Some Results on the Derivatives of the Gamma and Incomplete Gamma Function for Non-positive Integers

Zhongfeng Sun and Huizeng Qin

Abstract—**This paper is concerned with some recursive relations of the derivatives of the Gamma function $\Gamma(\alpha)$ and incomplete Gamma function $\Gamma(\alpha, z)$ for the complex value of α. In particular, $\frac{d^n\Gamma}{d\alpha^n}(-m)(n, m = 0, 1, 2, \ldots)$ can be expressed as linear forms in $\frac{d^j\Gamma}{d\alpha^j}(1)(j = 0, 1, \ldots, n+1)$ while $\frac{\partial^n\Gamma}{\partial\alpha^n}(-m, z)$ can be represented as the combination of $\frac{\partial^j\Gamma}{\partial\alpha^j}(1, z)(j = 0, 1, \ldots, n+1)$ and the elementary functions. With the aid of these results, we can establish the closed forms of some special integrals associated with $\Gamma(\alpha)$ and $\Gamma(\alpha, z)$, which can be expressed by the Riemann zeta functions and some special constants.**

Index Terms—**Incomplete Gamma function, Gamma function, Neutrix limit, Hurwitz zeta function, Digamma function.**

I. Introduction

THE incomplete Gamma function $\Gamma(\alpha, z)$ was defined by the following integral [1]

$$\Gamma(\alpha, z) = \int_z^{\infty} t^{\alpha-1} e^{-t} dt, \qquad (1)$$

where $\alpha \in \mathbb{C},\ |arg(z)| < \pi,\ z \neq 0$ and the Gamma function $\Gamma(\alpha)$ was defined by

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt, \quad Re(\alpha) > 0, \qquad (2)$$

where α can be extended to all complex numbers except non-positive integers after an appropriate analytic continuation. In this paper, we assume that $|arg(z)| < \pi,\ z \neq 0$ and $\alpha \in \mathbb{C}$.

For convenience, we introduce the following notations

$$\mathbb{N} := \{1, 2, 3, \ldots\},\ \ \mathbb{N}_0 := \mathbb{N} \cup \{0\},\ \ \mathbb{N}_0^- := \{0, -1, -2, \ldots\}.$$

Denote

$$\begin{aligned} \Gamma^{(n)}(\alpha) &= \frac{d^n}{d\alpha^n}\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} \ln^n t dt, \\ \Gamma^{(n)}(\alpha, z) &= \frac{\partial^n}{\partial\alpha^n}\Gamma(\alpha, z) = \int_z^{\infty} t^{\alpha-1} e^{-t} \ln^n t dt, \end{aligned} \qquad (3)$$

for $n \in \mathbb{N}_0$.

In recent years the issue of the neutrix limit for dealing with special functions such as the Gamma and incomplete Gamma function [2]∼[7], the Beta and incomplete Beta function [8]∼[10] and the Hurwitz zeta function [11] for non-positive integers have attracted much attention. Using the neutrix limit, $\Gamma^{(n)}(-m)\ (n, m \in \mathbb{N}_0)$ can be defined by the following neutrix calculus [2]∼[5]

$$\Gamma^{(n)}(-m) = N - \lim_{\varepsilon \to 0+0} \int_{\varepsilon}^{\infty} t^{-m-1} e^{-t} \ln^n t dt, \qquad (4)$$

where N is the neutrix [12]. Fisher and Kılıçman [2] discussed some recursive relations of the derivatives of the Gamma function for non-positive integers. However, there are some mistakes expressed in Theorem 4, 5 in [2] and the corresponding corrections will be shown in Remark 2.4 and 2.5 in this paper. Fisher *et al.* [6], [7] used the neutrix limit to establish the definition of the lower incomplete Gamma function $\gamma(\alpha, x)$ and the locally summable function $\gamma(\alpha, x_+) = H(x)\gamma(\alpha, x)$ for the non-positive integer α, where $H(x)$ denotes the Heaviside's function. Subsequently, Özçağ *et al.* [3] studied the partial derivatives of $\gamma(\alpha, x)$ and $\gamma(\alpha, x_+)$ for non-positive integers. Lin *et al.* [4] proved that $\frac{\partial^n\gamma}{\partial\alpha^n}(0, x)$ can be expressed by $\frac{\partial^{n+1}}{\partial\alpha^{n+1}}\gamma(1, x)(n \in \mathbb{N}_0)$ and the elementary functions.

The paper is structured as follows. Section II establishes some recursive relations of the derivatives of the Gamma function. Specially, $\Gamma^{(n)}(-m)(n, m \in \mathbb{N}_0)$ can be represented as linear forms in $\Gamma^{(j)}(1)(j = 0, 1, \ldots, n+1)$. Section III describes the recurrence formula for the partial derivatives of the incomplete Gamma function. In particular, $\Gamma^{(n)}(-m, z)$ can be expressed by $\Gamma^{(j)}(1, z)(j = 0, 1, \ldots, n+1)$ and the elementary functions. In Section IV, we present some examples to investigate the closed forms for some special integrals and the arbitrary precision calculation of $\Gamma^{(n)}(\alpha)$ and $\Gamma^{(n)}(\alpha, z)$. A final conclusion is drawn in Section V.

II. The recursive formulas of the derivatives of the Gamma function

Theorem 2.1 Let $n \geq 1$ be an integer. Then the recurrence relation of $\Gamma^{(n)}(\alpha)(\alpha \in \mathbb{C}\backslash\mathbb{N}_0^-)$ can be expressed as follows,

$$\Gamma^{(n)}(\alpha) = \begin{cases} \psi(\alpha)\Gamma(\alpha), & n = 1, \\ (n-1)! \sum_{k=0}^{n-2} \frac{(-1)^{n-k}}{k!} & \\ \times\zeta(n-k, \alpha)\Gamma^{(k)}(\alpha), & n > 1, \\ +\psi(\alpha)\Gamma^{(n-1)}(\alpha) & \end{cases} \qquad (5)$$

where $\zeta(s, \alpha)$ is the Hurwitz zeta function defined by

$$\zeta(s, \alpha) = \sum_{l=0}^{\infty} \frac{1}{(l+\alpha)^s} (Re(s) > 1), \qquad (6)$$

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Zhongfeng Sun is with School of Science, Shandong University of Technology, Zibo, Shandong, 255000, P. R. China.

Huizeng Qin is with School of Science, Shandong University of Technology, Zibo, Shandong, 255000, P. R. China. Huizeng Qin is the corresponding author. (e-mail: qin_hz@163.com(H.Z.Qin))

and the Digamma function $\psi(\alpha)$ is defined by

$$\psi(\alpha)=-\gamma-\frac{1}{\alpha}+\sum_{l=1}^{\infty}\left(\frac{1}{l}-\frac{1}{l+\alpha}\right), \qquad (7)$$

where γ denotes Euler's constant.

Proof. The Digamma function $\psi(\alpha)$ and its $k-$order derivatives $\psi^{(k)}(\alpha)$ can be expressed as follows([1], Sect. 12.3),

$$\psi(\alpha)=\frac{d}{d\alpha}\ln\Gamma(\alpha)=-\gamma-\frac{1}{\alpha}+\sum_{l=1}^{\infty}\left(\frac{1}{l}-\frac{1}{l+\alpha}\right), \quad (8)$$

for $\alpha\in\mathbb{C}\setminus\mathbb{N}_0^-$, and

$$\psi^{(k)}(\alpha)=\frac{d^k}{d\alpha^k}\psi(\alpha)=k!(-1)^{k+1}\sum_{l=0}^{\infty}\frac{1}{(l+\alpha)^{k+1}}, \quad (9)$$

for $k\in\mathbb{N}$, where γ denotes Euler's constant. From (8), we have

$$\Gamma'(\alpha)=\psi(\alpha)\Gamma(\alpha),\quad \alpha\in\mathbb{C}\setminus\mathbb{N}_0^-. \qquad (10)$$

Calculating $(n-1)$-order derivatives on α for (10) by using the Leibniz's rule, we get

$$\Gamma^{(n)}(\alpha)=\sum_{k=0}^{n-1}\binom{n-1}{k}\psi^{(n-1-k)}(\alpha)\Gamma^{(k)}(\alpha), \qquad (11)$$

for $\alpha\in\mathbb{C}\setminus\mathbb{N}_0^-$. Inserting (9) into (11), we can yield (5). ■

For the derivation of $\Gamma^{(n)}(-m)(n,m\in\mathbb{N}_0)$, Theorem 1 and 3 in [2] can be expressed in the following Lemma.

Lemma 2.2 Let $n,m\geq 0$ be integers. Then

$$\Gamma^{(n)}(0)=\frac{\Gamma^{(n+1)}(1)}{n+1}, \qquad (12)$$

and

$$\Gamma(-m)=\frac{(-1)^m}{m!}(H_m-\gamma), \qquad (13)$$

where $H_m=\sum\limits_{l=1}^{m}\frac{1}{l}$ and $H_0=0$. In particular, $\Gamma(0)=\Gamma'(1)=-\gamma$.

Theorem 2.3 Let $n,m\geq 1$ be integers. Then $\Gamma^{(n)}(-m)$ can be represented as follows,

$$\Gamma^{(n)}(-m)=\frac{(-1)^m n!}{m!}\sum_{j=0}^{n+1}\frac{\Gamma^{(n+1-j)}(1)H_m(1_j)}{(n+1-j)!}, \qquad (14)$$

where $\Gamma^{(j)}(1)(j=1,2,\ldots,n+1)$ is given by (5) and $H_m(1_j)(j\in\mathbb{N}_0)$ is the multiple harmonic sum defined by

$$\begin{aligned}&H_m(1_0)=1,\\ &H_m(1_j)=\sum_{m\geqslant m_1\geqslant\cdots\geqslant m_j\geqslant 1}\tfrac{1}{m_1m_2\cdots m_j}(j>0).\end{aligned} \qquad (15)$$

Proof. Integrating by parts for (4), we obtain

$$\begin{aligned}&\Gamma^{(n)}(-m)\\ =\;&-\frac{1}{m}N-\lim_{\varepsilon\to 0+0}\int_{\varepsilon}^{\infty}e^{-t}\ln^n t\,dt^{-m}\\ =\;&\frac{1}{m}N-\lim_{\varepsilon\to 0+0}\Big(e^{-\varepsilon}\varepsilon^{-m}\ln^n\varepsilon\\ &-\int_{\varepsilon}^{\infty}t^{-m}e^{-t}\ln^n t\,dt\\ &+n\int_{\varepsilon}^{\infty}t^{-m-1}e^{-t}\ln^{n-1}t\,dt\Big)\\ =\;&\frac{1}{m}\Big(n\Gamma^{(n-1)}(-m)-\Gamma^{(n)}(1-m)\Big).\end{aligned} \qquad (16)$$

Reusing (16), we have

$$\begin{aligned}&\Gamma^{(n)}(-m)\\ =\;&\frac{1}{m!}\Big[(m-2)!\Gamma^{(n)}(2-m)\\ &-n(m-2)!\Gamma^{(n-1)}(1-m)\\ &+n(m-1)!\Gamma^{(n-1)}(-m)\Big]\\ =\;&\cdots\\ =\;&\frac{1}{m!}\Big[(-1)^m\Gamma^{(n)}(0)\\ &+n\sum_{j=0}^{m-1}(-1)^j(m-1-j)!\Gamma^{(n-1)}(j-m)\Big].\end{aligned} \qquad (17)$$

Combining (12) with (17), we get

$$\begin{aligned}\Gamma^{(n)}(-m)\;&=\frac{(-1)^m}{m!}\times\Big[\frac{\Gamma^{(n+1)}(1)}{n+1}\\ &+n\sum_{k=1}^{m}(-1)^k(k-1)!\Gamma^{(n-1)}(-k)\Big].\end{aligned} \qquad (18)$$

Now let $n=1$ in (18). It follows by (13) and (18) that

$$\begin{aligned}&\Gamma'(-m)\\ =\;&\frac{(-1)^m}{m!}\left(\frac{\Gamma''(1)}{2}+\sum_{k=1}^{m}\frac{1}{k}(H_k-\gamma)\right)\\ =\;&\frac{(-1)^m}{m!}\left(\frac{\Gamma''(1)}{2}-\gamma H_m+H_m(1_2)\right).\end{aligned} \qquad (19)$$

Repeating use (18), we proceed to get

$$\begin{aligned}&\frac{(-1)^m m!}{n!}\Gamma^{(n)}(-m)\\ =\;&\frac{\Gamma^{(n+1)}(1)}{(n+1)!}+\sum_{k=1}^{m}\frac{1}{k}\Bigg(\frac{\Gamma^{(n)}(1)}{n!}\\ &+\frac{1}{(n-2)!}\sum_{l=1}^{k}(-1)^l(l-1)!\Gamma^{(n-2)}(-l)\Bigg)\\ =\;&\frac{\Gamma^{(n+1)}(1)}{(n+1)!}+\frac{\Gamma^{(n)}(1)}{n!}H_m\\ &+\frac{1}{(n-2)!}\sum_{m_1=1}^{m}\sum_{l=1}^{m_1}\frac{(-1)^l(l-1)!}{m_1}\Gamma^{(n-2)}(-l)\\ =\;&\cdots\\ =\;&\sum_{j=0}^{n-2}\frac{\Gamma^{(n+1-j)}(1)}{(n+1-j)!}H_m(1_j)+\\ &\sum_{m\geqslant m_1\geqslant\cdots\geqslant m_{n-2}\geqslant 1}\sum_{l=1}^{m_{n-2}}\frac{(-1)^l(l-1)!}{m_1\cdots m_{n-2}}\Gamma'(-l).\end{aligned} \qquad (20)$$

Due to $\Gamma'(1)=-\gamma$ and $\Gamma(1)=1$, we can yield (14) after inserting (19) into (20). ■

Remark 2.4 From (5), we can obtain that $\Gamma''(1) = \psi(1)\Gamma'(1) + \zeta(2,1)\Gamma(1) = \gamma^2 + \zeta(2)$, where $\zeta(2)$ is the Riemann zeta function. Comparing with (19), we know that $\frac{(-1)^m}{2m!}\Gamma''(1)$ is lost in the following formula(Theorem 4 in [2])

$$\Gamma'(-m) = \frac{(-1)^m}{m!}\left(H_m(1_2) - H_m\gamma\right), \tag{21}$$

which is resulting from the case $m = 1$ of (21) is ignored in the process of the mathematical induction. Therefore, (21) should be corrected by (19).

Remark 2.5 Setting $m = 1$ in (14), we have

$$\Gamma^{(n)}(-1) = -n!\sum_{j=2}^{n+1}\frac{\Gamma^{(j)}(1)}{j!} + n!(\gamma - 1). \tag{22}$$

Comparing with (22), the minus-sign " $-$ " is lost in the left side of the first item of the following formula(Theorem 5 in [2])

$$\Gamma^{(n)}(-1) = n!\sum_{j=2}^{n+1}\frac{\Gamma^{(j)}(1)}{j!} + n!(\gamma - 1). \tag{23}$$

The reason is similar to Remark 2.4. So, (23) should be corrected by (22).

III. The partial derivatives of the incomplete Gamma function

Theorem 3.1 Let n be a non-negative integer. Then

$$\begin{aligned}&\Gamma^{(n)}(\alpha,z) = \Gamma^{(n)}(\alpha)\\ &-n!\sum_{k=0}^{\infty}\frac{(-1)^k z^{k+\alpha}}{k!}\sum_{j=0}^{n}\frac{(-1)^j \ln^{n-j} z}{(\alpha+k)^{j+1}(n-j)!}\end{aligned} \tag{24}$$

for $\alpha \in \mathbb{C}\setminus\mathbb{N}_0^-$.

Proof. For the incomplete Gamma function $\Gamma(\alpha, z)$ ([13], Sect. 9.2), we have

$$\Gamma(\alpha,z) = \Gamma(\alpha) - \sum_{k=0}^{\infty}\frac{(-1)^k z^{k+\alpha}}{k!\,(\alpha+k)}, \tag{25}$$

where $\alpha \in \mathbb{C}\setminus\mathbb{N}_0^-$. Calculating n-order partial derivatives on α for (25) by using the Leibniz's rule, we can obtain (24). ■

Lemma 3.2 For $n \in \mathbb{N}_0$, we have

$$\Gamma^{(n)}(0,z) = \frac{\Gamma^{(n+1)}(1,z) - e^{-z}\ln^{n+1} z}{n+1}. \tag{26}$$

Proof. Setting $\alpha = 0$ in (3) and integrating by parts, we have

$$\begin{aligned}&\Gamma^{(n)}(0,z)\\ =\ & \int_z^{\infty} t^{-1}e^{-t}\ln^n t dt\\ =\ & \int_z^{\infty} e^{-t}\ln^n t d\ln t\\ =\ & -e^{-z}\ln^{n+1} z + \int_z^{\infty} e^{-t}\ln^{n+1} t dt\\ & -n\int_z^{\infty} t^{-1}e^{-t}\ln^n t dt\\ =\ & \Gamma^{(n+1)}(1,z) - e^{-z}\ln^{n+1} z - n\Gamma^{(n)}(0,z),\end{aligned} \tag{27}$$

which implies that (26) holds. ■

Lemma 3.3 For $m \in \mathbb{N}$, $\Gamma(-m, z)$ is given by

$$\begin{aligned}\Gamma(-m,z) = \ & \frac{(-1)^m}{m!}\Big[\Gamma^{(1)}(1,z) - e^{-z}\ln z\\ & +e^{-z}\sum_{k=1}^{m}(-1)^k(k-1)!z^{-k}\Big].\end{aligned} \tag{28}$$

Proof. With the aid of (26), we note that (28) holds if and only if the following formula

$$\begin{aligned}\Gamma(0,z) = \ & (-1)^m m!\Gamma(-m,z)\\ & -e^{-z}\sum_{k=1}^{m}(-1)^k(k-1)!z^{-k},\end{aligned} \tag{29}$$

holds for $m \in \mathbb{N}$.

Therefore, we aim to prove (29) holds for $m \in \mathbb{N}$ by using the mathematical induction.

(I). When $m = 1$, (29) reduces to

$$\Gamma(0,z) = -\Gamma(-1,z) + z^{-1}e^{-z}, \tag{30}$$

which can be obtained by using the following recursive relation([13], Sect. 9.2)

$$\Gamma(\alpha+1,z) = \alpha\Gamma(\alpha,z) + z^{\alpha}e^{-z}. \tag{31}$$

(II). Now assume that (29) holds for $m = p(p \in \mathbb{N})$, *i.e.*,

$$\begin{aligned}\Gamma(0,z) = \ & (-1)^p p!\Gamma(-p,z)\\ & -e^{-z}\sum_{k=1}^{p}(-1)^k(k-1)!z^{-k}.\end{aligned} \tag{32}$$

Combining (31) with (32), we yield

$$\begin{aligned}&\Gamma(0,z)\\ =\ & (-1)^p p!\left[-(p+1)\Gamma(-p-1,z) + z^{-p-1}e^{-z}\right]\\ & -e^{-z}\sum_{k=1}^{p}(-1)^k(k-1)!z^{-k}\\ =\ & (-1)^{p+1}(p+1)!\Gamma(-p-1,z)\\ & -e^{-z}\sum_{k=1}^{p+1}(-1)^k(k-1)!z^{-k},\end{aligned} \tag{33}$$

which means that (29) holds for $m = p + 1$. According to the mathematical induction, we conclude that (29) holds for $m \in \mathbb{N}$. ■

Theorem 3.4 Let n and m be positive integers. Then $\Gamma^{(n)}(-m, z)$ can be expressed as follows,

$$\begin{aligned}&\Gamma^{(n)}(-m,z) = \frac{(-1)^m n!}{m!}\times\\ &\left(e^{-z}\sum_{j=0}^{n}\frac{H_{m,j+1}(z)\ln^{n-j} z}{(n-j)!} + \right.\\ &\left.\sum_{j=0}^{n} H_m(1_j)\frac{\Gamma^{(n+1-j)}(1,z) - e^{-z}\ln^{n+1-j} z}{(n+1-j)!}\right),\end{aligned} \tag{34}$$

where

$$\begin{aligned}H_{m,1}(z) = \ & \sum_{k=1}^{m}(-1)^k(k-1)!z^{-k},\\ H_{m,l+1}(z) = \ & \sum_{m\ge m_1\ge\cdots\ge m_l\ge 1}\frac{1}{m_1\cdots m_l}\times\\ & \sum_{k=1}^{m_l}(-1)^k(k-1)!z^{-k},\quad l \in \mathbb{N}.\end{aligned} \tag{35}$$

Proof. Similar to the proof of Theorem 2.3, one has

$$\begin{aligned} & \Gamma^{(n)}(-m,z) \\ = & \int_z^{\infty} t^{-m-1}e^{-t}\ln^n t dt \\ = & \frac{(-1)^m}{m!}\Big[\Gamma^{(n)}(0,z)+H_{m,1}(z)e^{-z}\ln^n z \\ & +n\sum_{m_1=1}^{m}(-1)^{m_1}(m_1-1)!\Gamma^{(n-1)}(-m_1,z)\Big]. \end{aligned} \tag{36}$$

With the help of (26), (36) can be rewritten as follows,

$$\begin{aligned} & \Gamma^{(n)}(-m,z) \\ = & \frac{(-1)^m}{m!}\left[n\sum_{m_1=1}^{m}(-1)^{m_1}(m_1-1)!\right. \\ & \times\Gamma^{(n-1)}(-m_1,z)+H_{m,1}(z)e^{-z}\ln^n z \\ & \left.+\frac{\Gamma^{(n+1)}(1,z)-e^{-z}\ln^{n+1}z}{n+1}\right]. \end{aligned} \tag{37}$$

Inserting (28) into (37) as $n=1$, we obtain

$$\begin{aligned} & \Gamma^{(1)}(-m,z) \\ = & \frac{(-1)^m}{m!}\left[H_m\left(\Gamma^{(1)}(1,z)-e^{-z}\ln z\right)\right. \\ & +e^{-z}H_{m,2}(z)+e^{-z}H_{m,1}(z)\ln z \\ & \left.+\frac{1}{2}\left(\Gamma^{(2)}(1,z)-e^{-z}\ln^2 z\right)\right]. \end{aligned} \tag{38}$$

Reusing (37), we have

$$\begin{aligned} & \Gamma^{(n)}(-m,z)=\frac{(-1)^m n!}{m!}\times \\ & \left[\sum_{j=0}^{n-2}H_m(1_j)\frac{\Gamma^{(n+1-j)}(1,z)-e^{-z}\ln^{n+1-j}z}{(n+1-j)!}\right. \\ & +e^{-z}\sum_{j=0}^{n-2}\frac{H_{m,j+1}(z)\ln^{n-j}z}{(n-j)!}+\sum_{m\geqslant m_1\geqslant\cdots\geqslant m_{n-2}\geqslant 1} \\ & \left.\frac{1}{m_1m_2\cdots m_{n-2}}\sum_{l=1}^{m_{n-2}}(-1)^l(l-1)!\Gamma^{(1)}(-l,z)\right]. \end{aligned} \tag{39}$$

Combining (38) and (39), we finally obtain (34). ■

IV. Applications

A. The expressions of the closed forms for some special integrals

With the aid of (5), we can obtain the closed forms of the following integrals

$$\int_0^{\infty} t^{\alpha-1}e^{-t}\ln^n t dt=\Gamma^{(n)}(\alpha), \tag{40}$$

for $\alpha=m$ or $\frac{2m-1}{2}(m\in\mathbb{N})$ and $n\in\mathbb{N}_0$, which can be expressed by the Riemann zeta functions and some special constants such as Euler's constant γ and π in the following examples by using the Mathematica software.

Example 4.1 Setting $\alpha=5,\quad n=3$ in (40), we have

$$\begin{aligned} & \int_0^{\infty} t^4e^{-t}\ln^3 t dt \\ = & 60-210\gamma+150\gamma^2-24\gamma^3 \\ & +\Big[25-12\gamma\Big]\pi^2-48\zeta(3). \end{aligned} \tag{41}$$

Example 4.2 Setting $\alpha=3,\quad n=4$ in (40), we have

$$\begin{aligned} & \int_0^{\infty} t^2e^{-t}\ln^4 t dt \\ = & 12\gamma^2-12\gamma^3+2\gamma^4+\Big[2-6\gamma+2\gamma^2\Big]\pi^2 \\ & +\frac{3}{10}\pi^4+8\Big[2\gamma-3\Big]\zeta(3). \end{aligned} \tag{42}$$

Example 4.3 Setting $\alpha=\frac{1}{2},\quad n=3$ in (40), we have

$$\begin{aligned} & \frac{1}{\sqrt{\pi}}\int_0^{\infty} t^{-\frac{1}{2}}e^{-t}\ln^3 t dt \\ = & -\frac{3}{2}\gamma\pi^2-\gamma^3-3\Big[2\gamma^2+\pi^2\Big]\ln 2 \\ & -12\gamma\ln^2 2-8\ln^3 2-14\zeta(3). \end{aligned} \tag{43}$$

Example 4.4 Setting $\alpha=\frac{5}{2},\quad n=4$ in (40), we have

$$\begin{aligned} & \frac{1}{\sqrt{\pi}}\int_0^{\infty} t^{\frac{3}{2}}e^{-t}\ln^4 t dt \\ = & 12\gamma^2-8\gamma^3+\frac{3\gamma^4}{4} \\ & +3\Big[2-4\gamma+\frac{3\gamma^2}{4}\Big]\pi^2+\frac{21\pi^4}{16} \\ & +3\Big[16\gamma-16\gamma^2+2\gamma^3-8\pi^2+3\gamma\pi^2\Big]\ln 2 \\ & +3\Big[16-32\gamma+6\gamma^2+3\pi^2\Big]\ln^2 2+12\ln^4 2 \\ & +8\Big[3\gamma-8\Big]\ln^3 2+14\Big[6\ln 2-8+3\gamma\Big]\zeta(3). \end{aligned} \tag{44}$$

Calculating $n-$order derivatives on β for the following integral [14](pp.657),

$$\int_0^{\infty} e^{-\alpha t}\Gamma(\beta,t)dt=\frac{\Gamma(\beta)}{\alpha}\Big[1-(1+\alpha)^{-\beta}\Big],\quad \beta>0, \tag{45}$$

we yield

$$\begin{aligned} & \int_0^{\infty} e^{-\alpha t}\Gamma^{(n)}(\beta,t)dt \\ = & \frac{1}{\alpha}\Gamma^{(n)}(\beta)\Big[1-(1+\alpha)^{-\beta}\Big]-\frac{1}{\alpha}\sum_{k=0}^{n-1}C_n^k \\ & \times\Gamma^{(k)}(\beta)\ln^{n-k}[(1+\alpha)^{-1}](1+\alpha)^{-\beta}. \end{aligned} \tag{46}$$

Similarly, we can obtain the closed forms of integrals (46) in the following examples.

Example 4.5 Setting $\alpha=\frac{1}{2},\quad n=3,\quad \beta=3$ in (46), we have

$$\begin{aligned} & 27\int_0^{\infty} e^{-\frac{1}{2}t}\Gamma^{(3)}(3,t)dt \\ = & 38\Big[-6\gamma+9\gamma^2-2\gamma^3\Big]+19\Big[3-2\gamma\Big]\pi^2 \\ & +16\Big[6-18\gamma+6\gamma^2+\pi^2\Big]\ln\frac{3}{2} \\ & +48\Big[2\gamma-3\Big]\ln^2\frac{3}{2}+32\ln^3\frac{3}{2}-152\zeta(3). \end{aligned} \tag{47}$$

Example 4.6 Setting $\alpha=1,\quad n=2,\quad \beta=\frac{3}{2}$ in (46), we have

$$\begin{aligned} & \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-t}\Gamma^{(2)}\Big(\frac{3}{2},t\Big)dt \\ = & (1-2\sqrt{2})\Big[\gamma-\frac{\gamma^2}{4}-\frac{\pi^2}{8}\Big] \\ & +(3-4\sqrt{2})\Big[1-\frac{\gamma}{2}\Big]\ln 2+\Big[2\sqrt{2}-\frac{9}{4}\Big]\ln^2 2. \end{aligned} \tag{48}$$

B. Arbitrary precision calculation of $\Gamma^{(n)}(\alpha), \Gamma^{(n)}(\alpha, z)$

In this part, we consider the calculation of $\Gamma^{(n)}(\alpha)$ and $\Gamma^{(n)}(\alpha, z)$ at any specified precision by using (3), (5) and (24).

Here we denote the algorithm of (5) and (24) by FHP, while NIntegrate is still used to represent the numerical integration of (3) by the internal command NIntegrate in Mathemaitca software.

The comparison between the FHP and the NIntegrate algorithm by using Mathematica are listed in Table I, II, III and IV (*Computer Systems: Intel(R) Core(TM) i5-3470 CPU@3.20GHz 3.20GHz RAM 3.47GB*). It's important to note that T_P and R_P represent the running time*(unit: second)* and the relative error of the two numerical algorithms with the precision P, respectively.

Seen from Tables I, II, III and IV, the relative error of NIntegrate is much larger than FHP while the computing speed of NIntegrate is more slowly than FHP. Furthermore, the calculation speed of FHP is about $10 \sim 10^2$ times faster than NIntegrate in Tables II, III and IV.

TABLE I
CALCULATION OF $\Gamma^{(n)}(\alpha)$

(n, α)	$(4, \frac{3}{2})$		$(8, \frac{3}{2} + i)$	
Algorithm	NIntegrate	FHP	NIntegrate	FHP
T_{64}	0.750	0.	1.781	0.
R_{64}	10^{-65}	10^{-80}	10^{-64}	10^{-79}
T_{128}	2.953	0.	6.796	0.
R_{128}	10^{-129}	10^{-160}	10^{-128}	10^{-159}
T_{256}	15.343	0.	31.625	0.
R_{256}	10^{-257}	10^{-320}	10^{-256}	10^{-319}

TABLE II
CALCULATION OF $\Gamma^{(n)}(\alpha)$

(n, α)	$(16, \frac{22}{9} + 3i)$	
Algorithm	NIntegrate	FHP
T_{64}	1.265	0.156
R_{64}	10^{-47}	10^{-75}
T_{128}	3.437	0.156
R_{128}	10^{-76}	10^{-155}
T_{256}	8.750	0.156
R_{256}	10^{-134}	10^{-315}

V. CONCLUSION

In this work, we have constructed some recursive relations for the derivatives of the Gamma function $\Gamma(\alpha)$ and incomplete Gamma function $\Gamma(\alpha, z)$ for $\alpha \in \mathbb{C}$. With the help of those results, the closed forms of some special integrals are established in Examples 4.1$\sim$ 4.6, which can be expressed by the special constants and the Riemann zeta functions. Using the neutrix limit, we show that $\Gamma^{(n)}(-m)(n, m \in \mathbb{N}_0)$ can be expressed as linear forms in $\Gamma^{(j)}(1)(j = 0, 1, \ldots, n+1)$. By comparing with Theorem 2.3, we find that there are some mistakes in the expression of $\Gamma'(-m)$ and $\Gamma^{(n)}(-1)$ shown in Theorem 4 and 5 in [2], respectively. Furthermore, the corresponding corrections have been given in Remark 2.4 and 2.5, respectively. Finally, $\Gamma^{(n)}(-m, z)$ are represented as the combination of $\Gamma^{(j)}(1, z)(j = 0, 1, \ldots, n + 1)$ and the elementary functions. Numerical results in Tables I $\sim$ IV show that the FHP algorithm does not only improve the accuracy up to the specified precision usually, but also reduces the time-consuming effectively.

TABLE III
CALCULATION OF $\Gamma^{(n)}(\alpha, z)$

(n, α, z)	$(4, \frac{3}{2}, \frac{1}{2})$		$(8, -\frac{3}{2}, 2 + i)$	
Algorithm	NIntegrate	FHP	NIntegrate	FHP
T_{64}	0.406	0.015	1.062	0.031
R_{64}	10^{-65}	10^{-79}	10^{-64}	10^{-71}
T_{128}	1.859	0.015	4.062	0.031
R_{128}	10^{-129}	10^{-159}	10^{-128}	10^{-151}
T_{256}	9.625	0.031	20.700	0.062
R_{256}	10^{-257}	10^{-319}	10^{-256}	10^{-311}

TABLE IV
CALCULATION OF $\Gamma^{(n)}(\alpha, z)$

(n, α, z)	$(16, -\frac{13}{5} + i, 4 + 3i)$		$(16, -\frac{13}{5}, \frac{1}{2}i)$	
Algorithm	NIntegrate	FHP	NIntegrate	FHP
T_{64}	1.218	0.125	2.031	0.046
R_{64}	10^{-64}	10^{-66}	10^{-64}	10^{-64}
T_{128}	5.203	0.203	7.484	0.171
R_{128}	10^{-128}	10^{-146}	10^{-128}	10^{-144}
T_{256}	27.406	0.390	37.578	0.328
R_{256}	10^{-256}	10^{-306}	10^{-256}	10^{-304}

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