The Representations on the Partial Derivatives of the Extended, Generalized Gamma and Incomplete Gamma Functions and Their Applications

Aijuan Li, and Huizeng Qin

***Abstract*—In this paper, some recursive relations of the derivatives of the gamma function $\Gamma(\alpha)$ and incomplete gamma function $\Gamma(\alpha, z)$ are obtained for complex number $\alpha \neq 0, -1, -2, \cdots$. Moreover, the recurrence relation of the derivative of the $k-$gamma function $\Gamma_k(x)$ are also given. Thus, the partial derivative $\frac{\partial^{p+q}}{\partial\nu^p\partial\mu^q}B(\nu,\mu)$ of the Beta function $B(\nu, \mu)$ can also be represented for $p, q = 0, 1, 2, \cdots$. Based on these results, the partial derivatives of the extended, generalized complete and incomplete gamma functions are obtained. Furthermore, the partial derivative of confluent hypergeometric function is also considered.**

***Index Terms*—incomplete gamma function, gamma function, confluent hypergeometric function, Digamma function.**

I. Introduction

THE gamma function $\Gamma(\alpha)$ and incomplete gamma function $\Gamma(\alpha, z)$ were defined by the following integrals in [1]

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt, \quad Re(\alpha) > 0, \tag{1}$$

$$\gamma(\alpha, z) = \int_0^{z} t^{\alpha-1}e^{-t}dt, \;\; \alpha, z \in C, \;\; |arg(z)| < \pi, \;\; z \neq 0, \tag{2}$$

and

$$\Gamma(\alpha, z) = \int_z^{\infty} t^{\alpha-1}e^{-t}dt, \;\; \alpha, z \in C, \;\; |arg(z)| < \pi, \;\; z \neq 0, \tag{3}$$

where α can be extended to all complex numbers except non-positive integers and C denotes a complex set. These functions were found to be useful in heat conduction, probability theory and in the study of Fourier and Laplace transforms in [1]. Moreover, the gamma distribution which is formulated in terms of the gamma function is used in statistics to model a wide range of processes. For example, the time between occurrences of earthquakes in [2]. Furthermore, these functions can be expanded. For example, the extended complete and incomplete gamma functions were defined by the following integrals in [1,3-5]:

$$\begin{array}{c}\Gamma(\alpha, 0; b) = \int_0^{\infty} t^{\alpha-1}e^{-t-bt^{-1}}dt, \\ \gamma(\alpha, x; b) = \int_0^{x} t^{\alpha-1}e^{-t-bt^{-1}}dt, \\ \Gamma(\alpha, x; b) = \int_x^{\infty} t^{\alpha-1}e^{-t-bt^{-1}}dt,\end{array} \tag{4}$$

where $\alpha \in R, x \geq 0, b \geq 0$, but not both $x = 0, b = 0$, if $\alpha \leq 0$. The function $\Gamma(\alpha, x; b)$ can be used in closed form solutions to several problems in heat conduction with time-dependent boundary conditions. Moreover, the gamma function $\Gamma(\alpha)$ has played a role in cumulative probability functions in [1,6,7]. Similarly, the generalized complete and incomplete gamma functions were also defined by

$$\begin{array}{c}\Gamma_\beta(\alpha, c) = \int_0^{\infty} \frac{t^{\alpha-1}e^{-t}}{(t+c)^\beta}dt, \gamma_\beta(\alpha, x, c) = \int_0^{x} \frac{t^{\alpha-1}e^{-t}}{(t+c)^\beta}dt, \\ \Gamma_\beta(\alpha, x, c) = \int_x^{\infty} \frac{t^{\alpha-1}e^{-t}}{(t+c)^\beta}dt,\end{array} \tag{5}$$

where $\alpha = n + \varepsilon \geq 1(n = 1, 2, \cdots, 0 \leq \varepsilon < 1), \beta = m + \delta \geq 0(m = 1, 2, \cdots, -1 \leq \delta \leq 0)$ and $x, c > 0$ in [8,9]. The generalized gamma functions have been widely used in the solution of many problems of wave scattering and diffraction theory in [9-11].

In this paper, we assume that $\alpha, z \in C$, $|arg(z)| < \pi$ and $z \neq 0$ for (1)-(3). Since $\gamma(\alpha, z) = \Gamma(\alpha) - \Gamma(\alpha, z)$, we only consider $\Gamma^{(n)}(\alpha)$ and $\Gamma^{(n)}(\alpha, z)(n = 0, 1, 2, \cdots)$, i.e.

$$\begin{array}{c}\Gamma^{(n)}(\alpha) = \frac{d^n}{d\alpha^n}\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}\ln^n t dt, \\ \Gamma^{(n)}(\alpha, z) = \frac{\partial^n}{\partial\alpha^n}\Gamma(\alpha, z) = \int_z^{\infty} t^{\alpha-1}e^{-t}\ln^n t dt,\end{array} \tag{6}$$

Similarly, we also consider the following derivatives for (4) and (5), i.e.

$$\begin{array}{c}\Gamma^{(n)}(\alpha, 0; b) = \frac{d^n}{d\alpha^n}\Gamma(\alpha, 0; b) = \int_0^{\infty} t^{\alpha-1}e^{-t-bt^{-1}}\ln^n t dt, \\ \Gamma^{(n)}(\alpha, x; b) = \frac{\partial^n}{\partial\alpha^n}\Gamma(\alpha, x; b) = \int_x^{\infty} t^{\alpha-1}e^{-t-bt^{-1}}\ln^n t dt,\end{array} \tag{7}$$

and

$$\begin{aligned}\Gamma_\beta^{(n,m)}(\alpha, c) &= \frac{\partial^{n+m}}{\partial\alpha^n\partial\beta^m}\Gamma_\beta(\alpha, c) \\ &= (-1)^m \int_0^{\infty} \frac{t^{\alpha-1}e^{-t}\ln^n t\ln^m(t+c)}{(t+c)^\beta}dt, \\ \Gamma_\beta^{(n,m)}(\alpha, x, c) &= \frac{\partial^{n+m}}{\partial\alpha^n\partial\beta^m}\Gamma_\beta(\alpha, x, c) \\ &= (-1)^m \int_x^{\infty} \frac{t^{\alpha-1}e^{-t}\ln^n t\ln^m(t+c)}{(t+c)^\beta}dt.\end{aligned} \tag{8}$$

for $n, m = 0, 1, 2, \cdots$.

The structure of this paper is organized as follows. In Section 2, some recursive relations of the derivatives of the gamma function and $k-$gamma function are derived. Thus, $\widehat{\Gamma}^{(n)}(\alpha) = \frac{d^n}{d\alpha^n}\widehat{\Gamma}(\alpha)\left(\widehat{\Gamma}(\alpha) = \frac{1}{\Gamma(\alpha)}\right)$ and partial derivative $\frac{\partial^{p+q}}{\partial\nu^p\partial\mu^q}B(\nu,\mu)$ of the Beta function $B(\nu, \mu)$ can also be represented for $p, q = 0, 1, 2, \cdots$. In Section 3, some recursive relations of the partial derivatives of the incomplete gamma function are established. In Section 4, using the results of Section 2 and 3, we consider series expansions of (7) and (8). Moreover, the partial derivative of confluent hypergeometric

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Aijuan Li is with School of Science, Shandong University of Technology, Zibo, Shandong, 255049, P. R. China.

Huizeng Qin is with School of Science, Shandong University of Technology, Zibo, Shandong, 255049, P. R. China. Huizeng Qin is the corresponding author. (e-mail: qin_hz@163.com(H.Z.Qin))

function is also considered. The conclusion is given in the last section of the paper.

II. The recursive formulas of the derivatives of the Gamma function

Theorem 2.1 Let $n \geq 1$ be an integer. Then the recurrence relation of $\Gamma^{(n)}(\alpha)(\alpha \neq 0, -1, -2, \cdots)$ can be expressed as follows,

$$\Gamma^{(n)}(\alpha) = H_n^{\psi}(\alpha)\Gamma(\alpha), \tag{9}$$

where

$$\begin{array}{c} H_0^{\psi}(\alpha) = 1, \quad H_n^{\psi}(\alpha) = \sum\limits_{l=0}^{n-1} H_{n,l}^{\psi}(\alpha), \\ H_{n,0}^{\psi}(\alpha) = \psi^{(n-1)}(\alpha), \end{array} \tag{10}$$

and

$$\begin{array}{c} H_{n,l}^{\psi}(\alpha) = \sum\limits_{i=1}^{n-1} \left(\begin{array}{c} n-1 \\ i \end{array} \right) \psi^{(n-1-i)}(\alpha) H_{i,l-1}^{\psi}(\alpha), \\ n = 2, 3, \cdots, \quad l = 1, 2, \cdots. \end{array} \tag{11}$$

where $\psi(\alpha)$ is the Digamma function defined by

$$\psi(\alpha) = -\gamma - \frac{1}{\alpha} + \sum_{l=1}^{\infty} \left(\frac{1}{l} - \frac{1}{l+\alpha} \right),$$

and γ denotes Euler's constant.

Proof. The Digamma function $\psi(\alpha)$ and its $k-$order derivatives $\psi^{(k)}(\alpha)(k = 1, 2, \cdots)$ can be expressed as follows,

$$\psi(\alpha) = \frac{d}{d\alpha} \ln\Gamma(\alpha) = -\gamma - \frac{1}{\alpha} + \sum_{l=1}^{\infty} \left(\frac{1}{l} - \frac{1}{l+\alpha} \right), \tag{12}$$

for $\alpha \neq 0, -1, -2, \cdots$, and

$$\psi^{(k)}(\alpha) = \frac{d^k}{d\alpha^k} \psi(\alpha) = k!(-1)^{k+1} \sum_{l=0}^{\infty} \frac{1}{(l+\alpha)^{k+1}}, \tag{13}$$

where γ denotes Euler's constant. From (12), we have

$$\Gamma'(\alpha) = \psi(\alpha)\Gamma(\alpha), \quad \alpha \neq 0, -1, -2, \cdots. \tag{14}$$

Calculating $(n-1)(n = 2, 3, \cdots)$-order derivatives on α for (14) by using the Leibniz's derivation rule, we get

$$\Gamma^{(n)}(\alpha) = \sum_{k=0}^{n-1} \left(\begin{array}{c} n-1 \\ k \end{array} \right) \psi^{(n-1-k)}(\alpha)\Gamma^{(k)}(\alpha), \tag{15}$$

for $\alpha \neq 0, -1, -2, \cdots$. Repeated use of this recursive formula we can see that (9)-(11) hold.

Remark 2.1 K. S. Kolbig considered the following Laplace (or Mellin) integral in [12]

$$R_m(\mu, \nu) = \int_0^{\infty} t^{\nu-1} e^{-\mu t} \ln^m t dt, \ Re\mu, Re\nu > 0, \tag{16}$$

and gave a recurrence formula

$$\begin{array}{c} G_k(\mu, \nu) = \frac{d}{d\nu} G_{k-1}(\mu, \nu) + G_1(\mu, \nu) G_{k-1}(\mu, \nu), \\ G_0(\mu, \nu) = 1, G_1(\mu, \nu) = \psi(\nu) - \ln\mu, \end{array} \tag{17}$$

where $R_m(\mu, \nu) = \frac{\Gamma(\nu)}{\mu^{\nu}} G_m(\mu, \nu)$. By the variable substitution for (16) we have

$$R_0(\mu, \nu) = \mu^{-\nu} \int_0^{\infty} t^{\nu-1} e^{-t} dt = \mu^{-\nu}\Gamma(\nu), \tag{18}$$

Moreover, using the Leibniz's derivation rule on ν for (18), we have

$$\begin{array}{rl} R_m(\mu, \nu) & = \frac{\partial^m}{\partial \nu^m} R_0(\mu, \nu) \\ & = \mu^{-\nu} \sum\limits_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) (-1)^{m-k} \ln^{m-k} \mu \Gamma^{(k)}(\nu), \end{array} \tag{19}$$

Thus, $R_m(\mu, \nu)$ can also be calculated by Theorem 2.1.

Moreover, the $k-$gamma function $\Gamma_k(x)$ is the generalization of $\Gamma(x)$ and it was given by the following integral in [13]

$$\Gamma_k(x) = \int_0^{\infty} t^{x-1} e^{-\frac{t^k}{k}} dt, \quad x, k > 0, \tag{20}$$

and $\Gamma(x) = \Gamma_1(x)$.

Through variable substitution we obtain that $\Gamma_k(x) = k^{\frac{x-k}{k}} \Gamma(\frac{x}{k})$ and

$$\psi_k(x) = (\ln\Gamma_k(x))' = \frac{1}{k} \left(\ln k + \psi(\tfrac{x}{k}) \right). \tag{21}$$

Corresponding to Theorem 2.1, we have the following theorem.

Theorem 2.2 Let $n \geq 1$ be an integer. Then the recurrence relation of $\Gamma_k^{(n)}(x)(x \neq 0, -1, -2, \cdots)$ can be expressed as follows,

$$\Gamma_k^{(n)}(x) = H_n^{\psi_k}(x)\Gamma_k(x), \tag{22}$$

where

$$\begin{array}{c} H_0^{\psi_k}(x) = 1, H_n^{\psi_k}(x) = \sum\limits_{l=0}^{n-1} H_{n,l}^{\psi_k}(x), n = 1, 2, \cdots. \\ H_{n,0}^{\psi_k}(x) = \psi_k^{(n-1)}(x), n = 1, 2, \cdots. \\ H_{n,l}^{\psi_k}(x) \\ = \sum\limits_{i=1}^{n-1} \left(\begin{array}{c} n-1 \\ i \end{array} \right) \psi_k^{(n-1-i)}(x) H_{i,l-1}^{\psi_k}(x), n = 2, 3, \cdots. \\ \psi_k^{(n)}(x) = \left\{ \begin{array}{ll} \frac{1}{k} \left(\ln k + \psi(\frac{x}{k}) \right), & n = 0, \\ \frac{1}{k^n} \psi^{(n)}(\frac{x}{k}), & n = 1, 2, \cdots. \end{array} \right. \end{array} \tag{23}$$

Similarly, according to the proof of Theorem 2.1, the recurrence relation of $\widehat{\Gamma}^{(n)}(\alpha)(\alpha \neq 0, -1, -2, \cdots)$ can be obtained .

Theorem 2.3 Let $n \geq 1$ be an integer and $\widehat{\Gamma}(\alpha) = \frac{1}{\Gamma(\alpha)}$. Then the recurrence relation of $\widehat{\Gamma}^{(n)}(\alpha)(\alpha \neq 0, -1, -2, \cdots)$ can be expressed as follows,

$$\widehat{\Gamma}^{(n)}(\alpha) = H_n^{\widehat{\psi}}(\alpha)\widehat{\Gamma}(\alpha) \tag{24}$$

where

$$\begin{array}{c} H_0^{\widehat{\psi}}(\alpha) = 1, \quad H_n^{\widehat{\psi}}(\alpha) = \sum\limits_{l=0}^{n-1} H_{n,l}^{\widehat{\psi}}(\alpha), \\ H_{n,0}^{\widehat{\psi}}(\alpha) = \widehat{\psi}^{(n-1)}(\alpha), \end{array} \tag{25}$$

and

$$\begin{array}{c} H_{n,l}^{\widehat{\psi}}(\alpha) = \sum\limits_{i=1}^{n-1} \left(\begin{array}{c} n-1 \\ i \end{array} \right) \widehat{\psi}^{(n-1-i)}(\alpha) H_{i,l-1}^{\widehat{\psi}}(\alpha), \\ n = 2, 3, \cdots, \quad l = 1, 2, \cdots. \end{array} \tag{26}$$

and $\widehat{\psi} = -\psi(\alpha)$.

Moreover, we note that the Beta function is defined by

$$B(\nu, \mu) = \int_0^1 t^{\nu-1}(1-t)^{\mu-1} dt, Re\nu, Re\mu > 0.$$

Its partial derivatives $B_{p,q}(\nu,\mu)=\frac{\partial^{p+q}}{\partial\nu^p\partial\mu^q}B(\nu,\mu)(p,q=0,1,2,\cdots)$ have been discussed for $\nu,\mu,\nu+\mu\neq 0,\pm1,\pm2,\cdots$ in [14,15].

Furthermore, by

$$B(\nu,\mu)=\frac{\Gamma(\nu)\Gamma(\mu)}{\Gamma(\nu+\mu)}=\Gamma(\nu)\Gamma(\mu)\widehat{\Gamma}(\nu+\mu)$$

and Leibniz's derivation rule we have the following theorem.

Theorem 2.4 Let $p,q\geq 0$ be integers. Then

$$\begin{aligned}&B_{p,q}(\nu,\mu)\\ =&\sum_{k=0}^{p}\left(\begin{array}{c}p\\k\end{array}\right)\sum_{l=0}^{q}\left(\begin{array}{c}q\\l\end{array}\right)\widehat{\Gamma}^{(p+q-k-l)}(\nu+\mu)\\ &*\Gamma^{(k)}(\nu)\Gamma^{(l)}(\mu),\end{aligned}\tag{27}$$

for $\nu,\mu,\nu+\mu\neq 0,\pm1,\pm2,\cdots$, where $\widehat{\Gamma}(\nu+\mu)=\frac{1}{\Gamma(\nu+\mu)}$.

By (27), we notice that the partial derivatives of the Beta function can also be calculated by Theorem 2.1 and Theorem 2.3.

Note that

$$\begin{aligned}\Gamma(n)=(n-1)!,\Gamma(n-\tfrac{1}{2})=\tfrac{(2n-3)!!\sqrt{\pi}}{2^{n-1}},\\ \Gamma(\tfrac{1}{2}-n)=\tfrac{(-2)^n\sqrt{\pi}}{(2n-1)!!},\end{aligned}\tag{28}$$

and

$$\begin{aligned}\psi(n)=H_{n-1}-\gamma,\\ \psi(n-\tfrac{1}{2})=-\gamma-2\ln 2+2H_{2n-2}-H_{n-1}\end{aligned}\tag{29}$$

and

$$\begin{aligned}\zeta(s,n)=\zeta(s)-H_{n-1,s},\\ \zeta(s,n-\tfrac{1}{2})=(2^s-1)\zeta(s)-2^sH_{2n-2,s}+H_{n-1,s}\end{aligned}\tag{30}$$

for $n=1,2,\cdots$, where $H_n=\sum\limits_{l=1}^{n}\frac{1}{l}$ and $H_{n,s}=\sum\limits_{l=1}^{n}\frac{1}{l^s}$. Using (9), (24) and (27), we obtain that $\Gamma^{(n)}(\alpha),\widehat{\Gamma}^{(n)}(\alpha)(n=1,2,\cdots)$ and $B_{p,q}(\alpha,\beta)(p,q=0,1,2,\cdots)$ exist the close forms for $\alpha,\beta=k,k+\frac{1}{2}$ and $k=0,\pm1,\pm2,\cdots$. For Pochhammer symbol $(x)_n$, i.e. $(x)_n=x(x+1)\cdots(x+n-1)$, we have the following result: $[(x)_n]'=(x)_n\sum\limits_{k=0}^{n-1}\frac{1}{x+k}$. According to the method to prove Theorem 2.1 and 2.4, we easily get the following recurrence formula.

Corollary 2.5 Let $n,m\geq 1$ be integers and $x\neq 0,-1,-2,\cdots$ be a complex number. Then

$$(x)_n^{(m)}=(m-1)!\sum_{k=0}^{m-1}\frac{(-1)^k(x)_n^{(m-k)}H_{n-1,k}(x)}{(m-1-k)!}\tag{31}$$

and

$$\overline{(x)}_n^{(m)}=-(m-1)!\sum_{k=0}^{m-1}\frac{(-1)^kH_{n-1,k}(x)}{(m-1-k)!}\overline{(x)}_n^{(m-1-k)},\tag{32}$$

where $\overline{(x)}_n=\frac{1}{(x)_n},H_{n,p}(x)=\sum\limits_{k=0}^{n}\frac{1}{(x+k)^p},(x)_n^{(m)}=\frac{d^m}{dx^m}(x)_n$ and $\overline{(x)}_n^{(m)}=\frac{d^m}{dx^m}\overline{(x)}_n$.

However, the derivatives of $(x)_n$ and $\overline{(x)}_n$ can also be calculated by the following method.

Since $(x)_n=\sum\limits_{k=0}^{n}(-1)^{n-k}s(n,k)x^k$ and $\overline{(x)}_n=\frac{1}{(x)_n}=\frac{1}{(n-1)!}\sum\limits_{j=0}^{n-1}C_{n-1}^j\frac{(-1)^j}{x+j}$, we have

$$(x)_n^{(m)}=m!\sum_{k=m}^{n}\left(\begin{array}{c}k\\m\end{array}\right)(-1)^{n-k}s(n,k)x^{k-m},\tag{33}$$

and

$$\overline{(x)}_n^{(m)}=\frac{(-1)^m m!}{(j-1)!}\sum_{k=0}^{n-1}C_{n-1}^k\frac{(-1)k}{(x+k)^{m+1}}.\tag{34}$$

where $s(n,k)$ is the Stirling number of the first kind and $n,m=1,2,\cdots$.

In the following section, the partial derivatives of the incomplete gamma function are given.

III. The partial derivatives of the incomplete gamma function

Theorem 3.1 Let n be a non-negative integer. Then

$$\begin{aligned}&\Gamma^{(n)}(\alpha,z)\\ =&\Gamma^{(n)}(\alpha)-n!\sum_{k=0}^{\infty}\frac{(-1)^kz^{k+\alpha}}{k!}\sum_{j=0}^{n}\frac{(-1)^j\ln^{n-j}z}{(\alpha+k)^{j+1}(n-j)!},\end{aligned}\tag{35}$$

for $\alpha\neq 0,-1,-2,\cdots$.

Proof. For the incomplete gamma function $\Gamma(\alpha,z)$, we have

$$\Gamma(\alpha,z)=\Gamma(\alpha)-\sum_{k=0}^{\infty}\frac{(-1)^kz^{k+\alpha}}{k!(\alpha+k)},\quad \alpha\neq 0,-1,-2,\cdots.\tag{36}$$

Calculating n-order partial derivatives on α for (36) by using the Leibniz's derivatives rule, we can obtain (35).

Note that

$$\begin{aligned}\Gamma(\alpha,z)&=\tfrac{\Gamma(\alpha+1,z)-z^{\alpha}e^{-z}}{\alpha}\\ &=\tfrac{\Gamma(\alpha+2,z)-z^{\alpha+1}e^{-z}}{\alpha(\alpha+1)}-\tfrac{z^{\alpha}e^{-z}}{\alpha}\\ &=\cdots\\ &=\tfrac{\Gamma(\alpha+m,z)}{(\alpha)_m}-e^{-z}\sum_{k=1}^{m}\tfrac{z^{\alpha+k-1}}{(\alpha)_k}\\ &=\tfrac{\Gamma(\alpha+m,z)}{(m-1)!}\sum_{j=0}^{m-1}C_{m-1}^j\tfrac{(-1)^j}{\alpha+j}\\ &\quad-e^{-z}\sum_{k=1}^{m}\tfrac{z^{\alpha+k-1}}{(k-1)!}\sum_{j=0}^{k-1}C_{k-1}^j\tfrac{(-1)^j}{\alpha+j}\end{aligned}\tag{37}$$

By using Leibniz's derivation rule on α for (37), we have

$$\begin{aligned}\Gamma^{(n)}(\alpha,z)&=n!\sum_{l=0}^{n}\tfrac{\Gamma^{(n-l)}(\alpha+m,z)}{(n-l)!(m-1)!}\sum_{j=0}^{m-1}C_{m-1}^j\tfrac{(-1)^{j+l}}{(\alpha+j)^{l+1}}\\ &\quad-n!e^{-z}\sum_{k=1}^{m}\tfrac{z^{\alpha+k-1}}{(k-1)!}\sum_{l=0}^{n}\tfrac{\ln^{n-l}z}{(n-l)!}\\ &\quad\times\sum_{j=0}^{k-1}C_{k-1}^j\tfrac{(-1)^{j+l}}{(\alpha+j)^{l+1}}.\end{aligned}\tag{38}$$

However, $\Gamma(\alpha,z)$ and $\Gamma(\alpha)$ are not defined for $\alpha=0$. Thus, we give their complementary definitions. According to the neutrix limit in [16] we have

$$\begin{aligned}\Gamma^{(n)}(0)&=N-\lim_{\varepsilon\to0}\int_{\varepsilon}^{\infty}t^{-1}e^{-t}\ln^n t\,dt\\ &=N-\lim_{\varepsilon\to0}\left(-\tfrac{1}{n+1}e^{-\varepsilon}\ln^{n+1}\varepsilon\right)\\ &\quad+N-\lim_{\varepsilon\to0}\left(\tfrac{1}{n+1}\int_{\varepsilon}^{\infty}e^{-t}\ln^{n+1}t\,dt\right)\\ &=\tfrac{1}{n+1}\int_0^{\infty}e^{-t}\ln^{n+1}t\,dt=\tfrac{1}{n+1}\Gamma^{(n+1)}(1)\end{aligned}\tag{39}$$

and

$$
\begin{aligned}
&N-\lim_{\alpha\to 0}\sum_{k=0}^{\infty}\frac{(-1)^k z^{k+\alpha}}{k!}\sum_{j=0}^{n}\frac{(-1)^j\ln^{n-j}z}{(\alpha+k)^{j+1}(n-j)!}\\
&=N-\lim_{\alpha\to 0}\sum_{j=0}^{n}\frac{(-1)^j\ln^{n-j}z}{(n-j)!}\frac{z^\alpha}{\alpha^{j+1}}\\
&+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\sum_{j=0}^{n}\frac{(-1)^j\ln^{n-j}z}{(n-j)!}\frac{z^k}{k^{j+1}}\\
&=\sum_{j=0}^{n}\frac{(-1)^j\ln^{n-j}z}{(n-j)!}\frac{z^{j+1}}{(j+1)!}\\
&+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\sum_{j=0}^{n}\frac{(-1)^j\ln^{n-j}z}{(n-j)!}\frac{z^k}{k^{j+1}}\\
&=\frac{\ln^{n+1}z}{(n+1)!}+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\sum_{j=0}^{n}\frac{(-1)^j\ln^{n-j}z}{(n-j)!}\frac{z^k}{k^{j+1}}
\end{aligned}
\tag{40}
$$

Let $\alpha\to -m$ in (38), we have the following theorem.
Theorem 3.2 Let $n\geq 0$ and $m\geq 1$ be integers. Then

$$
\begin{aligned}
&\Gamma^{(n)}(-m,z)\\
&=n!\sum_{l=0}^{n}\frac{\Gamma^{(n+1-l)}(1,z)-e^{-z}\ln^{n+1-l}z}{(n+1-l)!(m-1)!}\sum_{j=0}^{m-1}C_{m-1}^{j}\frac{(-1)^{j-1}}{(m-j)^{l+1}}\\
&-n!e^{-z}\sum_{k=1}^{m}\frac{z^{-m+k-1}}{(k-1)!}\sum_{l=0}^{n}\frac{\ln^{n-l}z}{(n-l)!}\sum_{j=0}^{k-1}C_{k-1}^{j}\frac{(-1)^{j-1}}{(m-j)^{l+1}}.
\end{aligned}
\tag{41}
$$

In the following section, the applications of the partial derivatives of the complete and incomplete gamma functions are given.

IV. Applications of the partial derivatives of complete and incomplete gamma functions

A. The partial derivatives of generalized complete and incomplete gamma functions

The series expansions for generalized complete and incomplete gamma functions were given in [14] as follows:

$$
\begin{aligned}
&\Gamma_\beta(\alpha,c)\\
&=\begin{cases}\sum_{i=0}^{\infty}(c^{-\beta-i}\gamma(\alpha+i,c)+c^i\Gamma(\alpha-i-\beta,c))\\ \quad\times\begin{pmatrix}-\beta\\ i\end{pmatrix},\quad for\ \beta-\alpha+i<0,\\ \sum_{i=0}^{\infty}(c^{-\beta-i}\gamma(\alpha+i,c)+c^i c^{\alpha-i-\beta}E_{\beta-\alpha+i+1}(c))\\ \quad\times\begin{pmatrix}-\beta\\ i\end{pmatrix},\quad for\ \ \beta-\alpha+i\geq 0.\end{cases}
\end{aligned}
\tag{42}
$$

and

$$
\begin{aligned}
&\Gamma_\beta(\alpha,x,c)\\
&=\begin{cases}\Gamma_\beta(\alpha,c)-\sum_{i=0}^{\infty}\begin{pmatrix}-\beta\\ i\end{pmatrix}\frac{\Gamma(\alpha+i)-\Gamma(\alpha+i,x)}{c^{\beta+i}},for\ x<c,\\ \sum_{i=0}^{\infty}\begin{pmatrix}-\beta\\ i\end{pmatrix}c^i\Gamma(\alpha-\beta-i,x),\qquad for\ x>c.\end{cases}
\end{aligned}
\tag{43}
$$

where $E_\beta(c)$ is the exponential integral function defined by

$$E_\beta(c)=\int_1^\infty t^{-\beta}e^{-ct}dt \tag{44}$$

However, if we calculate $\Gamma_\beta(\alpha,c)$ and $\Gamma_\beta(\alpha,x,c)$ by using (42)-(44), then the error is caused by the relatively large. In particular, when c decreases, the error becomes more prominent. The reason for the error is using the following series expansion of (42) and (43).

$$
\frac{1}{(t+c)^\beta}=\begin{cases}\sum_{i=0}^{\infty}\frac{(-\beta-i+1)_i t^i}{i!c^{i+\beta}},|t|<|c|,\\ \sum_{i=0}^{\infty}\frac{(-\beta-i+1)_i c^i}{i!t^{i+\beta}},|t|>|c|,\end{cases}
\tag{45}
$$

The convergence speed of the series (45) is very slow at near c.

Since

$$t^{\alpha-1}=(t+c)^{\alpha-1}\sum_{i=0}^{\infty}\frac{(1-\alpha)_i}{i!}\frac{c^i}{(t+c)^i}, \tag{46}$$

we give a slightly different form power series expansion as follows:

$$
\begin{aligned}
\Gamma_\beta(\alpha,c)&=\sum_{i=0}^{\infty}\frac{(-1)^i(\beta)_i}{i!c^{i+\beta}}\int_0^{c_1}t^{\alpha+i-1}e^{-t}dt\\
&+\sum_{i=0}^{\infty}\frac{(1-\alpha)_i c^i}{i!}\int_{c_1}^{\infty}(t+c)^{\alpha-\beta-i-1}e^{-t}dt\\
&=\sum_{i=0}^{\infty}\frac{(-1)^i(\beta)_i(\Gamma(\alpha+i)-\Gamma(\alpha+i,c_1))}{i!c^{i+\beta}}\\
&+e^c\sum_{i=0}^{\infty}\frac{(1-\alpha)_i c^i\Gamma(\alpha-\beta-i,c+c_1)}{i!}
\end{aligned}
\tag{47}
$$

and

$$
\Gamma_\beta(\alpha,x,c)=\begin{cases}\sum_{i=0}^{\infty}\frac{(-1)^i(\beta)_i(\Gamma(\alpha+i,x)-\Gamma(\alpha+i,c_1))}{i!c^{i+\beta}}\\ +e^c\sum_{i=0}^{\infty}\frac{(1-\alpha)_i c^i\Gamma(\alpha-\beta-i,c+c_1)}{i!},|x|\leq|c_1|,\\ e^c\sum_{i=0}^{\infty}\frac{(1-\alpha)_i c^i\Gamma(\alpha-\beta-i,c+x)}{i!},\quad |x|>|c_1|.\end{cases}
\tag{48}
$$

where $c_1=\frac{c}{2}$(Values suggest that the best such how to choose), α,β,c,x can be complex numbers satisfying $c>0$ or $Im\ c\neq 0, Re\ x>0$. By numerical experiment, we find that $\sum_{i=0}^{\infty}\frac{(-1)^i(\beta)_i}{i!c^{i+\beta}}\int_0^{c_1}t^{\alpha+i-1}e^{-t}dt$ has a good convergence rate when $|\frac{t}{c}|<\frac{1}{20}$ and $\sum_{i=0}^{\infty}\frac{(1-\alpha)_i c^i}{i!}\int_{c_1}^{\infty}(t+c)^{\alpha-\beta-i-1}e^{-t}dt$ also has better convergence rate when $|\frac{c}{t+c}|<\frac{2}{3}$.

When $\alpha=1,2,\cdots$, (46) becomes

$$t^{\alpha-1}=\sum_{i=0}^{\alpha-1}\begin{pmatrix}\alpha-1\\ i\end{pmatrix}(-c)^{\alpha-1-i}(t+c)^i, \tag{49}$$

Therefore, we have

$$\Gamma_\beta(\alpha,c)=e^c\sum_{i=0}^{\alpha-1}\begin{pmatrix}\alpha-1\\ i\end{pmatrix}(-c)^i\Gamma(\alpha-\beta-i,c) \tag{50}$$

and

$$\Gamma_\beta(\alpha,x,c)=e^c\sum_{i=0}^{\alpha-1}\begin{pmatrix}\alpha-1\\ i\end{pmatrix}(-c)^i\Gamma(\alpha-\beta-i,c+x). \tag{51}$$

It follows from (50) and (51) that $\Gamma_\beta(\alpha,c)$ and $\Gamma_\beta(\alpha,x,c)$ have finite linear combinations of $\Gamma(\alpha,c)$.

In the following, the results of computing $\Gamma_\beta(\alpha,c)$ are given in Table I-III by different methods.

Table I The results of computing $\Gamma_\beta(\alpha, c)$ by using (47) or (50)

β	α	c	c_1	using (47) or (50)	relative error
1	$\frac{1}{2}$	$\frac{35}{2}$	$\frac{35}{4}$	$0.0986075675030\cdots$	10^{-14}
1	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{5}{4}$	$0.6135473689567\cdots$	10^{-10}
5	$\frac{7}{2}$	5	$\frac{5}{2}$	$0.0001329445251\cdots$	10^{-8}
10	13	84	21	$6.97537804\cdots * 10^{-12}$	10^{-32}
10	13	10	5	$0.0000399525536\cdots$	10^{-32}
$\frac{7}{2}$	$\frac{7}{5}$	5	$\frac{5}{2}$	$0.0016229475288\cdots$	10^{-10}

Table II The results of computing $\Gamma_\beta(\alpha, c)$ by using (42)

β	α	c	c_1	using (42)	relative error
1	$\frac{1}{2}$	$\frac{35}{2}$	$\frac{35}{4}$	$0.0986075676782\cdots$	10^{-9}
1	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{5}{4}$	$0.6148162192735\cdots$	10^{-2}
5	$\frac{7}{2}$	5	$\frac{5}{2}$	$2.0049305783581\cdots$	*
10	13	84	21	$6.97537808\cdots * 10^{-12}$	10^{-8}
10	13	10	5	$2.250843430\cdots * 10^{6}$	*
$\frac{7}{2}$	$\frac{7}{5}$	5	$\frac{5}{2}$	$0.0208284837555\cdots$	*

Table III The real values of $\Gamma_\beta(\alpha, c)$

β	α	c	c_1	values of $\Gamma_\beta(\alpha, c)$
1	$\frac{1}{2}$	$\frac{35}{2}$	$\frac{35}{4}$	$0.0986075675031\cdots$
1	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{5}{4}$	$0.6135473690055\cdots$
5	$\frac{7}{2}$	5	$\frac{5}{2}$	$0.0001329445244\cdots$
10	13	84	21	$6.975378\cdots * 10^{-12}$
10	13	10	5	$0.0000399525536\cdots$
$\frac{7}{2}$	$\frac{7}{5}$	5	$\frac{5}{2}$	$0.0016229475288\cdots$

Now the results of computing $\Gamma_\beta(\alpha, x, c)$ are given in Table IV-VI by different methods.

Table IV The results of computing $\Gamma_\beta(\alpha, x, c)$ by using (48) or (51)

β	α	c	x	c_1	using (48) or (51)	relative error
1	$\frac{1}{2}$	$\frac{35}{2}$	$\frac{5}{2}$	$\frac{35}{4}$	$0.00215450226\cdots$	10^{-14}
1	$\frac{1}{2}$	$\frac{5}{2}$	1	$\frac{5}{4}$	$0.06652798250\cdots$	10^{-11}
5	$\frac{7}{2}$	5	2	$\frac{5}{2}$	$0.00006039480\cdots$	10^{-10}
10	13	84	10	21	$6.97537804\cdots * 10^{-12}$	10^{-32}
10	13	10	2	5	$0.0000399525536\cdots$	10^{-32}
$\frac{7}{2}$	$\frac{7}{5}$	3	2	$\frac{3}{2}$	$0.0016229475288\cdots$	10^{-10}

Table V The results of computing $\Gamma_\beta(\alpha, x, c)$ by using (43)

β	α	c	x	c_1	using (43)	relative error
1	$\frac{1}{2}$	$\frac{35}{2}$	$\frac{5}{2}$	$\frac{35}{4}$	$0.00215450243\cdots$	10^{-7}
1	$\frac{1}{2}$	$\frac{5}{2}$	1	$\frac{5}{4}$	$0.06754729288\cdots$	10^{-2}
5	$\frac{7}{2}$	5	2	$\frac{5}{2}$	$3.74875444829\cdots$	*
10	13	84	10	21	$4.765912\cdots * 10^{-12}$	10^{-8}
10	13	10	2	5	$2.250843430\cdots * 10^{6}$	*
$\frac{7}{2}$	$\frac{7}{5}$	3	2	$\frac{3}{2}$	$0.5652668755473\cdots$	*

Table VI The real values of $\Gamma_\beta(\alpha, x, c)$

β	α	c	x	c_1	values of $\Gamma_\beta(\alpha, x, c)$
1	$\frac{1}{2}$	$\frac{35}{2}$	$\frac{5}{2}$	$\frac{35}{4}$	$0.00215450226\cdots$
1	$\frac{1}{2}$	$\frac{5}{2}$	1	$\frac{5}{4}$	$0.06652798255\cdots$
5	$\frac{7}{2}$	5	2	$\frac{5}{2}$	$0.00006039480\cdots$
10	13	84	10	21	$4.765912\cdots * 10^{-12}$
10	13	10	2	5	$0.00003995069\cdots$
$\frac{7}{2}$	$\frac{7}{5}$	3	2	$\frac{3}{2}$	$0.00043577030\cdots$

Seen from Table I - Table VI, the algorithms (47)-(51) have better accuracy. However, the algorithms (42) and (43) are not only poor accuracy, but also the numerical results are away from the true value for small c(see "*" in Table II and Table V).

In the following, we consider the partial derivatives of $\Gamma_\beta(\alpha, c)$ and $\Gamma_\beta(\alpha, x, c)$.

By (47), (48) and Leibniz's derivation rule, we have

$$\begin{aligned}\Gamma_\beta^{(n,m)}(\alpha, c) &= \sum_{i=0}^{\infty} \frac{(-1)^i\left(\Gamma^{(n)}(\alpha+i)-\Gamma^{(n)}(\alpha+i,c_1)\right)}{i!c^{i+\beta}} \\ &\quad * \sum_{j=0}^{m}(-1)^j \binom{m}{j} (\beta)_i^{(m-j)} \ln^j c \\ &\quad +(-1)^{m+n}e^c \sum_{i=0}^{\infty}\frac{c^i}{i!}\sum_{k=0}^{n}(-1)^k \binom{n}{k} \\ &\quad *(1-\alpha)_i^{(n-k)}\Gamma^{(m+k)}(\alpha-\beta-i, c+c_1),\end{aligned} \tag{52}$$

and

$$\Gamma_\beta^{(n,m)}(\alpha, x, c) = \begin{cases} (-1)^{m+n}e^c \sum_{i=0}^{\infty}\frac{c^i}{i!}\sum_{k=0}^{n}(-1)^k \binom{n}{k} \\ \times(1-\alpha)_i^{(n-k)}\Gamma^{(m+k)}(\alpha-\beta-i, c+c_1) \\ +\sum_{i=0}^{\infty}\frac{(-1)^i\left(\Gamma^{(n)}(\alpha+i,x)-\Gamma^{(n)}(\alpha+i,c_1)\right)}{i!c^{i+\beta}} \\ \times\sum_{j=0}^{m}\binom{m}{j}(-1)^j(\beta)_i^{(m-j)}\ln^j c, \quad for\ |x| \le |c_1|, \\ (-1)^{m+n}e^c\sum_{i=0}^{\infty}\frac{c^i}{i!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1-\alpha)_i^{(n-k)} \\ \times\Gamma^{(m+k)}(\alpha-\beta-i, c+x), \quad for\ |x| > |c_1|. \end{cases} \tag{53}$$

for $n, m = 0, 1, 2, \cdots$.

In the following, the partial derivatives of extended incomplete gamma functions are considered.

B. The partial derivatives of extended incomplete gamma functions

For (4), the following result was given in [1]

$$\Gamma(\alpha, 0; b) = \gamma(\alpha, x; b)+\Gamma(\alpha, x; b) = 2b^{\alpha/2}K_\alpha(2\sqrt{b}),\ b>0, \tag{54}$$

where K_α is the modified Bessel function of the second kind.

Moreover, for $\Gamma(\alpha, 0; b)$ we have

$$\begin{aligned}\Gamma(\alpha, 0; b) &= \int_0^\infty t^{\alpha-1}e^{-\left(t+\frac{b}{t}\right)}dt \\ &= \int_0^{\sqrt{b}} t^{\alpha-1}e^{-\left(t+\frac{b}{t}\right)}dt + \int_{\sqrt{b}}^\infty t^{\alpha-1}e^{-\left(t+\frac{b}{t}\right)}dt \\ &= b^\alpha\int_{\sqrt{b}}^\infty u^{-\alpha-1}e^{-\left(u+\frac{b}{u}\right)}du \\ &\quad +\int_{\sqrt{b}}^\infty t^{\alpha-1}e^{-\left(t+\frac{b}{t}\right)}dt \\ &= b^\alpha\sum_{i=0}^{\infty}\frac{(-b)^i}{i!}\int_{\sqrt{b}}^\infty u^{-\alpha-i-1}e^{-u}du \\ &\quad +\sum_{i=0}^{\infty}\frac{(-b)^i}{i!}\int_{\sqrt{b}}^\infty t^{\alpha-i-1}e^{-t}dt \\ &= b^\alpha\sum_{i=0}^{\infty}\frac{(-b)^i\Gamma(-\alpha-i,\sqrt{b})}{i!}+\sum_{i=0}^{\infty}\frac{(-b)^i\Gamma(\alpha-i,\sqrt{b})}{i!}\end{aligned} \tag{55}$$

Similarly, the following series expansions of $\Gamma(\alpha, x; b)$ are obtained

$$\Gamma(\alpha, x; b) = \int_x^\infty t^{\alpha-1}e^{-\left(t+\frac{b}{t}\right)}dt = \begin{cases} \sum_{i=0}^{\infty}\frac{(-b)^i\Gamma(\alpha-i,x)}{i!}, & x \ge \sqrt{b}, \\ b^\alpha\sum_{i=0}^{\infty}\frac{(-b)^i\left(\Gamma(-\alpha-i,\sqrt{b})-\Gamma(-\alpha-i,\frac{b}{x})\right)}{i!} \\ +\sum_{i=0}^{\infty}\frac{(-b)^i\Gamma(\alpha-i,\sqrt{b})}{i!}, & x < \sqrt{b}, \end{cases} \tag{56}$$

For $\alpha \neq 0, \pm 1, \pm 2, \cdots$ and by (55) and (56), we have

$$\begin{aligned}\Gamma^{(n)}(\alpha,0;b) &= \frac{d^n}{d\alpha^n}\Gamma(\alpha,0;b)\\ &= b^\alpha \sum_{i=0}^{\infty} \frac{(-b)^i}{i!} \sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} \ln^{n-k} b\\ &\times \Gamma^{(k)}(-\alpha-i,\sqrt{b}) + \sum_{i=0}^{\infty} \frac{(-b)^i \Gamma^{(n)}(\alpha-i,\sqrt{b})}{i!},\end{aligned} \tag{57}$$

and

$$\Gamma^{(n)}(\alpha,x;b) = \frac{d^n}{d\alpha^n}\Gamma(\alpha,x;b)$$
$$= \begin{cases} \sum_{i=0}^{\infty} \frac{(-b)^i \Gamma^{(n)}(\alpha-i,x)}{i!}, & x \geq \sqrt{b},\\ b^\alpha \sum_{i=0}^{\infty} \frac{(-b)^i}{i!} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^k \ln^{n-k} b & \\ \times \left(\Gamma^{(k)}(-\alpha-i,\sqrt{b}) - \Gamma^{(k)}(-\alpha-i,\frac{b}{x})\right) & \\ + \sum_{i=0}^{\infty} \frac{(-b)^i \Gamma^{(n)}(\alpha-i,\sqrt{b})}{i!}, & x < \sqrt{b}, \end{cases} \tag{58}$$

for $n = 0, 1, 2, \cdots$.

In the following, the partial derivatives of confluent hypergeometric functions are obtained.

C. The partial derivatives of confluent hypergeometric function

The confluent hypergeometric function is analytic on C and it is defined by the following series expansion

$$\begin{aligned}\Phi(\nu,\mu;z) &= \sum_{n=0}^{\infty} \frac{(\nu)_n z^n}{(\mu)_n n!}\\ &= 1 + \sum_{n=1}^{\infty} \frac{(\nu)_n z^n}{n!(n-1)!} \sum_{j=0}^{n-1} C_{n-1}^{j} \frac{(-1)^j}{\mu+j},\end{aligned} \tag{59}$$

Moreover, the confluent hypergeometric function can also be represented as the following integral

$$\Phi(\nu,\mu;z) = \frac{\Gamma(\mu)}{\Gamma(\nu)\Gamma(\mu-\nu)} \int_0^1 t^{\nu-1}(1-t)^{\mu-\nu-1} e^{zt} dt, 0 < Re\nu < Re\mu. \tag{60}$$

By analytic continuation, $\Phi(\nu,\mu;z)$ has been defined in addition to $\mu = 0, -1, -2, \cdots$ from (59).

Moreover, by (60), we have

$$\begin{aligned}&\int_0^1 t^{\nu-1}(1-t)^{\mu-1} e^{zt} dt\\ &= B(\nu,\mu)\Phi(\nu,\mu+\nu;z), \qquad Re\nu, Re\mu > 0,\end{aligned} \tag{61}$$

Using the Leibniz's derivation rule on ν and μ for (61), we have

$$\begin{aligned}&\int_0^1 t^{\nu-1}(1-t)^{\mu-1} e^{zt} \ln^p t \ln^q(1-t) dt\\ &= \sum_{k=0}^{p} \begin{pmatrix} p \\ k \end{pmatrix} \sum_{l=0}^{q} \begin{pmatrix} q \\ l \end{pmatrix} B_{p-k,q-l}(\nu,\mu)\\ &\times \sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} \Phi_{i,k+l-i}(\nu,\mu+\nu;z),\end{aligned} \tag{62}$$

for $Re\nu + q, Re\mu + p > 0$.

Moreover, using the Leibniz's derivation rule on ν and μ for (59), we obtain

$$\begin{aligned}&\Phi_{p,q}(\nu,\mu;z)\\ &= \frac{\partial^{p+q}}{\partial\nu^p \partial\mu^q}\Phi(\nu,\mu;z)\\ &= (-1)^q q! \sum_{n=1}^{\infty} \frac{(\nu)_n^{(p)} z^n}{n!(n-1)!} \sum_{j=0}^{n-1} C_{n-1}^{j} \frac{(-1)^j}{(\mu+j)^{q+1}},\end{aligned} \tag{63}$$

for $p, q = 0, 1, 2, \cdots$.

Similarly, using the following identity

$$\begin{aligned}\Gamma_\mu(\nu,z) &= z^{\mu-\nu}B(\nu,\mu-\nu)\Phi(\nu,\nu+1-\mu;z)\\ &+\Gamma(\nu-\mu)\Phi(\mu,1+\mu-\nu;z),\end{aligned} \tag{64}$$

we have

$$\begin{aligned}\Gamma_\mu^{(p,q)}(\nu,z) &= \frac{\partial^{p+q}}{\partial\nu^p \partial\mu^q}\Gamma_\mu(\nu,z)\\ &= \sum_{k=0}^{p} \begin{pmatrix} p \\ k \end{pmatrix} (-1)^k \sum_{l=0}^{q} \begin{pmatrix} q \\ l \end{pmatrix} (-1)^{q-l}\\ &\times \Gamma^{(p+q-k-l)}(\nu-\mu) \sum_{v=0}^{l} \begin{pmatrix} l \\ v \end{pmatrix}\\ &\times \Phi_{l-v,k+v}(\mu,1+\mu-\nu;z) + z^{\nu-\mu}\\ &\times \sum_{l=0}^{q} \begin{pmatrix} q \\ l \end{pmatrix} (-1)^{q-l} \sum_{k=0}^{p} \begin{pmatrix} p \\ k \end{pmatrix}\\ &\times \sum_{u=0}^{p-k} \begin{pmatrix} p-k \\ u \end{pmatrix}\\ &\times \Phi_{p-k-u,q+u-l}(\nu,\nu+1-\mu;z)\\ &\times \sum_{u_1=0}^{k} \begin{pmatrix} k \\ u_1 \end{pmatrix} \sum_{u_2=0}^{u_1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\\ &\times (-1)^{u_2} \sum_{v=0}^{l} \begin{pmatrix} l \\ v \end{pmatrix} (-1)^v \ln^{k-u_1+v} z\\ &\times B_{u_1-u_2,l+u_2-u}(\nu,\mu-\nu),\end{aligned} \tag{65}$$

for $p, q = 0, 1, 2, \cdots$.

Whether computing speed or accuracy, (65) is much better than (52).

On the other hand, the confluent hypergeometric function of the second kind is defined by

$$\Psi(\nu,\mu;z) = \frac{1}{\Gamma(\nu)} \int_0^\infty t^{\nu-1}(1+t)^{\mu-\nu-1} e^{-zt} dt, \ Re\nu > 0. \tag{66}$$

Through variable substitution, (66) can be written as

$$\begin{aligned}\Psi(\nu,\mu;z) &= \frac{z^{1-\mu}}{\Gamma(\nu)} \int_0^\infty \frac{t^{\nu-1}e^{-t}}{(z+t)^{\nu+1-\mu}} dt\\ &= z^{1-\mu}\widehat{\Gamma}(\nu)\Gamma_{\nu+1-\mu}(\nu,z),\end{aligned} \tag{67}$$

where $\Gamma_\mu(\nu,z)$ is one of the generalized incomplete gamma functions (5). Using the following identities

$$\begin{aligned}\Psi(\nu,\mu;z) &= \frac{\Gamma(1-\mu)}{\Gamma(\nu+1-\mu)}\Phi(\nu,\mu;z)\\ &+\frac{z^{1-\mu}\Gamma(\mu-1)}{\Gamma(\nu)}\Phi(\nu+1-\mu,2-\mu;z),\end{aligned} \tag{68}$$

for $\nu, \mu, 1-\mu, \nu-\mu+1 \neq 0, -1, -2, \cdots$ and Leibniz's derivation rule on ν, μ, we have

$$\begin{aligned}\Psi_{p,q}(\nu,\mu;z) &= \frac{\partial^{p+q}}{\partial\nu^p \partial\mu^q}\Psi(\nu,\mu;z)\\ &= (-1)^q q! z^{1-\mu} \sum_{k=0}^{p} \begin{pmatrix} p \\ k \end{pmatrix} \widehat{\Gamma}^{(k)}(\nu)\\ &\times \sum_{l=0}^{q} \sum_{u=0}^{q-l} \frac{\Phi_{p-k+u,q-l-u}(\nu+1-\mu,2-\mu;z)}{(q-l-u)!u!}\\ &\times \sum_{v=0}^{l} \frac{(-1)^v \ln^{l-v} z \Gamma^{(v)}(\mu-1)}{(l-v)!v!}\\ &+q! \sum_{k=0}^{p} \begin{pmatrix} p \\ k \end{pmatrix} \sum_{l=0}^{q} \frac{(-1)^l \Phi_{p-k,q-l}(\nu,\mu;z)}{(q-l)!}\\ &\times \sum_{v=0}^{l} \frac{\Gamma^{(l-v)}(1-\mu)\widehat{\Gamma}^{(k+v)}(\nu+1-\mu)}{(l-v)!v!},\end{aligned} \tag{69}$$

for $p, q = 0, 1, 2, \cdots$.

Moreover, through variable substitution, (66) can also be written as

$$\Psi(\nu,1-\mu;z) = \frac{e^z}{\Gamma(\nu)}B(\nu,\mu,z), \qquad Re\nu > 0. \tag{70}$$

where

$$B(\nu,\mu,z)=\int_0^1 u^{\nu-1}(1-u)^{\mu-1}e^{-\frac{z}{1-u}}du \tag{71}$$

is an extension of the Beta function.

By (67) and (70), we have

$$\begin{aligned} B(\nu,\mu,z) &= e^{-z}B(\nu,\mu)\Phi(\nu,1-\mu;z) \\ &+e^{-z}z^{\mu}\Gamma(-\mu)\Phi(\nu+\mu,1+\mu;z) \end{aligned} \tag{72}$$

Therefore, by the Leibniz' derivation rule on ν,μ for (72), we can obtain the following the derivatives of the extension Beta function $B(\nu,\mu,z)$:

$$\begin{aligned} B_{p,q}(\nu,\mu,z) &= \frac{\partial^{p+q}}{\partial\nu^p\partial\mu^q}B(\nu,\mu,z) \\ &= e^{-z}\sum_{k=0}^{p}\begin{pmatrix} p \\ k \end{pmatrix}\sum_{l=0}^{q}\begin{pmatrix} q \\ l \end{pmatrix} \\ &\times(-1)^l B_{p-k,q-l}\Phi_{k,l}(\nu,1-\mu;z) \\ &+e^{-z}z^{\mu}\sum_{l=0}^{q}\begin{pmatrix} q \\ l \end{pmatrix}\sum_{v=0}^{q-l}\begin{pmatrix} q-l \\ v \end{pmatrix} \\ &\times\Phi_{p+q-l-v,v}(\nu+\mu,1+\mu;z)\sum_{u=0}^{l}\begin{pmatrix} l \\ u \end{pmatrix} \\ &\times(-1)^{l-u}\Gamma^{(l-u)}(-\mu)\ln^u z, \end{aligned} \tag{73}$$

for $p,q=0,1,2,\cdots$.

V. Conclusion

In this paper, some recursive relations of the derivatives of the gamma function $\Gamma(\alpha)$ and incomplete gamma function $\Gamma(\alpha,z)$ are obtained for complex number $\alpha\neq 0,-1,-2,\cdots$. Thus, the partial derivative $\frac{\partial^{p+q}}{\partial\nu^p\partial\mu^q}B(\nu,\mu)$ of the Beta function $B(\nu,\mu)$ can also be represented for $p,q=0,1,2,\cdots$. Based on these results, the partial derivatives of the extended, generalized complete and incomplete gamma functions are obtained. Moreover, the partial derivative of confluent hypergeometric function is also considered.

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