The Representations on the Partial Derivatives of the Extended, Generalized Gamma and Incomplete Gamma Functions and Their Applications

Aiju Li, and Huizeng Qin

Abstract—In this paper, some recursive relations of the derivatives of the gamma function $\Gamma(\alpha)$ and incomplete gamma function $\Gamma(\alpha, z)$ are obtained for complex number $\alpha \neq 0, -1, -2, \cdots$. Moreover, the recurrence relation of the derivative of the $k$-gamma function $\Gamma_k(x)$ are also given. Thus, the partial derivative $\frac{\partial^{p+q}}{\partial p \partial q} B(\nu, \mu)$ of the Beta function $B(\nu, \mu)$ can also be represented for $p, q = 0, 1, 2, \cdots$. Based on these results, the partial derivatives of the extended, generalized complete and incomplete gamma functions are obtained. Furthermore, the partial derivative of confluent hypergeometric function is also considered.

Index Terms—incomplete gamma function, gamma function, confluent hypergeometric function, Digamma function.

I. INTRODUCTION

The gamma function $\Gamma(\alpha)$ and incomplete gamma function $\Gamma(\alpha, z)$ were defined by the following integrals in [1]

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt, \quad Re(\alpha) > 0, \quad (1)$$

$$\gamma(\alpha, z) = \int_0^z t^{\alpha-1}e^{-t}dt, \quad \alpha, z \in C, \quad |\arg(z)| < \pi, \quad z \neq 0,$$

and

$$\Gamma(\alpha, z) = \int_z^\infty t^{\alpha-1}e^{-t}dt, \quad \alpha, z \in C, \quad |\arg(z)| < \pi, \quad z \neq 0,$$

where $\alpha$ can be extended to all complex numbers except non-positive integers and $C$ denotes a complex set. These functions were found to be useful in heat conduction, probability theory and in the study of Fourier and Laplace transforms in [1]. Moreover, the gamma distribution which is formulated in terms of the gamma function is used in statistics to model a wide range of processes. For example, the time between occurrences of earthquakes in [2]. Furthermore, these functions can be expanded. For example, the extended complete and incomplete gamma functions were defined by the following integrals in [1,3-5]:

$$\Gamma(\alpha, 0; b) = \int_0^\infty t^{\alpha-1}e^{-t-bt^{-1}}dt,$$

$$\gamma(\alpha, x; b) = \int_x^\infty t^{\alpha-1}e^{-t-bt^{-1}}dt,$$

$$\Gamma(\alpha, x; b) = \int_0^\infty t^{\alpha-1}e^{-t-bt^{-1}}dt,$$

where $\alpha \in R, x \geq 0, b \geq 0$, but not both $x = 0, b = 0$, if $\alpha \leq 0$. The function $\Gamma(\alpha; x; b)$ can be used in closed form solutions to several problems in heat conduction with time-dependent boundary conditions. Moreover, the gamma function $\Gamma(\alpha)$ has played a role in cumulative probability functions in [1,6,7]. Similarly, the generalized complete and incomplete gamma functions were also defined by

$$\Gamma_\beta(\alpha, c) = \int_0^\infty e^{-c \cdot \frac{t^{\alpha-1}e^{-t}}{\Gamma(\alpha)}}dt, \quad \gamma_\beta(\alpha, x, c) = \int_0^x e^{-c \cdot \frac{t^{\alpha-1}e^{-t}}{\Gamma(\alpha)}}dt,$$

where $\alpha = n + \epsilon \geq 1(n = 1, 2, \cdots, 0 \leq \epsilon < 1), \beta = m + \delta \geq 0(m = 1, 2, \cdots, -1 \leq \delta \leq 0)$ and $x, c > 0$ in [8,9]. The generalized gamma functions have been widely used in the solution of many problems of wave scattering and diffraction theory in [9-11].

In this paper, we assume that $\alpha, z \in C, |\arg(z)| < \pi$ and $z \neq 0$ for (1)-(3). Since $\gamma(\alpha, z) = \Gamma(\alpha) - \Gamma(\alpha, z)$, we only consider $\Gamma^{(n)}(\alpha)$ and $\Gamma^{(n)}(\alpha, z)(n = 0, 1, 2, \cdots)$, i.e.

$$\Gamma^{(n)}(\alpha) = \frac{d^n}{d\alpha^n} \Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}ln^ndt,$$

$$\Gamma^{(n)}(\alpha, z) = \int_z^\infty t^{\alpha-1}e^{-t}ln^ndt,$$

Similarly, we also consider the following derivatives for (4) and (5), i.e.

$$\Gamma^{(n)}(\alpha, 0; b) = \frac{d^n}{d\alpha^n} \Gamma(\alpha, 0; b) = \int_0^\infty t^{\alpha-1}e^{-t-bt^{-1}}ln^ndt,$$

$$\Gamma^{(n)}(\alpha, x; b) = \frac{d^n}{d\alpha^n} \Gamma(\alpha, x; b) = \int_x^\infty t^{\alpha-1}e^{-t-bt^{-1}}ln^ndt,$$

and

$$\Gamma^{(n,m)}(\alpha, c) = \frac{d^{m+n}}{d\alpha^md\alpha^n} \Gamma(\alpha, c) = (-1)^m \int_0^\infty e^{-t-c \cdot \frac{t^{\alpha-1}e^{-t}}{\Gamma(\alpha)}} \cdot ln^m(t+c)dt,$$

$$\Gamma^{(n,m)}(\alpha, x, c) = \frac{d^{m+n}}{d\alpha^md\alpha^n} \Gamma(\alpha, x, c) = (-1)^m \int_0^\infty e^{-t-c \cdot \frac{t^{\alpha-1}e^{-t}}{\Gamma(\alpha)}} \cdot ln^m(t+c)dt,$$

for $n, m = 0, 1, 2, \cdots$.

The structure of this paper is organized as follows. In Section 2, some recursive relations of the derivatives of the gamma function and $k$-gamma function are derived. Thus, $\check{\Gamma}(n)(\alpha) = \frac{d^n}{d\alpha^n} \check{\Gamma}(\alpha) \cdot \frac{1}{\check{\Gamma}(\alpha)}$ and partial derivative $\frac{\partial^{p+q}}{\partial p \partial q} B(\nu, \mu)$ of the Beta function $B(\nu, \mu)$ can also be represented for $p, q = 0, 1, 2, \cdots$. In Section 3, some recursive relations of the partial derivatives of the incomplete gamma function are established. In Section 4, using the results of Section 2 and 3, we consider series expansions of (7) and (8). Moreover, the partial derivative of confluent hypergeometric

(Advance online publication: 23 August 2017)
function is also considered. The conclusion is given in the last section of the paper.

II. THE RECURSIVE FORMULAS OF THE DERIVATIVES OF THE GAMMA FUNCTION

**Theorem 2.1** Let $n \geq 1$ be an integer. Then the recurrence relation of $\Gamma^{(n)}(\alpha)(\alpha \neq 0, -1, -2, \ldots)$ can be expressed as follows,

$$\Gamma^{(n)}(\alpha) = H_n^{\psi}(\alpha) \Gamma(\alpha),$$

where

$$H_n^{\psi}(\alpha) = 1, \quad H_n^{\psi}(\alpha) = \sum_{i=0}^{n-1} H_{n,i}^{\psi}(\alpha),$$

and

$$H_{n,i}^{\psi}(\alpha) = \psi(n-1+i)(\alpha),$$

and $\psi(\alpha)$ is the Digamma function defined by

$$\psi(\alpha) = -\gamma - \frac{1}{\alpha} + \sum_{i=1}^{\infty} \left( \frac{1}{i} - \frac{1}{i + \alpha} \right),$$

and $\gamma$ denotes Euler’s constant.

**Proof.** The Digamma function $\psi(\alpha)$ and its $k$–order derivatives $\psi^{(k)}(\alpha)(k = 1, 2, \ldots)$ can be expressed as follows,

$$\psi(\alpha) = \frac{d}{d\alpha} \ln \Gamma(\alpha) = -\gamma - \frac{1}{\alpha} + \sum_{i=1}^{\infty} \left( \frac{1}{i} - \frac{1}{i + \alpha} \right),$$

for $\alpha \neq 0, -1, -2, \ldots,$ and

$$\psi^{(k)}(\alpha) = \frac{d^k}{d\alpha^k} \psi(\alpha) = k!(-1)^{k+1} \sum_{i=0}^{\infty} \frac{1}{(i + \alpha)^{k+1}},$$

where $\gamma$ denotes Euler’s constant. From (12), we have

$$\Gamma^{(n)}(\alpha) = \psi(n-1+i)(\alpha) \Gamma^{(k)}(\alpha), \quad \alpha \neq 0, -1, -2, \ldots.$$ (14)

Calculating $(n-1)(n = 2, 3, \ldots)$-order derivatives on $\alpha$ for (14) by using the Leibniz’s derivation rule, we get

$$\Gamma^{(n)}(\alpha) = \sum_{i=0}^{n-1} \binom{n-1}{i} \psi(n-1-k)(\alpha) \Gamma^{(k)}(\alpha),$$

for $\alpha \neq 0, -1, -2, \ldots.$ Repeated use of this recursive formula we can see that (9)-(11) hold.

**Remark 2.1** K. S. Kolbig considered the following Laplace (or Mellin) integral in [12]

$$R_m(\mu, \nu) = \int_0^\infty t^{\nu-1} e^{-at} \ln \Gamma(\nu), \quad Re\mu, Re\nu > 0,$$

and gave a recurrence formula

$$G_k(\mu, \nu) = \frac{d}{d\nu} G_{k-1}(\mu, \nu) + G_1(\mu, \nu) G_{k-1}(\mu, \nu),$$

$$G_0(\mu, \nu) = 1, \quad G_1(\mu, \nu) = \psi(\nu) - \ln \mu,$$

where $R_m(\mu, \nu) = \frac{\Gamma(\nu)}{\mu^\nu} G_m(\mu, \nu).$ By the variable substitution for (16) we have

$$R_0(\mu, \nu) = \mu^{-\nu} \int_0^{\infty} t^{\nu-1} e^{-t} dt = \mu^{-\nu} \Gamma(\nu),$$

Moreover, using the Leibniz’s derivation rule on $\nu$ for (18), we have

$$R_m(\mu, \nu) = \frac{d^m}{d\nu^m} R_0(\mu, \nu) = \mu^{-\nu} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \ln \mu^{m-k} \Gamma^{(k)}(\nu),$$

Thus, $R_m(\mu, \nu)$ can also be calculated by Theorem 2.1.

Moreover, the $k$–gamma function $\Gamma_k(x)$ is the generalization of $\Gamma(x)$ and it was given by the following integral in [13]

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{k}{t}} dt, \quad x, k > 0,$$

and $\Gamma(x) = \Gamma_1(x).$

Through variable substitution we obtain that $\Gamma_k(x) = k^{-x} \Gamma\left( \frac{x}{k} \right)$ and

$$\psi_k(x) = (\ln \Gamma_k(x))' = \frac{1}{k} (\ln k + \psi(\frac{x}{k})).$$

Corresponding to Theorem 2.1, we have the following theorem.

**Theorem 2.2** Let $n \geq 1$ be an integer. Then the recurrence relation of $\Gamma_k^{(n)}(\alpha)(\alpha \neq 0, -1, -2, \ldots)$ can be expressed as follows,

$$\Gamma_k^{(n)}(\alpha) = H_n^{\psi}(\alpha) \Gamma_k^{(n)}(\alpha),$$

where

$$H_n^{\psi}(\alpha) = 1, \quad H_n^{\psi}(\alpha) = \sum_{i=0}^{n-1} H_{n,i}^{\psi}(\alpha),$$

and

$$H_{n,i}^{\psi}(\alpha) = \psi(n-1+i)(\alpha) H_k^{\psi}(\alpha),$$

for $\alpha \neq 0, -1, -2, \ldots.$

Similiarly, according to the proof of Theorem 2.1, the recurrence relation of $\Gamma_k^{(n)}(\alpha)(\alpha \neq 0, -1, -2, \ldots)$ can be obtained.

**Theorem 2.3** Let $n \geq 1$ be an integer and $\tilde{\Gamma}(\alpha) = \frac{1}{\Gamma(\alpha)}.$

Then the recurrence relation of $\tilde{\Gamma}(n)(\alpha)(\alpha \neq 0, -1, -2, \ldots)$ can be expressed as follows,

$$\tilde{\Gamma}^{(n)}(\alpha) = \tilde{H}_n^{\tilde{\psi}}(\alpha) \tilde{\Gamma}(\alpha)$$

where

$$\tilde{H}_0^{\tilde{\psi}}(\alpha) = 1, \quad \tilde{H}_n^{\tilde{\psi}}(\alpha) = \sum_{i=0}^{n-1} \tilde{H}_{n,i}^{\tilde{\psi}}(\alpha),$$

and

$$\tilde{H}_{n,i}^{\tilde{\psi}}(\alpha) = \tilde{\psi}(n-1+i)(\alpha) \tilde{H}_{k+1}^{\tilde{\psi}}(\alpha),$$

for $\alpha \neq 0, -1, -2, \ldots.$

Moreover, we note that the Beta function is defined by

$$B(\nu, \mu) = \int_0^1 t^{\nu-1}(1-t)^{\mu-1} dt, \quad Re\nu, Re\mu > 0.$$
Its partial derivatives $B_{p,q}(\nu, \mu) = \frac{\partial^p \partial^q}{\partial \nu^p \partial \mu^q} B(\nu, \mu)(p, q = 0, 1, 2, \cdots)$ have been discussed for $\nu, \mu, \nu + \mu \neq 0, \pm 1, \pm 2, \cdots$ in [14,15].

Furthermore, by

$$B(\nu, \mu) = \frac{\Gamma(\nu) \Gamma(\mu)}{\Gamma(\nu + \mu)} = \Gamma(\nu) \Gamma(\mu) \Gamma(\nu + \mu)$$

and Leibniz’s derivation rule we have the following theorem.

**Theorem 2.4** Let $p, q \geq 0$ be integers. Then

$$\frac{d^n}{dx^n} \left( \frac{B_{p,q}(\nu, \mu)}{1+x} \right)$$

for $\nu, \mu, \nu + \mu \neq 0, \pm 1, \pm 2, \cdots$, where $\Gamma(\nu + \mu) = \frac{1}{\Gamma(\nu + \mu)}$.

By (27), we notice that the partial derivatives of the Beta function can also be calculated by Theorem 2.1 and Theorem 2.3.

Note that

$$\Gamma(n) = (n-1)!, \Gamma(n) = \frac{2n-1)!}{2n-1}, \Gamma(\frac{1}{2} - n) = (-2n)! \Gamma(\frac{1}{2} - n),$$

and

$$\psi(n) = H_n - 1 - \gamma, \quad \psi(n - \frac{1}{2}) = -\gamma - 2 \ln 2 + 2H_{2n-2} - H_{n-1}.$$}

for $n = 1, 2, \cdots$, where $H_n = \sum_{l=1}^{n} \frac{1}{l}$ and $H_{n,s} = \sum_{l=1}^{n} \frac{1}{l^s}$. Using (9), (24) and (27), we obtain that $\Gamma^{(n)}(\alpha), \Gamma^{(n)}(\beta)(n = 1, 2, \cdots)$ and $B_{p,q}(\alpha, \beta)(p, q, 0, 1, 2, \cdots)$ exist the close forms for $\alpha, \beta = k, k + \frac{1}{2}$ and $k = 0, \pm 1, \pm 2, \cdots$. For Pochhammer symbol $(\alpha)_n$, i.e. $x_n = x(x+1)(x+2)\cdots(x+n-1)$, we have the following result: $(x_n)^{m} = (x_n) \sum_{i=0}^{m-1} \frac{1}{(x+i)^{m}}$.

According to the method to prove Theorem 2.1 and 2.4, we easily get the following recurrence formula.

**Corollary 2.5** Let $n, m \geq 1$ be integers and $x \neq 0, -1, -2, \cdots$ be a complex number. Then

$$(x_n)^{(m)} = (m-1)! \sum_{k=0}^{m-1} \frac{(-1)^k}{(m-k)!} H_{m-k-1} s(n, k)x^k.$$ (31)

and

$$(\overline{x}_n)^{(m)} = -(m-1)! \sum_{k=0}^{m-1} \frac{(-1)^k}{(m-k)!} H_{m-k-1} \overline{x}_n (x_n)^{(m-k)}.$$ (32)

where $\overline{x}_n = \frac{1}{\Gamma(n+1)} H_{n,p}(x)$ and $\overline{C}_n^{(m)} = \frac{d^n}{dx^n} \overline{x}_n$.

However, the derivatives of $x_n$ and $\overline{x}_n$ can also be calculated by the following method.

Since $x_n = \sum_{k=0}^{n} \frac{(-1)^{n-k} s(n,k) x^k}{k!}$ and $\overline{x}_n = \frac{1}{\Gamma(n+1)} C_n^{(-1)/n}$, we have

$$(x_n)^{(m)} = m! \sum_{k=m}^{n} \frac{k}{m} \frac{(-1)^{n-k} s(n,k) x^{k-m}}{k!},$$ (33)

and

$$\overline{x}_n^{(m)} = \frac{(-1)^{m+1} m!}{(n-1)!} \sum_{k=0}^{n-1} C_k^{n-1} \frac{(-1)^{k}}{(x+k)^{n-1}}.$$ (34)

where $s(n,k)$ is the Stirling number of the first kind and $n, m = 1, 2, \cdots$.

In the following section, the partial derivatives of the incomplete gamma function are given.

III. THE PARTIAL DERIVATIVES OF THE INCOMPLETE GAMMA FUNCTION

**Theorem 3.1** Let $n$ be a non-negative integer. Then

$$\Gamma^{(n)}(\alpha, z) = n! \sum_{k=0}^{\infty} (-1)^k \frac{k!}{(\alpha+k)!} \sum_{j=0}^{n} \frac{(-1)^j \ln^{n-j} z}{(n-j)!},$$ (35)

for $\alpha \neq -1, -2, \cdots$.

Proof. For the incomplete gamma function $\Gamma(\alpha, z)$, we have

$$\Gamma(\alpha, z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{k+\alpha}}{k!}$$ (36)

Calculating $n$-order partial derivatives on $\alpha$ for (36) by using the Leibniz’s derivatives rule, we can obtain (35).

Note that

$$\Gamma(\alpha, z) = \frac{\Gamma(\alpha+1, z) - e^{-z}}{\alpha+1} = \frac{\Gamma(\alpha+2, z) - e^{-z}}{\alpha+2} = \cdots = \frac{\Gamma(\alpha+m, z) - e^{-z} \sum_{k=0}^{m} \frac{z^k}{(\alpha+k)!}}{\alpha+m}$$ (37)

By using Leibniz’s derivation rule on $\alpha$ for (37), we have

$$\Gamma^{(n)}(\alpha, z) = n! \sum_{l=0}^{\infty} \frac{\Gamma^{(n-l)}(\alpha+m, z)}{(\alpha+l)(\alpha+l-1)(\alpha+l-2)\cdots(\alpha+1)} \sum_{j=0}^{m} \frac{(-1)^j \ln^{n-l} z}{(n-l)!}$$

However, $\Gamma(\alpha, z)$ and $\Gamma(\alpha)\alpha$ are not defined for $\alpha = 0$. Thus, we give their complementary definitions. According to the neutrix limit in [16] we have

$$\Gamma^{(n)}(0) = N = -\lim_{\varepsilon \to 0} \int_{0}^{\infty} \frac{t^{n} \Gamma^{(n)}(1+t)}{e^{t} - 1} dt$$

$$= -\lim_{\varepsilon \to 0} \left( \frac{1}{\varepsilon^n} e^{-\ln^{n+1} \varepsilon} \right)$$

$$+ N = -\lim_{\varepsilon \to 0} \left( \frac{1}{\varepsilon^n} e^{-\ln^{n+1} \varepsilon} \right)$$

$$= \frac{1}{n+1} \int_{0}^{\infty} e^{-t} \ln^{n+1} t dt = \frac{1}{n+1} \Gamma^{(n+1)}(0)$$ (39)
and

\[ N = \lim_{n \to 0} \sum_{k=0}^{\infty} \frac{(-1)^{j+k} n^{j+i}}{k!} \sum_{j=0}^{n} \frac{(-1)^{j} \ln^{n-j} z}{(n-j)!} \]

\[ = N \lim_{n \to 0} \sum_{k=1}^{\infty} \frac{(-1)^{j} \ln^{n-j} z}{(n-j)!} \frac{z^{k+1}}{k^{1+\tau}} \]

\[ + \sum_{k=1}^{\infty} \frac{(-1)^{j} k^{1+\tau}}{k^{1+\tau}} \sum_{j=0}^{n} \frac{(-1)^{j} \ln^{n-j} z}{(n-j)!} \frac{z^{k}}{k^{1+\tau}} \]

\[ = \sum_{k=1}^{\infty} \frac{(-1)^{j} \ln^{n-j} z}{(n-j)!} \frac{z^{k+1}}{k^{1+\tau}} \]

\[ + \sum_{k=1}^{\infty} \frac{(-1)^{j} k^{1+\tau}}{k^{1+\tau}} \sum_{j=0}^{n} \frac{(-1)^{j} \ln^{n-j} z}{(n-j)!} \frac{z^{k}}{k^{1+\tau}} \]

\[ = \frac{\ln^{n+1} z}{(n+1)!} + \sum_{k=1}^{\infty} \frac{(-1)^{j} k^{1+\tau}}{k^{1+\tau}} \sum_{j=0}^{n} \frac{(-1)^{j} \ln^{n-j} z}{(n-j)!} \frac{z^{k}}{k^{1+\tau}} \]

Let \( \alpha \to m \) in (38), we have the following theorem.

**Theorem 3.2** Let \( n \geq 0 \) and \( m \geq 1 \) be integers.

\[ \Gamma(n) \left( -m, z \right) = n! \sum_{i=0}^{m} \frac{(-1)^{i} \Gamma(n-i) \ln^{n-i} z}{(n-i)!} \frac{z^{i}}{i^{1+\tau}} \]

\[ - n! e^{-z} \sum_{k=1}^{m} \frac{(-1)^{k} k^{1+\tau}}{k^{1+\tau}} \sum_{j=0}^{n-k} \frac{(-1)^{j} \ln^{n-j} z}{(n-j)!} \frac{z^{k}}{k^{1+\tau}} \]

\[ \text{(41)} \]

In the following section, the applications of the partial derivatives of the complete and incomplete gamma functions are given.

**IV. APPLICATIONS OF THE PARTIAL DERIVATIVES OF COMPLETE AND INCOMPLETE GAMMA FUNCTIONS**

**A. The partial derivatives of generalized complete and incomplete gamma functions**

The series expansions for generalized complete and incomplete gamma functions were given in [14] as follows:

\[ \Gamma_{\beta}(\alpha, c) = \sum_{i=0}^{\infty} \left( \frac{(-\beta)^{i}}{i!} \right) \alpha^{i+1} e^{\alpha i} \Gamma(\alpha - i, c) \]

\[ \times \left( \left( -\frac{\beta}{i} \right) \right), \quad \text{for } \beta - \alpha + i < 0, \]

\[ \sum_{i=0}^{\infty} \left( \frac{(-\beta)^{i}}{i!} \right) \alpha^{i+1} e^{\alpha i} \Gamma(\alpha - i, c) \]

\[ \times \left( \left( -\frac{\beta}{i} \right) \right), \quad \text{for } \beta - \alpha + i \geq 0. \]  

\[ \text{(42)} \]

and

\[ \Gamma_{\beta}(\alpha, x, c) = \sum_{i=0}^{\infty} \left( \frac{(-\beta)^{i}}{i!} \right) \frac{\Gamma(\alpha + i, x) - \Gamma(\alpha + i, c)}{e^{\alpha i}} \]

\[ \times \left( \left( -\frac{\beta}{i} \right) \right), \quad \text{for } x < c, \]

\[ \sum_{i=0}^{\infty} \left( \frac{(-\beta)^{i}}{i!} \right) e^{\alpha i} \Gamma(\alpha - i, x), \quad \text{for } x > c. \]

\[ \text{(43)} \]

where \( E_{\beta}(c) \) is the exponential integral function defined by

\[ E_{\beta}(c) = \int_{1}^{\infty} t^{-\beta} e^{-ct} \, dt \]

\[ \text{(44)} \]

However, if we calculate \( \Gamma_{\beta}(\alpha, c) \) and \( \Gamma_{\beta}(\alpha, x, c) \) by using (42)-(44), then the error is caused by the relatively large. In particular, when \( c \) decreases, the error becomes more prominent. The reason for the error is using the following series expansion of (42) and (43).

\[ \frac{1}{(\tau + m)} = \sum_{i=0}^{\infty} \frac{(-\beta - i + 1)^{i}}{i!} \left( \frac{\alpha}{\tau + m} \right)^{i}, \quad |\tau| < |c|, \]

\[ \text{(45)} \]

The convergence speed of the series (45) is very slow at near c.

Since

\[ t^{\alpha - 1} = (t + c)^{\alpha - 1} \sum_{i=0}^{\infty} \frac{(-\alpha)^{i}}{i!} \left( \frac{t + c}{t + c} \right)^{i}, \]

\[ \text{(46)} \]

we give a slightly different form power series expansion as follows:

\[ \Gamma_{\beta}(\alpha, c) = \sum_{i=0}^{\infty} \frac{(-\beta)^{i} \Gamma(\alpha + i, c)}{i!} \frac{t^{\alpha + i - 1} e^{-t}}{t^{i+\tau}}, \quad |\alpha - i| \leq |c|, \]

\[ + \sum_{i=0}^{\infty} \frac{(-\beta)^{i} \Gamma(\alpha + i, c)}{i!} \frac{t^{\alpha + i - 1} e^{-t}}{t^{i+\tau}}, \quad |\alpha - i| > |c|. \]

\[ \text{(47)} \]

and

\[ \Gamma_{\beta}(\alpha, x, c) = \sum_{i=0}^{\infty} \frac{(-\beta)^{i} \Gamma(\alpha + i, x) - \Gamma(\alpha + i, c)}{i!} \frac{t^{\alpha + i - 1} e^{-t}}{t^{i+\tau}}, \quad |x| \leq |c|, \]

\[ + \sum_{i=0}^{\infty} \frac{(-\beta)^{i} \Gamma(\alpha + i, x) - \Gamma(\alpha + i, c)}{i!} \frac{t^{\alpha + i - 1} e^{-t}}{t^{i+\tau}}, \quad |x| > |c|. \]

\[ \text{(48)} \]

where \( c_{1} = \frac{1}{2} \) (Values suggest that the best such how to choose), \( \alpha, \beta, c, x \) can be complex numbers satisfying \( c > 0 \) or \( Im c \neq 0, Re x > 0 \). By numerical experiment, we find that \( \sum_{i=0}^{\infty} \frac{(-\beta)^{i} \Gamma(\alpha + i, x) - \Gamma(\alpha + i, c)}{i!} \frac{t^{\alpha + i - 1} e^{-t}}{t^{i+\tau}} \) has a good convergence rate when \( |\frac{1}{\tau}| < \frac{1}{\sqrt{b}} \) and \( \sum_{i=0}^{\infty} \frac{(-\beta)^{i} \Gamma(\alpha + i, x) - \Gamma(\alpha + i, c)}{i!} \frac{t^{\alpha + i - 1} e^{-t}}{t^{i+\tau}} \)

also has better convergence rate when \( |\frac{1}{\tau + m}| < \frac{3}{5} \).

When \( \alpha = 1, 2, \ldots, 46 \) becomes

\[ t^{\alpha - 1} = \sum_{i=0}^{\infty} \frac{(-\alpha - 1)^{i}}{i!} \left( -c \right)^{\alpha - 1 - i} (t + c)^{i}, \]

\[ \text{(49)} \]

Therefore, we have

\[ \Gamma_{\beta}(\alpha, c) = e^{\alpha - \alpha} \sum_{i=0}^{\infty} \left( \frac{-\alpha - 1}{i} \right) \left( -c \right)^{\alpha - 1 - i} \Gamma(\alpha - \beta - i, c), \]

\[ \text{(50)} \]

and

\[ \Gamma_{\beta}(\alpha, x, c) = e^{\alpha - \alpha} \sum_{i=0}^{\infty} \left( \frac{-\alpha - 1}{i} \right) \left( -c \right)^{\alpha - 1 - i} \Gamma(\alpha - \beta - i, c + x). \]

\[ \text{(51)} \]

It follows from (50) and (51) that \( \Gamma_{\beta}(\alpha, c) \) and \( \Gamma_{\beta}(\alpha, x, c) \) have finite linear combinations of \( \Gamma(\alpha, c) \).

In the following, the results of computing \( \Gamma_{\beta}(\alpha, c) \) are
given in Table I-III by different methods.

<table>
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<th>Table I</th>
<th>The results of computing $\Gamma_\beta(\alpha, c)$ by using (47) or (50)</th>
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<table>
<thead>
<tr>
<th>Table II</th>
<th>The results of computing $\Gamma_\beta(\alpha, c)$ by using (42)</th>
<th>$\beta$</th>
<th>$\alpha$</th>
<th>$c_1$</th>
<th>$\text{using (42)}$</th>
<th>$\text{relative error}$</th>
</tr>
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<tbody>
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<td>0.61436219273560 · · ·</td>
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<td>0.000099529536 · · ·</td>
<td>$10^{-10}$</td>
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<thead>
<tr>
<th>Table III</th>
<th>The real values of $\Gamma_\beta(\alpha, c)$</th>
<th>$\beta$</th>
<th>$\alpha$</th>
<th>$c_1$</th>
<th>$\text{values of } \Gamma_\beta(\alpha, c)$</th>
<th>$\text{relative error}$</th>
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<td>0.000099529536 · · ·</td>
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</table>

Now the results of computing $\Gamma_\beta(\alpha, x, c)$ are given in Table IV-VI by different methods.

<table>
<thead>
<tr>
<th>Table IV</th>
<th>The results of computing $\Gamma_\beta(\alpha, x, c)$ by using (48) or (51)</th>
<th>$\beta$</th>
<th>$\alpha$</th>
<th>$c_1$</th>
<th>$\text{using (48) or (51)}$</th>
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<td>5/2</td>
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<thead>
<tr>
<th>Table V</th>
<th>The results of computing $\Gamma_\beta(\alpha, x, c)$ by using (43)</th>
<th>$\beta$</th>
<th>$\alpha$</th>
<th>$c_1$</th>
<th>$\text{using (43)}$</th>
<th>$\text{relative error}$</th>
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<td>$10^{-10}$</td>
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<table>
<thead>
<tr>
<th>Table VI</th>
<th>The real values of $\Gamma_\beta(\alpha, x, c)$</th>
<th>$\beta$</th>
<th>$\alpha$</th>
<th>$c_1$</th>
<th>$\text{values of } \Gamma_\beta(\alpha, x, c)$</th>
<th>$\text{relative error}$</th>
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<td>4/5</td>
<td>5/2</td>
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<td>0.0016229475288 · · ·</td>
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</table>

Seen from Table I - Table VI, the algorithms (47)-(51) have better accuracy. However, the algorithms (42) and (43) are not only poor accuracy, but also the numerical results are away from the true value for small $c$ (see "**" in Table II and Table V).

In the following, we consider the partial derivatives of $\Gamma_\beta(\alpha, c)$ and $\Gamma_\beta(\alpha, x, c)$.

By (47), (48) and Leibniz’s derivation rule, we have

$$\Gamma_\beta^{(n,m)}(\alpha, c) = \sum_{i=0}^{\infty} (-1)^i \left( \Gamma^{(n)}(\alpha+i) - \Gamma^{(n)}(\alpha+i,c) \right)$$

$$\sum_{j=0}^{m} (-1)^j \left( \frac{m}{j} \right) (\beta)^{(m-j)} \ln^j c$$

$$+ (-1)^{n+m} e^c \sum_{k=0}^{\infty} \frac{e}{n+k} \sum_{i=0}^{\infty} (-1)^{(n+k)} \left( \frac{n}{k} \right) \Gamma(m+k)(\alpha - \beta - i, c + c_1),$$

and

$$= \sum_{j=0}^{m} \left( \frac{m}{j} \right) (\beta)^{(m-j)} \ln^j c, \text{ for } |x| \leq |c_1|,$$

$$+ (-1)^{n+m} e^c \sum_{k=0}^{\infty} \frac{e}{n+k} \sum_{i=0}^{\infty} (-1)^{(n+k)} \left( \frac{n}{k} \right) (1-\alpha)^{(n-k)} \Gamma(m+k)(\alpha - \beta - i, c + x), \text{ for } |x| > |c_1|.$$
For \( \alpha \neq 0, \pm 1, \pm 2, \cdots \) and by (55) and (56), we have

\[
\Gamma^{(n)}(\alpha; b) = \frac{d^n}{dn^0} \Gamma(\alpha; b) = \ln^{n-k} b \times \Gamma^{(k)}(-\alpha - i, \sqrt{b}) + \sum_{i=0}^{\infty} \frac{(-b)^i \Gamma^{(n)}(\alpha - i, \sqrt{b})}{i!},
\]

and

\[
\Gamma^{(n)}(\alpha, x; b) = \frac{d^n}{dx^0} \Gamma(\alpha, x; b) = \left\{ \begin{array}{ll}
\sum_{i=0}^{\infty} \frac{(-b)^i}{i!} \sum_{k=0}^{\infty} \left( \frac{n}{k} \right) \ln^{n-k} b \\
\times \Gamma^{(k)}(-\alpha - i, \sqrt{b}) - \Gamma^{(k)}(-\alpha - i, \sqrt{b}) \\
+ \sum_{i=0}^{\infty} \frac{(-b)^i \Gamma^{(n)}(\alpha - i, \sqrt{b})}{i!},
\end{array} \right.
\]

for \( n = 0, 1, 2, \cdots \).

In the following, the partial derivatives of confluent hypergeometric functions are obtained.

C. The partial derivatives of confluent hypergeometric function

The confluent hypergeometric function is analytic on \( C \) and it is defined by the following series expansion

\[
\Phi(\nu, \mu; z) = \sum_{n=0}^{\infty} \frac{(\nu)_n}{(\mu)_n} \frac{z^n}{n!},
\]

Moreover, the confluent hypergeometric function can also be represented as the following integral

\[
\Phi(\nu, \mu; z) = \frac{\Gamma(\mu)}{\Gamma(\nu) \Gamma(\nu - \mu)} \int_0^1 t^{\nu-1}(1-t)^{\mu-1}e^{-zt}dt, 0 < Rev < Re\mu.
\]

By analytic continuation, \( \Phi(\nu, \mu; z) \) has been defined in addition to \( \mu = 0, -1, -2, \cdots \) from (59).

Moreover, by (60), we have

\[
\int_0^1 t^{\nu-1}(1-t)^{\mu-1}e^{-zt}dt = B(\nu, \mu) \Phi(\nu, \mu; z), \quad Rev, Re\mu > 0.
\]

Using the Leibniz’s derivation rule on \( \nu \) and \( \mu \) for (61), we have

\[
\int_0^1 t^{\nu-1}(1-t)^{\mu-1}e^{-zt}dt = \sum_{k=0}^{\infty} \frac{p^k}{k!} B_p(\nu, \mu) \Phi(\nu, \mu; z),
\]

for \( Rev + q, Re\mu + p > 0 \).

Moreover, using the Leibniz’s derivation rule on \( \nu \) and \( \mu \) for (59), we obtain

\[
\Phi_{p, q}(\nu, \mu; z) = \frac{d^{p+q}}{d\nu^{p}d\mu^{q}} \Phi(\nu, \mu; z) = (-1)^q q! \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!(\nu+j)^{p+q}},
\]

for \( p, q = 0, 1, 2, \cdots \).

Similarly, using the following identity

\[
\Gamma(\nu, \mu; z) = z^{\mu-1} B(\nu, \mu - \nu) \Phi(\nu, \mu; z) = z^{\mu-1} B(\nu, \mu - \nu) \Phi(\nu, \mu; z),
\]

we have

\[
\Gamma(\nu, \mu; z) = z^{\mu-1} B(\nu, \mu - \nu) \Phi(\nu, \mu; z) = z^{\mu-1} B(\nu, \mu - \nu) \Phi(\nu, \mu; z).
\]

(Advance online publication: 23 August 2017)
where
\[ B(\nu, \mu, z) = \int_0^1 u^{\nu-1}(1-u)^{\mu-1} e^{-ze} du \] (71)
is an extension of the Beta function.

By (67) and (70), we have
\[ B(\nu, \mu, z) = e^{-z} B(\nu, \mu) \Phi(\nu, 1-\mu; z) \]
\[ + e^{-z} z^\mu \Gamma(-\mu) \Phi(\nu + \mu, 1 + \mu; z) \] (72)

Therefore, by the Leibniz’ derivation rule on \( \nu, \mu \) for (72), we can obtain the following the derivatives of the extension Beta function \( B(\nu, \mu, z) \):
\[ B_{p,q}(\nu, \mu, z) = \frac{\partial^{p+q}}{\partial \nu^p \partial \mu^q} B(\nu, \mu, z) \]
\[ = e^{-z} \sum_{k=0}^{p} \binom{p}{k} \sum_{l=0}^{q} \binom{q}{l} \frac{(-1)^k \sum_{k=0}^{p} \binom{p}{k} \sum_{l=0}^{q} \binom{q}{l} \Phi_p,q-1(\nu, 1-\mu; z)}{l} \]
\[ + e^{-z} z^\mu \sum_{l=0}^{q} \binom{q}{l} \frac{(-1)^l \sum_{u=0}^{l} \binom{l}{u} \Phi_{p+q-l-1,v}(\nu + \mu, 1 + \mu; z)}{l} \]
\[ \times \frac{1}{\Gamma(\nu) \Gamma(\mu)} \times(-1)^{-u} \Gamma(-u) (\mu) \ln^u z, \] (73)
for \( p, q = 0, 1, 2, \cdots \).

V. CONCLUSION

In this paper, some recursive relations of the derivatives of the gamma function \( \Gamma(\alpha) \) and incomplete gamma function \( \Gamma(\alpha, z) \) are obtained for complex number \( \alpha \neq 0, -1, -2, \cdots \).

Thus, the partial derivative \( \frac{\partial^{p+q}}{\partial \nu^p \partial \mu^q} B(\nu, \mu) \) of the Beta function \( B(\nu, \mu) \) can also be represented for \( p, q = 0, 1, 2, \cdots \).

Based on these results, the partial derivatives of the extended, generalized complete and incomplete gamma functions are obtained. Moreover, the partial derivative of confluent hypergeometric function is also considered.

ACKNOWLEDGMENT

The authors are grateful to the anonymous referees for constructive and helpful comments that improved this paper.

REFERENCES
