le \|z_n-p\|^2=\|a_n\xi f(x_n)+(I-a_nD)Sx_n-p\|^2= \|(I-a_nD)Sx_n-(I-a_nD)p-a_n\xi f(x_n)-a_nDp\|^2\le (I-a_n\overline{\gamma})\|x_n-p\|^2+2a_n\langle\xi f(x_n)-Dp,z_n-p\rangle\le \|x_n-p\|^2+2a_n\|\xi f(x_n)-Dp\|\|z_n-p\| \quad (21)$$

(15) and (21) deduce

$$\begin{aligned}\|x_{n+1}-p\|^2&=\|y_n-p\|^2+\tau(L_1\tau-1)\left\|(J_\lambda^{B_2}-I)Ay_n\right\|^2\\&\le(I-a_n\overline{\gamma})\|x_n-p\|^2\\&+2a_n\|\xi f(x_n)-Dp\|\|z_n-p\|+\tau(L_1\tau\\&-1)\left\|(J_\lambda^{B_2}-I)Ay_n\right\|^2\\&\le\|x_n-p\|^2\\&+2a_n\|\xi f(x_n)-Dp\|\|z_n-p\|+\tau(L_1\tau\\&-1)\left\|(J_\lambda^{B_2}-I)Ay_n\right\|^2\end{aligned}$$

Moreover,

$$\begin{aligned}&\tau(1-L_1\tau)\left\|(J_\lambda^{B_2}-I)Ay_n\right\|^2\\&\le 2a_n\|\xi f(x_n)-Dp\|\|z_n-p\|+(\|x_n-p\|^2\\&\qquad-\|x_{n+1}-p\|^2)\\&\le 2a_n\|\xi f(x_n)-Dp\|\|z_n-p\|+(\|x_n-x_{n-1}\|)(\|x_n-p\|+\|x_{n+1}-p\|)\end{aligned} \quad (22)$$

Since $\tau(1-L\tau)>0$, $a_n\to 0$ and (22), we have

$\lim_{n\to\infty}\left\|(J_\lambda^{B_2}-I)Ay_n\right\|=0$
(23)

Furthermore, it follows from (10) that

$$\begin{aligned}\|x_{n+1}-p\|^2&=\left\|J_\lambda^{B_1}(y_n+\tau A^*(J_\lambda^{B_2}-I)Ay_n)-p\right\|^2\\&=\left\|J_\lambda^{B_1}(y_n+\tau A^*(J_\lambda^{B_2}-I)Ay_n)-J_\lambda^{B_1}p\right\|^2\\&\le\langle x_{n+1}-p,y_n+\tau A^*(J_\lambda^{B_2}-I)Ay_n)-p\rangle\\&=\frac{1}{2}(\|x_{n+1}-p\|^2\\&+-\left\|x_{n+1}-p-(y_n+\tau A^*(J_\lambda^{B_2}-I)Ay_n\right.\\&\left.-p)\right\|^2=\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}(\|x_{n+1}-p\|^2+\left\|y_n+\tau A^*(J_\lambda^{B_2}-I)Ay_n-p\right\|^2\\&\quad-\left\|x_{n+1}-y_n-\tau A^*(J_\lambda^{B_2}-I)Ay_n\right\|^2)\\&\le\frac{1}{2}(\|x_{n+1}-p\|^2)+\|y_n-p\|^2+\tau(L\tau\\&\quad-1)\left\|(J_\lambda^{B_2}-I)Ay_n-p\right\|^2\\&\quad-\left\|x_{n+1}-y_n-\tau A^*(J_\lambda^{B_2}-I)Ay_n\right\|^2)\\&\le\frac{1}{2}(\|x_{n+1}-p\|^2+\|y_n-p\|^2\\&\quad-(\|x_{n+1}-y_n\|^2+\tau^2\left\|A^*(J_\lambda^{B_2}-I)Ay_n\right\|^2\\&\quad-2\tau\langle x_{n+1}-y_n,A^*(J_\lambda^{B_2}-I)Ay_n\rangle))\\&\le\frac{1}{2}(\|x_{n+1}-p\|^2+\|y_n-p\|^2\\&\quad-\|x_{n+1}-y_n\|^2\\&\quad+2\tau\|A(x_{n+1}-y_n)\|\left\|(J_\lambda^{B_2}-I)Ay_n\right\|)\end{aligned}$$

which implies that

$$\|x_{n+1}-p\|^2\le\|y_n-p\|^2-\|x_{n+1}-y_n\|^2+2\tau\|A(x_{n+1}-y_n)\|\left\|(J_\lambda^{B_2}-I)Ay_n\right\| \quad (24)$$

Furthermore,

$$\|x_{n+1}-y_n\|^2\le\|y_n-p\|^2-\|x_{n+1}-p\|^2+2\tau\|A(x_{n+1}-y_n)\|\left\|(J_\lambda^{B_2}-I)Ay_n\right\|\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+2a_n\|\xi f(x_n)-Dp\|\|y_n-p\|+2\tau\|A(x_{n+1}-y_n)\|\left\|(J_\lambda^{B_2}-I)Ay_n\right\|\le\|x_n-x_{n-1}\|(\|x_n-p\|-\|x_{n+1}-p\|)+2a_n\|\xi f(x_n)-Dp\|\|y_n-p\|+2\tau\|A(x_{n+1}-y_n)\|\left\|(J_\lambda^{B_2}-I)Ay_n\right\| \quad (25)$$

Following the fact $a_n\to 0$ and (20), (23), we have

$\lim_{n\to\infty}\|x_{n+1}-y_n\|=0$
(26)

which means that

$\lim_{n\to\infty}\|x_n-y_n\|=0$
(27)

Since $I-J_\lambda^{B_2}$ is a firmly nonexpansive mapping, we obtain

$$\begin{aligned}&\left\|(I-J_\lambda^{B_2})Ax_n-(I-J_\lambda^{B_2})Ay_n\right\|\\&\le\langle Ax_n-Ay_n,(I-J_\lambda^{B_2})Ax_n\\&\quad-(I-J_\lambda^{B_2})Ay_n\rangle\\&\le\langle x_n-y_n,A^*(I-J_\lambda^{B_2})Ax_n\\&\quad-A^*(I-J_\lambda^{B_2})Ay_n\rangle\\&\le\|x_n-y_n\|\left\|A^*(I-J_\lambda^{B_2})Ax_n\right.\\&\quad\left.-A^*(I-J_\lambda^{B_2})Ay_n\right\|\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\left\|(I-J_\lambda^{B_2})Ax_n-(I-J_\lambda^{B_2})Ay_n\right\|=0$$

This together with (23) implies

$\lim_{n\to\infty}\left\|\left(I-J_\lambda^{B_2}\right)Ax_n\right\|=0$
(28)

On the other hand, we note that

$$\begin{aligned}\left\|x_{n+1}-J_\lambda^{B_1}y_n\right\| &\le \left\|J_\lambda^{B_1}\left(y_n+\tau A^*\left(J_\lambda^{B_2}-I\right)Ay_n\right)-J_\lambda^{B_1}y_n\right\|\\ &\le \left\|\left(y_n+\tau A^*\left(J_\lambda^{B_2}-I\right)Ay_n\right)-y_n\right\|\\ &= \left\|\tau A^*\left(J_\lambda^{B_2}-I\right)Ay_n\right\|\\ &\le \tau\|A^*\|\left\|\left(J_\lambda^{B_2}-I\right)Ay_n\right\|\end{aligned}$$

And

$$\begin{aligned}\left\|x_n-J_\lambda^{B_1}x_n\right\| &\le \left\|x_n-x_{n+1}+x_{n+1}-J_\lambda^{B_1}y_n\right.\\ &\quad \left.+J_\lambda^{B_1}y_n-J_\lambda^{B_1}x_n\right\|\\ &\le \|x_n-x_{n+1}\|+\left\|x_{n+1}-J_\lambda^{B_1}y_n\right\|\\ &\quad +\left\|J_\lambda^{B_1}y_n-J_\lambda^{B_1}x_n\right\|\end{aligned}$$

From (20), (23) and (27), we have

$$\lim_{n\to\infty}\left\|x_n-J_\lambda^{B_1}x_n\right\|=0 \quad (29)$$

It follows from lemma 3.1 and (21) that

$$\|y_n-p\|^2=\left\|J_\lambda^{B_1}(a_n\xi f(x_n)+(I-a_nD)Sx_n)-p\right\|^2=\left\|J_\lambda^{B_1}z_n-p\right\|^2\le\|z_n-p\|^2-\left\|J_\lambda^{B_1}z_n-z_n\right\|^2=\|z_n-p\|^2-\|y_n-z_n\|^2\le\|x_n-p\|^2+2a_n\|\xi f(x_n)-Dp\|\|z_n-p\|-\|y_n-z_n\|^2 \quad (30)$$

Therefore,

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|y_n-p\|^2\\ &\le \|x_n-p\|^2\\ &\quad +2a_n\|\xi f(x_n)-Dp\|\|z_n-p\|\\ &\quad -\|y_n-z_n\|^2\end{aligned}$$

Furthermore,

$$\begin{aligned}\|y_n-z_n\|^2 &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\quad +2a_n\|\xi f(x_n)-Dp\|\|z_n-p\|\\ &\le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)\\ &\quad +2a_n\|\xi f(x_n)-Dp\|\|z_n-p\|\end{aligned}$$

which means that

$$\lim_{n\to\infty}\|y_n-z_n\|=0 \quad (31)$$

Consequently,

$$\lim_{n\to\infty}\|x_n-z_n\| \le \lim_{n\to\infty}(\|x_n-y_n\|+\|y_n-z_n\|)=0$$
(32)

Since

$$\begin{aligned}z_n-Sx_n &= a_n\xi f(x_n)+(I-a_nD)Sx_n-Sx_n\\ &= a_n(\xi f(x_n)-DSx_n)\end{aligned}$$

By $a_n\to 0$, we have

$$\lim_{n\to\infty}\|z_n-Sx_n\|=0 \quad (33)$$

By (32), one has

$\lim_{n\to\infty}\|Sx_n-x_n\|=0$
(34)

Step 4. Next, we will prove that

$\lim_{n\to\infty}\sup\langle(D-\xi f)z,z-y_n\rangle\le 0$
(35)

To obtain this inequality, we need to show the following inequality $\lim_{n\to\infty}\sup\langle(D-\xi f)z,z-x_n\rangle\le 0$ holds, where $z=P_\Omega(I-D+\xi f)(z)$ is a unique solution of the variational inequality (11).

We choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\lim_{n\to\infty}\sup\langle(D-\xi f)z,z-x_n\rangle=\lim_{j\to\infty}\sup\langle(D-\xi f)z,z-x_{n_j}\rangle$$

Since $\{x_{n_j}\}$ is bounded, then there exists a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$ which converges weakly to q. Without loss of generality, we assume that $x_{n_j}\to q$. Thus, from (28), (29) and (34), we obtain $q\in\Omega$. Since $z=P_\Omega(I-D+\xi f)z$, it follows that

$$\lim_{n\to\infty}\sup\langle(D-\xi f)z,z-x_n\rangle=\lim_{j\to\infty}\sup\langle(D-\xi f)z,z-x_{n_j}\rangle=\langle(D-\xi f)z,z-q\rangle\le 0.$$

This together with (27) means that (35) holds.

Step 5. Finally, we show that $x_n\to z$.

Indeed, from (10), we have

$$\begin{aligned}\|y_n-z\|^2=\left\|J_\lambda^{B_1}z_n-z\right\|^2 &\le \langle z_n-z,J_\lambda^{B_1}z_n-z\rangle\\ &= \langle z_n-z,y_n-z\rangle\\ &= \langle a_n\xi f(x_n)+(I-a_nD)S_nx_n-z,y_n-z\rangle\\ &\quad +a_n\langle\xi f(z)-Dz,y_n-z\rangle\\ &\le (1-a_n(\bar\gamma-\xi\beta))\|x_n-z\|\|y_n-z\|\\ &\quad +a_n\langle\xi f(z)-Dz,y_n-z\rangle\\ &\le \frac{(1-a_n(\bar\gamma-\xi\beta))}{2}\|x_n-z\|^2+\frac{1}{2}\|y_n-z\|^2\\ &\quad +a_n\langle\xi f(z)-Dz,y_n-z\rangle\end{aligned}$$

It follows that

$$\|y_n-z\|^2\le(1-a_n(\bar\gamma-\xi\beta))\|x_n-z\|^2+2a_n\langle\xi f(z)-Dz,y_n-z\rangle \quad (36)$$

From (36) and (10), we obtain

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \|y_n-z\|^2\\ &\le (1-a_n(\bar\gamma-\xi\beta))\|x_n-z\|^2\\ &\quad +2a_n\langle\xi f(z)-Dz,y_n-z\rangle\\ &= (1-a_n(\bar\gamma-\xi\beta))\\ &\quad +\frac{2a_n(\bar\gamma-\xi\beta)}{(\bar\gamma-\xi\beta)}\langle\xi f(z)-Dz,y_n-z\rangle\end{aligned}$$

Hence, all conditions of Lemma 2.2 are satisfied, we immediately deduce that $x_n \to x$. This completes the proof.

The following conclusions can be obtained from Algorithm 3.1 and Theorem 3.1 immediately.

**Algorithm 3.2** For an arbitrary initial point $x_0 \in H_1$, we define $\{x_n\}$ iteratively

$$\begin{cases} y_n = J_\lambda^{B_1}(a_n \xi f(x_n) + (I - a_n)x_n) \\ x_{n+1} = J_\lambda^{B_1}(y_n + \tau A^*(J_\lambda^{B_2} - I)Ay_n) \end{cases} \tag{37}$$

**Theorem 3.2** Let the sequences $\{x_n\}$ be generated by (37). Assume that the sequence $\{a_n\}$ satisfies the control conditions:

(i) $\lim_{n\to\infty} a_n = 0$.

(ii) $\sum_{n=0}^{\infty} a_n = \infty$.

(iii) $\sum_{n=0}^{\infty}|a_{n+1} - a_n| < \infty$.

Then, $\{x_n\}$ converges strongly to a point $z \in \Gamma$, which solves the variational inequalities (11).

**Algorithm 3.3** For an arbitrary initial point $x_0 \in H_1$, we define $\{x_n\}$ iteratively by

$$\begin{cases} y_n = J_\lambda^{B_1}((I - a_n)x_n) \\ x_{n+1} = J_\lambda^{B_1}(y_n + \tau A^*(J_\lambda^{B_2} - I)Ay_n) \end{cases} \tag{38}$$

where $\lambda > 0$, $a_n \in [0,1]$, and $\tau \in (0, \frac{1}{L_1})$, $L_1$ is the spectral radius of the operator $AA^*$ and $A^*$ is the adjoint of A.

**Theorem 3.3** Let the sequences $\{x_n\}$ be generated by (38). Assume that the sequence $\{a_n\}$ satisfies the following conditions:

(i) $\lim_{n\to\infty} a_n = 0$.

(ii) $\sum_{n=0}^{\infty} a_n = \infty$.

(iii) $\sum_{n=0}^{\infty}|a_{n+1} - a_n| < \infty$.

Then, $\{x_n\}$ converges strongly to a point $z \in \Gamma$, which is the minimum norm element in $\Gamma$.

## IV. APPLICATIONS

We now pay our attention to applying our iterative algorithms to split feasibility problem.

The split feasibility problem (SFP) was first introduced by Censor and Elfving [20] to look for

$$x \in C \text{ such that } Ax \in Q \tag{39}$$

where $A: H_1 \to H_1$ is a bounded linear operator, C and Q are nonempty closed convex subset of real Hilbert spaces $H_1$ and $H_2$, respectively. It is well known that SFP arise from phase retrievals and medical image reconstruction [21].

Define $B_1 = \partial\delta_C: H_1 \to 2^{H_1}$, where $\delta_C: H_1 \to [0, +\infty]$ is the indicator function of C, i.e,

$$\delta_C = \begin{cases} 0, & x \in C \\ +\infty, & x \notin C \end{cases}$$

and $B_2 = \partial\delta_Q: H_2 \to 2^{H_2}$, where $\delta_Q: H_2 \to [0, +\infty]$ is the indicator function of Q, i.e,

$$\delta_Q = \begin{cases} 0, & x \in Q \\ +\infty, & x \notin Q \end{cases}$$

Thus, Algorithm 3.1 becomes to be the following algorithm.

**Algorithm 4.1** For an arbitrary initial point $x_0 \in H_1$, we define $\{x_n\}$ iteratively by

$$\begin{cases} y_n = P_C(a_n \xi f(x_n) + (I - a_n D)Sx_n) \\ x_{n+1} = P_C(y_n + \tau A^*(P_Q - I)Ay_n) \end{cases} \tag{40}$$

where $\lambda > 0$, $a_n \in [0,1]$, and $\tau \in (0, \frac{1}{L_1})$, $L_1$ is the spectral radius of the operator $AA^*$ and $A^*$ is the adjoint of A.

Furthermore, if $S = I$, then Algorithm 4.1 can reduce to the following algorithm [22].

**Algorithm 4.2** For an arbitrary initial point $x_0 \in H_1$, we define $\{x_n\}$ iteratively by

$$\begin{cases} y_n = P_C(a_n \xi f(x_n) + (I - a_n D)x_n) \\ x_{n+1} = P_C(y_n + \tau A^*(P_Q - I)Ay_n) \end{cases} \tag{41}$$

where $\lambda > 0$, $a_n \in [0,1]$, and $\tau \in (0, \frac{1}{L_1})$, $L_1$ is the spectral radius of the operator $AA^*$ and $A^*$ is the adjoint of A.

## V. NUMERICAL EXAMPLES

We now propose a numerical example to demonstrate the performance and the convergence of our result. In the experiment, the stopping criterion is $\|x_n - x^*\| \le \varepsilon$, IT denotes the iterative number, and SOL denotes a solution of the test problem. Set $a_n = \frac{1}{n}$, $\lambda \in 0.5$, $\xi = 1$, $D = I$, and the initial point $(100,100)^T$.

**Example 5.1** Let A and $B_1, B_2: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B_1 = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}, B_2 = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}$$

We take a mapping $S: v = (v_1, v_2)^T: \to (\sin v_1, \sin v_2)^T$, and it is easily to see that S is nonexpansive. Since $\frac{1}{L_1} = 1$, so, we can take $\tau \in 0.8$.

**Example 5.2** Let A and $B_1, B_2: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$A = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}, B_1 = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}, B_2 = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}$$

Let S be the same as in Example 5.1. Since $\frac{1}{L_1} = 0.1910$, so, we can take $\tau \in 0.15$.

**Example 5.3** Let $B_1: \mathbb{R}^2 \to \mathbb{R}^2$ and $B_2: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 2 & 2 \end{pmatrix}, B_1 = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}, B_2 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

We define a mapping $S = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$, and it is easy to observe that S is nonexpansive. Since $\frac{1}{L_1} = 0.0588$, so, we can take $\tau \in 0.05$.

TABLE I
NUMERICAL RESULTS FOR EXAMPLE 5.1

| $\varepsilon = 10^{-4}$ | | | | $\varepsilon = 10^{-6}$ | | |
|---|---|---|---|---|---|---|
| Method | IT | Solution | Norm | IT | Solution | Norm |
| 3.1 | 6 | (0.0,0.2) | 2.0e-5 | 8 | (0.0, 0.2) | 1.5e-7 |
| [4] | 10 | (0.0,0.5) | 5.1e-5 | 13 | (0.0, 0.4) | 4.1e-7 |
| [12] | 9 | (0.0, 0.5) | 5.0e-5 | 12 | (0.0, 0.3) | 3.4e-7 |
| [14] | 9 | (0.0, 0.3) | 3.1e-5 | 12 | (0.0, 0.2) | 2.2e-7 |

TABLE II
NUMERICAL RESULTS FOR EXAMPLE 5.2

| $\varepsilon = 10^{-4}$ | | | | $\varepsilon = 10^{-6}$ | | |
|---|---|---|---|---|---|---|
| Method | IT | Solution | Norm | IT | Solution | Norm |
| 3.1 | 7 | (-0.1, 0.7) | 7.1e-5 | 10 | (-0.0, 0.4) | 4.0e-7 |
| [4] | 17 | (-0.1, 0.4) | 4.2e-5 | 22 | (-0.1, 0.5) | 4.9e-7 |
| [12] | 13 | (-0.1, 0.6) | 6.1e-5 | 18 | (-0.1, 0.5) | 5.5e-7 |
| [14] | 11 | (-0.1, 0.6) | 5.7e-5 | 16 | (-0.1, 0.5) | 5.0e-7 |

Table I, II and III at the initial point $(10, 10)^T$ with $\varepsilon=10^{10}$ show that the iteration process of the sequence is a monotone decreasing sequence and the iteration sequence converges to $(0, 0)^T$. Furthermore, it reveals that the more the iteration steps are, the more slowly the sequence converges to $(0, 0)^T$. From Table I, II and III, we can observe that our algorithm is more effective for SVIP (3)-(4) than the proposed algorithms in [4], [12] and [14].

TABLE III
NUMERICAL RESULTS FOR EXAMPLE 5.3

| $\varepsilon = 10^{-4}$ | | | | $\varepsilon = 10^{-6}$ | | |
|---|---|---|---|---|---|---|
| Method | IT | Solution | Norm | IT | Solution | Norm |
| 3.1 | 6 | (-0.0, 0.3) | 3.2e-5 | 8 | (-0.0, 0.2) | 2.2e-7 |
| [4] | 14 | (-0.2, 0.7) | 6.8e-5 | 19 | (-0.1, 0.4) | 4.5e-7 |
| [12] | 11 | (-0.2, 0.7) | 7.7e-5 | 18 | (-0.1, 0.4) | 3.8e-7 |
| [14] | 9 | (0.1, 0.1) | 1.6e-5 | 11 | (0.2, 0.4) | 4.2e-7 |

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